BRANCHING OF ELECTRON AND PHOTON GREEN'S FUNCTIONS

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The branchings of the electron Green's function G(p) and the photon Green's function D(k) at the respective points $p^2 = m^2$ and $k^2 = 0$ are discussed. In electrodynamics the branching of G(p) indicates a nonstationary behavior of the amplitude of the one-electron state, and with the usual gauge for the potentials this amplitude increases with the time; this is connected with the indefinite metric. A branching of D(k) at $k^2 = 0$ arises as a consequence of the weak three-photon and photon-neutrino interactions, but the singularity of the function at this point is a weak one, so that the one-photon state remains stationary.

In the nonrelativistic case the Fourier component of the Green's function

$$G(\mathbf{p}, \tau) = \frac{1}{2\pi} \int G(\mathbf{p})\, e^{-ip_0\tau}\, dp_0$$

is equal to the probability amplitude for finding the initial state $\Phi_{0\mathbf{p}} = a^+_{\mathbf{p}}|\mathrm{vac}\rangle$ in the state

$$\Phi_{\mathbf{p}}(\tau) = \exp(-iH\tau)\,\Phi_{0\mathbf{p}},$$

obtained from $\Phi_{0\mathbf{p}}$ after the time τ.[1] We shall derive analogous relations in the relativistic case. If the Schrödinger operator ψ of the spinor field has the form

$$\psi(\mathbf{x}) = \sum_{\mathbf{p}\lambda} (u_{\mathbf{p}\lambda} e^{i\mathbf{px}} a_{\mathbf{p}\lambda} + v^*_{-\mathbf{p}\lambda} e^{-i\mathbf{px}} b^+_{\mathbf{p}\lambda}),$$

then

$$G_1(\mathbf{p}, \tau) = \langle a_{\mathbf{p}\lambda} e^{-iH\tau} a^+_{\mathbf{p}\lambda}\rangle = \frac{1}{4\varepsilon_p}\,\mathrm{Sp}\, G(-\mathbf{p}, \tau)(\hat{p} + m),$$

$$G_2(\mathbf{p}, \tau) = \langle b^+_{\mathbf{p}\lambda} e^{-iH\tau} b_{\mathbf{p}\lambda}\rangle = \frac{1}{4\varepsilon_p}\,\mathrm{Sp}\, G(\mathbf{p}, \tau)(\hat{p} - m). \quad (1)$$

Here $\tau > 0$, the averaging is over the physical vacuum, and $p_0 = \epsilon_{\mathbf{p}} = (\mathbf{p}^2 + m^2)^{1/2}$; the notations are those of Feynman. In electrodynamics the function G(p) has a well known branching at the point $p^2 = m^2$,[2,3]

$$\underset{p^2 \to m^2}{G(p)} = i\,\frac{\hat{p} + m}{(p^2 + m^2)^{1+\beta}}, \qquad \beta = \frac{\alpha}{2\pi}(3 - d_{l0}). \quad (2)$$

For $d_{l0} < 3$ (for example, for the commonly used $d_{l0} = 1$ or $d_{l0} = 0$) we have $\beta > 0$, and according to Eqs. (1) and (2) this means an increase of the amplitude of the one-electron state with time:

$$G_1(\mathbf{p}, \tau) = \mathrm{const}\cdot\exp(-i\varepsilon_p\tau + \beta\ln\varepsilon_p\tau). \quad (3)$$

The increase of the amplitude, Eq. (3), is connected with the indefinite metric in electrodynamics. In addition to the "no-quantum" state $\Phi_{0\mathbf{p}\lambda} = a^+_{\mathbf{p}\lambda}|\mathrm{vac}\rangle$ the state vector contains states of negative norm with longitudinal and scalar quanta, which are physically indistinguishable from $\Phi_{0\mathbf{p}\lambda}$ by gauge invariance. The complete norm of the state $\exp(-iH\tau)\times\Phi_{0\mathbf{p}\lambda}$ is conserved, according to general theorems.[4]

In the usual case of states with positive norms the replacement of the pole of the Green's function by a branch point describes a damping of the one-particle state. Such a branching is due to the presence of a spectrum of two-particle excitations joining on continuously to the one-particle ones. Since through the weak interaction a photon can go over into two photons and two neutrinos, a replacement of the pole by a branch point must occur in its Green's function $D_{rn}(k)$. If the quantity $k^2D_{rn}(k)$ went to zero for $k^2 \to 0$, the photon would be in principle unstable, dissociating in time into a pair of massless particles with the same direction of motion. Let us find the form of the branching of D(k).

In lowest order in k, when we take into account CP invariance and symmetry in the particles, the three-photon vertex part must have the form

$$e_{1i}e_{2k}e_{3l}\Gamma_{ikl} = a\varepsilon_{iklm}\,[e_{1i}e_{2k}e_{3l}(k_{1m}(k_3^2 - k_2^2) + k_{2m}(k_1^2 - k_3^2) + k_{3m}(k_2^2 - k_1^2)) - 2\,(e_{1i}e_{2k}k_{1l}k_{2m}(e_3k_3) + e_{2i}e_{3k}k_{2l}k_{3m}(e_1k_1) + e_{3i}e_{1k}k_{3l}k_{1m}(e_2k_2))]. \quad (4)$$

Here ϵ_{iklm} is the unit antisymmetric tensor, e_i, k_i are the polarization and momentum vectors of the quanta, and $k_1 + k_2 + k_3 = 0$. The expression (4) is gauge invariant: in a longitudinal external field, for example, $e_3 = k_3 f(k_3^2)$, the scattering amplitude of a real photon with $k_1^2 = k_2^2 = 0$ is zero; the general case can be reduced to this by means of dispersion relations. If we derive the interaction (4) from the known weak and electromagnetic interactions, we get for the constant a the estimate $a \sim e^3G^2\Lambda^2\mu^{-4}$, where $G^2 \sim 10^{-13}$, μ is the mass of the π meson, and Λ is the momentum at which the four-fermion interaction is effectively cut off.[5]

Similarly, the amplitude for transition of a quantum into two neutrinos must have the form[6]

$$e_r \Gamma_r = b_1 e_r \sigma_r + b_2 \sigma_r e_n (\delta_{rn} k^2 - k_r k_n). \quad (5)$$

Here $\sigma_r = (\boldsymbol{\sigma}, 1)$ is the two-rowed matrix spin vector; we are using the two-component representation. The case $b_1 \neq 0$ corresponds to a charged massless particle, and we have treated it earlier;[7] we here take $b_1 = 0$. On the assumption of a direct $(e\nu, e\nu)$ interaction the quantity b_2 has the order of magnitude $b_2 \sim G e \mu^{-2} \ln(\Lambda/m)$.

Setting, as usual,

$$D_{rn}(k) = k^{-4}\,[d_t(\delta_{rn} k^2 - k_r k_n) + d_l k_r k_n],$$
$$d_t = [1 - 4\pi\Pi(k^2)]^{-1},$$

we find that the contribution of the processes in which we are interested to the imaginary part of Π is

$$2\,\mathrm{Im}\,\Pi(k^2) = \begin{cases} [4a^2 + b_2^2/12\pi]\,k^4, & k^2 > 0, \\ 0, & k^2 < 0. \end{cases}$$

From this we have

$$d_t^{-1}(k^2) = 1 - k^4 (8a^2 + b_2^2/6\pi)(\ln(-k^2) + \mathrm{const}). \quad (6)$$

Because transitions to two-particle states are strongly forbidden, the singularity in d_t is rather weak, and there is no "infrared" damping. Thus the hypothetical case $b_1 \neq 0$,[7] for which $d_t^{-1} = 1 - (b_1^2/6\pi)\ln(-k^2)$, is the only one in which the photon is not rigorously stable.

We note, finally, that the expression (4) can be interesting in itself as the phenomenological amplitude of a "purely electromagnetic" process that does not conserve parity.[8,6] For example, it describes a rotation of the plane of polarization of a photon in scattering by a Coulomb field. These effects are small, however, if we use the estimate for a given here.

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