SINGULARITY IN THE SCHWARZSCHILD SOLUTION OF THE GRAVITATION EQUATIONS

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Schwarzschild's solution of the gravitational field equations has a singularity at the gravitational radius. It is shown that this singularity can be removed by a suitable choice of coordinate system. Examples of such coordinate systems are given.

THE centrally symmetric solution of Einstein's field equations in empty space is (see reference 1, Sec. 97)

$$ds^2 = \left(1 - \frac{1}{r}\right) dt^2 - \left(1 - \frac{1}{r}\right)^{-1} dr^2 - r^2 d\sigma^2,$$
$$d\sigma^2 = d\vartheta^2 + \sin^2\vartheta \cdot d\varphi^2. \tag{1}$$

The units have been chosen so that $c = 1$ and $\alpha = 2kM/c^2 = 1$. The solution (1) has a singularity at $r = 0$ and also at $r = 1$, which is the so-called gravitational radius. Finkelstein has attempted to remove the singularity at the gravitational radius by a suitable choice of coordinate system.[2] In his coordinates the line element is

$$ds^2 = \left(1 - \frac{1}{r}\right) dt^2 + \frac{2}{r} dt dr + \left(1 + \frac{1}{r}\right) dr^2 - r^2 d\sigma^2. \tag{2}$$

This line element suffers from the disadvantage that g_{tt} changes sign at $r = 1$, which means that for $r > 1$ t is a time-like coordinate, while for $r < 1$, t is a space-like coordinate. It should also be pointed out that Finkelstein's coordinates are not orthogonal. Fronsdal[3] embedded the Scharzschild space in a flat six dimensional manifold and showed that for a stationary metric (i.e. independent of coordinate time), the singularity at $r = 1$ could not be removed by a coordinate transformation. If nonstationary metrics were allowed, then the singularity at $r = 1$ could be removed. Although Fronsdal showed how this could be done, he did not actually do it.

The purpose of the present note is to present some coordinate systems which do not have the singularity.

Consider the coordinate transformation

$$\tau = t - \int \frac{r dr}{(r-1) f(r)}, \qquad \xi = \int \frac{r f(r)}{r-1} dr - t, \tag{3}$$

where $f(r)$ satisfies the conditions $f'(1) \neq 0$, $f^2(1) = 1$, $f^2(r) > 1$ for $r > 1$, $f^2(r) < 1$ for $r < 1$.

In the new coordinate system the line element is

$$ds^2 = \frac{r-1}{r(1-f^{-2}(r))}\left(d\tau^2 - \frac{d\xi^2}{f^2(r)}\right) - r^2 d\sigma^2. \tag{4}$$

$f(r)$ has been chosen so that $1 - f^{-2}$ has a first order zero at $r = 1$, so that the line element (4) does not have a singularity at $r = 1$.

For the particular case $f(r) = r^2 - r + 1$, we have

$$\tau = t - \frac{1}{2}\ln\frac{(r-1)^2}{r^2 - r + 1} + \frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{2}{\sqrt{3}}\left(r - \frac{1}{2}\right)\right],$$
$$\xi = r^3/3 + r + \ln|r - 1| - t. \tag{5}$$

The transformation (5) is single valued (in both directions) for all values of (r, t). Upon carrying out the further transformation $\rho = (3\xi)^{1/3}$, the expression (4) takes the form

$$ds^2 = \frac{(\psi^2 - \psi + 1)^2}{(\psi^2 - \psi + 2)\psi^2} d\tau^2 - \frac{\rho^4 d\rho^2}{(\psi^2 - \psi + 2)\psi^2} - \psi^2 d\sigma^2, \tag{6}$$

in the coordinates (ρ, τ), where $\psi = \psi(\rho^3/3 + \tau)$ and $\psi(y)$ is defined by the equation

$$y = \frac{\psi^3}{3} + \psi + \frac{1}{2}\ln(\psi^2 - \psi + 1) + \frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{2}{\sqrt{3}}\left(\psi - \frac{1}{2}\right)\right]. \tag{7}$$

Since the right hand side of (7) is a monotonic function of ψ, $\psi(y)$ is a single valued function of its argument. For finite τ and $\rho \to \infty$ (6) may be written in the approximate form

$$ds^2 = (1 - 1/\rho) d\tau^2 + (1 + 1/\rho) d\rho^2 - \rho^2 d\sigma^2. \tag{8}$$

Formula (8) agrees with (1) for large values of $\rho = r$. Thus for $\rho \to \infty$ the coordinate system (ρ, τ) is the same as Schwarzschild's coordinate system, but differs from the latter in that it has no singularity at the gravitational radius.

Upon substituting $f(r) = \sqrt{r}$ in (3) we obtain

$$x^1 = \frac{2\sqrt{r}}{3}(r + 3) + \ln\left|\frac{\sqrt{r} - 1}{\sqrt{r} + 1}\right| - t,$$
$$x^0 = t - 2\sqrt{r} - \ln\left|\frac{\sqrt{r} - 1}{\sqrt{r} + 1}\right|; \tag{9}$$

$$ds^2 = (dx^0)^2 - \left[\tfrac{3}{2}(x^1 + x^0)\right]^{-2/3}(dx^1)^2 - \left[\tfrac{3}{2}(x^1 + x^0)\right]^{4/3} d\sigma^2; \tag{10}$$

$$r = [\tfrac{3}{2}(x^1 + x^0)]^{2/3},$$
$$t = x^0 + 2[\tfrac{3}{2}(x^1 + x^0)]^{1/3} + \ln\left|\frac{[3(x^1 + x^0)/2]^{1/3} - 1}{[3(x^1 + x^0)/2]^{1/3} + 1}\right|. \quad (11)$$

The coordinate system (x^0, x^1) has the following properties: 1) the time coordinate is the time read by a clock moving on the coordinate lines x^1, ϑ, φ = const, 2) the lines x^1, ϑ, φ = const are geodesics, so that the coordinate system (x^0, x^1) can be realized by freely falling bodies carrying clocks. It is a "freely falling" coordinate system. It follows from (11) that the transformation from (x^0, x^1) to (r, t) is single valued. The transformation from (r, t) to (x^0, x^1) is double valued, which corresponds to the two possible square roots of r. This can be explained as follows: in the coordinate system (r, t) a freely falling body $(x^1, \vartheta, \varphi = \text{const})$ will reach $r = 0$ in a finite proper time, and upon "reflection" will retrace its path. This explains the double-valuedness of the transformation (9), with a branch point at $r = 0$. In other words, the point (r, t) can correspond to an incoming particle or an outgoing particle.

In conclusion, we should like to point out (see also reference 4) that the singularity at $r = 1$ in Schwarzschild's coordinate system is associated with the fact that the coordinates change character upon passing through $r = 1$: the space-like coordinates become time-like, while the time-like ones become space-like. This is clear from (1), for if $r > 1$, then r is space-like $(g_{rr} < 0)$ and t is time-like $(g_{tt} > 0)$, while for $r < 1$, the situation is reversed $(g_{rr} > 1,\ g_{tt} < 0)$. The transformation (3) avoids this state of affairs; for suitable choice of $f(r)$ the coordinate ξ is space-like everywhere $(g_{\xi\xi} < 0$ everywhere$)$, while τ is time-like everywhere $(g_{\tau\tau} > 0$ everywhere$)$.

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[1] L. D. Landau and E. M. Lifshitz, Теория поля (Theory of Fields), 2d ed., Gostekhizdat (1960);

[2] D. Finkelstein, Phys. Rev. **110**, 965 (1958).

[3] C. Fronsdal, Phys. Rev. **116**, 778 (1959).

[4] I. D. Novikov, Астрономический журнал **38**, No. 3 (1961), Soviet Astronomy **5**, in press.

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