SOVIET PHYSICS JETP VOLUME 13, NUMBER 6 DECEMBER, 1961

# *THEORY OF THE ABSORPTION OF SOUND IN DILUTE* $He^3$ *IN* He *II SOLUTIONS*

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Submitted to JETP editor November 10, 1960; resubmitted February 18, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **40**, 1705-1709 (June, 1961)

We have evaluated the absorption coefficient of first sound due to second viscosity in dilute solutions of $He^3$ in He II. The processes that lead to the establishment of equilibrium in the phonon and roton numbers are regarded as the slow processes which cause second viscosity. We compare our results with experiments.

KHALATNIKOV[1] obtained phenomenologically the following expression for the coefficient of absorption of first sound of frequency $\omega$ in dilute solutions of $He^3$ in He II:

$$\alpha(x)=\frac{\omega^2}{2\rho c^3}\left\{\frac{4}{3}\eta+\zeta_2+\frac{Dc^2}{\rho}\left(\frac{\partial\rho}{\partial x}\right)^2\frac{m_3x}{kT}\right\}, \tag{1}$$

where $\rho$ is the density of the solution, c the sound velocity in it, $\eta$, $\zeta_2$, and D respectively the coefficients of first viscosity, second viscosity, and diffusion, x the $He^3$ concentration, and $m_3$ the mass of a $He^3$ atom.

The second term in (1), which is connected with the second viscosity, is the main contribution to the absorption coefficient. The processes that establish the number of excitations (see reference 2) are, as in the case of pure He II, the slow processes that cause second viscosity.

The measurements by Harding and Wilks[3] have shown that the absorption of sound decreases appreciably when the $He^3$ concentration is increased. This indicates that the coefficient of second viscosity $\zeta_2$ must depend strongly on the concentration. In the present paper we evaluate the sound absorption connected with second viscosity.

Equilibrium in solution for a given number of excitations is much faster established than equilibrium in the number of excitations. We shall therefore assume that the first equilibrium is established, i.e., that the excitation distribution functions are the equilibrium ones, but that the chemical potential is different from zero. As far as impurity excitations are concerned, their number is given (it is equal to the number of impurity atoms) and we shall thus assume the "impurity gas" to be in complete equilibrium.

The basic processes which in pure He II accompany a change in the number of excitations are the five-phonon process and the processes where a phonon-roton scattering changes a phonon into a roton, and vice versa. We must elucidate the influence of impurities on the speed with which the number of phonons and rotons changes. In the following we consider the most probable of the processes which are of interest to us: scattering by impurities involving three phonons or involving roton-phonon interchanges.

## IMPURITY SCATTERING INVOLVING THREE PHONONS

The Hamiltonian for the interaction between impurities and phonons is of the form (see reference 4)

$$V=-\frac{1}{2}(\mathbf{Pv}+\mathbf{vP})+\frac{\partial\Delta}{\partial\rho}\rho'+\frac{1}{2}\frac{\partial^2\Delta}{\partial\rho^2}\rho'^2+\frac{1}{6}\frac{\partial^3\Delta}{\partial\rho^3}\rho'^3, \tag{2}$$

where $\Delta$ is the zero energy of the impurity, $\rho'$ the change in density due to the presence of phonons, $\mathbf{v}$ the velocity of the medium, connected with the presence of phonons, and $\mathbf{P} = -i\hbar\nabla$ the momentum operator of the impurity.

Our problem consists in calculating the probability that an impurity makes a transition from a state i with momentum $\mathbf{P}$ into a state f with momentum $\mathbf{P}'$ during which a phonon with momentum $\mathbf{p}_1$ is absorbed and two phonons with momenta $\mathbf{p}_2$ and $\mathbf{p}_3$ are emitted.

We apply the method of approximate quantization of the nonlinear equations of hydrodynamics, developed by Landau and Khalatnikov,[5] and we perform our calculations in a manner similar to theirs, taking terms of fourth order in $\rho'$ into account (this is necessary because three phonons are involved in the process considered by us); we then get for the matrix element for the above-mentioned transition the expression

$$M\equiv V_{fi}=\frac{A}{c}\left(\frac{\rho}{2c}\right)^{3/2}(p_1p_2p_3)^{1/2}\{M_{12}+M_{13}+M_{23}\},$$

where

$$M_{12} = -\frac{(p_1 - p_2)^2}{p_1 p_2 [1 - \mathbf{n}_1\mathbf{n}_2 + 3\gamma (p_1 - p_2)^2]} \left\{ \frac{P}{\rho^2} (\mathbf{n}_1 + \mathbf{n}_3, \mathbf{m}) \right.$$
$$\left. \times (1 + \mathbf{n}_1\mathbf{n}_3) - \frac{\partial\Delta}{\partial\rho}\left(\frac{3\mathbf{n}_1\mathbf{n}_3}{\rho c'} - \frac{B}{c^3}\right) + \frac{1}{c}\frac{\partial^2\Delta}{\partial\rho^2} \right\};$$

$M_{13}$ is obtained from $M_{12}$ by the substitution $\mathbf{n}_2 \rightarrow \mathbf{n}_3$, $\mathbf{n}_3 \rightarrow \mathbf{n}_2$, $\mathbf{p}_2 \rightarrow \mathbf{p}_3$; $M_{23}$ is obtained from $M_{12}$ by the substitution $\mathbf{n}_1 \rightarrow \mathbf{n}_3$, $\mathbf{n}_3 \rightarrow \mathbf{n}_1$, $\mathbf{p}_1 \rightarrow -\mathbf{p}_3$;

$$A = c^2/\rho + {}^1/_2\, \partial c^2/\partial\rho, \qquad B = c^2/\rho - \partial c^2/\partial\rho;$$

the constant $\gamma$ is defined by the dependence of the phonon energy $\epsilon_{ph}$ on the momentum p: $\epsilon_{ph} = c(p - \gamma p^3)$: $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, and $\mathbf{m}$ are unit vectors in the directions of the momenta $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{P}$, respectively.

The absolute square of the matrix element M can easily be integrated over the directions of the vectors $\mathbf{m}$, $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$:

$$W \equiv \int |M|^2 do\, do_1\, do_2\, do_3 = \frac{A^2}{c^2}\left(\frac{\rho}{2c}\right)^3 \frac{(4\pi)^4}{6\gamma} p_1 p_2 p_3 \left(K + \frac{4P^2}{3\rho^4}\right)$$
$$\times \left\{ \frac{(p_1 - p_2)^2}{p_1^2 p_2^2} + \frac{(p_1 - p_3)^2}{p_1^2 p_3^2} + \frac{(p_2 + p_3)^2}{p_2^2 p_3^2} \right\},$$
$$K = \left(\frac{\partial\Delta}{\partial\rho}\right)^2 \left(\frac{3}{\rho^2 c^2} + \frac{B^2}{c^6}\right) + \frac{1}{c^2}\left(\frac{\partial^2\Delta}{\partial\rho^2}\right)^2 + \frac{2B}{c^4}\frac{\partial\Delta}{\partial\rho}\frac{\partial^2\Delta}{\partial\rho^2}. \tag{3}$$

In integrating we have used the fact that $\gamma p^2 \ll 1$ (p is a momentum of the order of the average phonon momentum).

The probability dw for the transition in which we are interested is determined by the perturbation-theory formula

$$dw = 2\pi\hbar^{-1} |M|^2 \delta(E_f - E_i)\, d\mathbf{p}_2\, d\mathbf{p}_3/(2\pi\hbar)^6,$$

where $E_f$ and $E_i$ are the energies of the final and initial states.

If one uses the momentum conservation law one sees easily that as P, $p_1$, $p_2$, $p_3 \ll mc$, where m is the effective impurity mass

$$E_f - E_i \approx c(p_1 - p_2 - p_3).$$

We are interested in the rate of change of the total number of phonons $N_{ph}$ per unit time, due to the process considered:

$$\dot{N}_{ph} = \iiint \{N(P)\, n(p_1)\, [1 + n(p_2)]\, [1 + n(p_3)]$$
$$- N(P')\, n(p_2)\, n(p_3)\, [1 + n(p_1)]\}\, dw\, d\mathbf{P}\, d\mathbf{p}_1\, (2\pi\hbar)^{-6}. \tag{4}$$

Here

$$N(P) = (2\pi\hbar)^3 \frac{x\rho}{m_3} (2\pi mkT)^{-3/2} \exp\left\{-\frac{P^2}{2mkT}\right\}$$

is the impurity distribution function and n(p) the phonon distribution function.

If the number of phonons is not equal to its equilibrium value, this means that n(p) contains a non-vanishing chemical potential $\mu_{ph}$

$$n(p) = [\exp\{[\varepsilon_{ph}(p) - \mu_{ph}]/kT\} - 1]^{-1}.$$

If the deviation from equilibrium is a small one,

$$n(p) = n_0(p) + n_0(p)\, [1 + n_0(p)]\, \mu_{ph}/kT, \tag{5}$$

where $n_0(p)$ is the equilibrium distribution function.

Using (5) we transform (4) to

$$\dot{N}_{ph} = -\frac{\mu_{ph}}{kT} \iiint N(p)\, n_0(p_1)\, [1 + n_0(p_2)]$$
$$\times [1 + n_0(p_3)]\, dw \frac{d\mathbf{P}\, d\mathbf{p}_1}{(2\pi\hbar)^6}. \tag{6}$$

Integrating in (6) over the angles, integrating over $p_1$ to get rid of the $\delta$-function, and using (3) we get

$$\dot{N}_{ph} = -(2\pi\hbar)^{-12} \frac{\mu_{ph}}{kT} \int_0^\infty dP \int_0^\infty dp_2 \int_0^\infty dp_3 N(P)\, P^2 (p_2 + p_3)^2 p_2^2 p_3^2$$
$$\times e^{c(p_2+p_3)/kT} [e^{c(p_2+p_3)/kT} - 1]^{-1}$$
$$\times [e^{cp_2/kT} - 1]^{-1} [e^{cp_3/kT} - 1]^{-1} W. \tag{7}$$

We can without great loss of accuracy neglect in the denominator of (7) unity as compared to $\exp\{c(p_2 + p_3)/kT\}$, and we can then again perform the integration in (7) and we get

$$\dot{N}_{ph} = -\Gamma_{phi}\mu_{ph}, \quad \Gamma_{phi} = x \frac{64\pi^2\beta\rho Q}{(2\pi\hbar)^9 m_3 c^{10}\hbar}\left(\frac{K}{4} + \frac{mkT}{\rho^4}\right)(kT)^8. \tag{8}$$

In (8) we have used the notation

$$\beta = 36.3, \qquad Q = \frac{A^2}{c^2}\left(\frac{\rho}{2c}\right)^3 \frac{(4\pi)^4}{6\gamma}.$$

The quantity K contains the derivatives $\partial\Delta/\partial\rho$ and $\partial^2\Delta/\partial\rho^2$ which we do not know and which we shall therefore in the following determine from the experiments.

The three-phonon process can occur also in pure He II through scattering by a roton, but calculations show that its contribution is small compared with the contribution from the five-phonon process (at low temperatures) or as compared to that from the roton-scattering process involving a roton-phonon transition (at higher temperatures).

## IMPURITY SCATTERING INVOLVING A ROTON-PHONON TRANSITION

We must now calculate the probability for a transition of the impurity from a state with momentum $\mathbf{P}$ to a state with momentum $\mathbf{P}'$ during which a phonon with momentum $\mathbf{p}$ is absorbed and a roton with momentum $\mathbf{P}_1$ is emitted.

It is clear that the phonon that takes part in this process must be very energetic. We shall assume that it interacts with the impurity in the same way as the roton, but that it has a different dispersion law. We get thus for the probability of the above mentioned process

$$dw = 2\pi\hbar^{-1} |v_{ir}|^2 \delta(E_1 - E_2)\, d\mathbf{P}_1/(2\pi\hbar)^3, \tag{9}$$

where $v_{ir} = 8 \times 10^{-38}$ erg-cm$^3$ is the impurity-roton interaction constant, and $E_1$ and $E_2$ are the energies of the initial and final states.

We are interested in the rate of change in the number of excitations due to this process; this rate of change is, moreover, determined by the formula

$$\dot{N}_r = -\dot{N}_{ph} = \frac{\mu_{ph} - \mu_r}{kT} \iiint n_0(p)\, N(P)\, dw \frac{dp dP}{(2\pi\hbar)^6}, \qquad (10)$$

where $\mu_r$ is the roton chemical potential.

Khalatnikov[2] has shown that we can estimate in this way only an upper limit for the quantities which we evaluate, i.e., we get from (10)

$$\dot{N}_r = -\dot{N}_{ph} = \Gamma_{phri}(\mu_{ph} - \mu_r), \qquad (11)$$

where

$$\Gamma_{phri} \lesssim x \frac{2\sqrt{2\pi}}{\hbar^4} |v_{ir}|^2 \frac{P_0^2 \Delta_r^2 \sqrt{m_r}}{(2\pi\hbar c)^3} \frac{\rho}{m_3} (kT)^{-1/2} e^{-\Delta_r/kT}$$

$$= 1.1 \cdot 10^{51} x T^{-1/2} e^{-\Delta_r/kT}. \qquad (11')$$

Here, $P_0$, $\Delta_r$ and $m_r$ are the roton spectrum parameters. The exact value of $\Gamma_{phri}$ will be determined from experiments.

## THE SOUND ABSORPTION COEFFICIENT

The following expressions hold for the rate of change of the number of phonons and rotons in pure He II:[2]

$$\dot{N}_{ph} = -(\Gamma_{ph} + \Gamma_{phr})\mu_{ph} + \Gamma_{phr}\mu_r,$$
$$\dot{N}_r = -\Gamma_{phr}\mu_r + \Gamma_{phr}\mu_{ph}, \qquad (12)$$

where one must take for $\Gamma_{ph}$ and $\Gamma_{phr}$ the values

$$\Gamma_{ph} = 6{,}3 \cdot 10^{42} T^{11}, \qquad \Gamma_{phr} = 6{,}5 \cdot 10^{50} e^{-2\Delta_r/T}. \qquad (13)$$

When there are impurities present we must according to (8) and (11) replace $\Gamma_{ph}$ by $\Gamma_{ph} + \Gamma_{phi}$ and $\Gamma_{phr}$ by $\Gamma_{phr} + \Gamma_{phri}$ in (12), i.e., we have now instead of (12)

$$\dot{N}_{ph} = -(\Gamma_{ph} + \Gamma_{phi} + \Gamma_{phr} + \Gamma_{phri})\mu_{ph} + (\Gamma_{phr} + \Gamma_{phri})\mu_r,$$
$$\dot{N}_r = -(\Gamma_{phr} + \Gamma_{phri})\mu_r + (\Gamma_{phri} + \Gamma_{phr})\mu_{ph}. \qquad (14)$$

The coefficient for the absorption of sound of not too high a frequency in pure He II, due to second viscosity, is given by the equation (see reference 2)

$$\tilde{\alpha}(0) = \frac{\omega^2}{2\rho c^3} \left[ \frac{1}{\Gamma_{ph}} \left( \frac{\partial p}{\partial \mu_r} + \frac{\partial p}{\partial \mu_{ph}} \right)^2 + \frac{1}{\Gamma_{phr}} \left( \frac{\partial p}{\partial \mu_r} \right)^2 \right]_{\rho, S_0},$$

where $S_0$ is the entropy per unit mass of pure He II. As the sound velocity is independent of the impurity concentration[1] at low concentrations we can compare (12) and (14) and immediately write down for the solution

$$\tilde{\alpha}(x) = \frac{\omega^2}{2\rho c^3} \left[ \frac{1}{\Gamma_{ph} + \Gamma_{phi}} \left( \frac{\partial p}{\partial \mu_r} + \frac{\partial p}{\partial \mu_{ph}} \right)^2 + \frac{1}{\Gamma_{phr} + \Gamma_{phri}} \left( \frac{\partial p}{\partial \mu_r} \right)^2 \right]_{\rho, S, x}, \qquad (15)$$

where S is the entropy per unit mass of the solution.

We transform this last expression in terms of p, T, x, $\mu_r$, and $\mu_{ph}$ using the fact that the derivative $(\partial\rho/\partial T)_{p,x}$ is small:

$$\tilde{\alpha}(x) = \frac{\omega^2 c}{2\rho} \left[ \frac{1}{\Gamma_{ph} + \Gamma_{phi}} \left( \frac{\partial \rho}{\partial \mu_r} + \frac{\partial \rho}{\partial \mu_{ph}} \right)^2 + \frac{1}{\Gamma_{phr} + \Gamma_{phri}} \left( \frac{\partial \rho}{\partial \mu_r} \right)^2 \right]_{p, T, x}. \qquad (16)$$

Using the thermodynamic identity (for constant concentration)

$$d\Phi = -S dT + \frac{1}{\rho} dp - \frac{N_{ph}}{\rho} d\mu_{ph} - \frac{N_r}{\rho} d\mu_r,$$

we get the following expressions for the derivatives occurring in (16)

$$\partial\rho/\partial\mu_{ph} = \rho\,(\partial N_{ph}/\partial p)_{T,x} - N_{ph}/c^2,$$
$$\partial\rho/\partial\mu_r = \rho\,(\partial N_r/\partial p)_{T,x} - N_r/c^2. \qquad (17)$$

One can conclude from the work of Dash and Taylor[6] that

$$\frac{\partial}{\partial x}\frac{\partial \rho}{\partial \mu_{ph}} \sim \frac{\partial \rho}{\partial \mu_{ph}}, \qquad \frac{\partial}{\partial x}\frac{\partial \rho}{\partial \mu_r} \sim \frac{\partial \rho}{\partial \mu_r},$$

but it follows from (8) that

$$\Gamma_{phi}/x > 2{,}7 \cdot 10^{44} T^9 \gg \Gamma_{ph} \text{ when } T \sim 1^\circ K.$$

The main concentration dependence of $\tilde{\alpha}(x)$ comes thus from taking $\Gamma_{phi}$ and $\Gamma_{phri}$ into account and we can thus in (16) put

$$\partial\rho/\partial\mu_r = (\partial\rho/\partial\mu_r)_0, \qquad \partial\rho/\partial\mu_{ph} = (\partial\rho/\partial\mu_{ph})_0,$$

where $(\partial\rho/\partial\mu)_0$ is the value for pure He II. The values for $(\partial\rho/\partial\mu_r)_0$ and $(\partial\rho/\partial\mu_{ph})_0$ were calculated by Khalatnikov:[2]

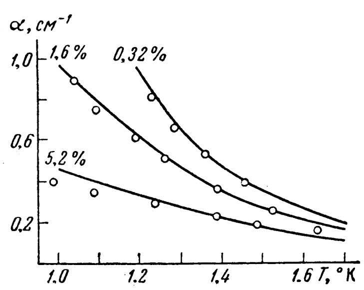
$$\left(\frac{\partial \rho}{\partial \mu_r}\right)_0 = -N_r \left[ \frac{\Delta_r}{T} \left( \frac{N_r}{N_{ph}} - 23.2 \right) - 22.5 \right] \Big/ c^2 \left[ \left( \frac{\Delta_r^2}{T^2} + \frac{\Delta_r}{T} + \frac{3}{4} \right) \frac{N_r}{N_{ph}} + \frac{\pi^4}{9} \right], \qquad (18)$$

$$\left(\frac{\partial \rho}{\partial \mu_{ph}}\right)_0 = -N_r \left[ 6.4 \frac{\Delta_r^2}{T^2} + 9.4 \frac{\Delta_r}{T} + 9.3 \right] \Big/ c^2 \left[ \left( \frac{\Delta_r^2}{T^2} + \frac{\Delta_r}{T} + \frac{3}{4} \right) \frac{N_r}{N_{ph}} + \frac{\pi^4}{9} \right].$$

Thus:

$$\tilde{\alpha}(x) = \frac{\omega^2 c}{2\rho} \left[ \frac{1}{\Gamma_{ph} + \Gamma_{phi}} \left[ \left(\frac{\partial \rho}{\partial \mu_r}\right)_0 + \left(\frac{\partial \rho}{\partial \mu_{ph}}\right)_0 \right]^2 + \frac{1}{\Gamma_{phr} + \Gamma_{phri}} \left(\frac{\partial \rho}{\partial \mu_r}\right)_0^2 \right]. \qquad (19)$$

In the figure we have given a comparison of the values of the sound absorption coefficient deter-



mined from Eq. (19) (adding to it the quantity $2\omega^2\eta/3\rho c^3$ which at high temperatures becomes appreciable) with the experimental data of Harding and Wilks,[3] where we have put

$$\Gamma_{\text{phi}} = 1.4 \cdot 10^{44} x T^8 (1.3 + 1.9T),$$

which corresponds to

$$K\rho^4 = 5.3 \cdot 10^{-39} \text{g}^2\,\text{cm}^2\,\text{sec}^{-2}$$

$$\Gamma_{\text{phri}} = x \cdot 0.7 \cdot 10^{50} T^{-1/2} e^{-\Delta_r/T};$$

the value of $\eta$ was evaluated by Zharkov.[7]

In conclusion the author expresses his gratitude to I. M. Khalatnikov for his assistance with this paper and to V. N. Zharkov for valuable remarks.

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Translated by D. ter Haar
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