ELECTROMAGNETIC WAVES IN A MEDIUM POSSESSING A CONTINUOUS ENERGY SPECTRUM. II.

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An electromagnetic-wave theory in which spatial dispersion is taken into account is applied to the case of exciton states. Propagation of waves in an infinite crystal and their transmission through a plane-parallel plate are considered. The absorption of electromagnetic waves in the presence of spatial dispersion is investigated.

IN a previous work[1] the equations of the electromagnetic field in a non-conducting medium in the presence of spatial dispersion were obtained:

$$\Delta\mathcal{E}^{\perp}_{x'}(\mathbf{r}) + \gamma\mathcal{E}^{\perp}_{x'}(\mathbf{r}) + 4\pi\gamma\int K^{\perp}_{x'y'}(\mathbf{r}, \mathbf{r}')\,\mathcal{E}^{\perp}_{y'}(\mathbf{r}')\,d\mathbf{r}' = 0, \quad (1)$$

$$\mathcal{E}^{\parallel}_{x'}(\mathbf{r}) = -4\pi\int K^{\parallel}_{x'y'}(\mathbf{r}, \mathbf{r}')\,\mathcal{E}^{\perp}_{y'}(\mathbf{r}')\,d\mathbf{r}', \quad (2)$$

where $\mathcal{E}^{\perp}(\mathbf{r})$, $\mathcal{E}^{\parallel}(\mathbf{r})$ are the solenoidal and irrotational parts of the electric field intensity, $\gamma = \mu\omega^2/c^2$, and summation over repeated indices is understood. The polarizability kernel is

$$K_{x'y'}(\mathbf{r}, \mathbf{r}') = K^{\perp}_{x'y'}(\mathbf{r}, \mathbf{r}') + K^{\parallel}_{x'y'}(\mathbf{r}, \mathbf{r}')$$

$$= \frac{1}{\hbar\omega}\sum_n (E_n - E_0)\left[\frac{(0\,|\,G_{x'}(\mathbf{r})\,|\,n)\,(n\,|\,G_{y'}(\mathbf{r}')\,|\,0)}{E_n - E_0 - \hbar\omega - i\varepsilon_n(\omega)}\right.$$

$$\left. - \frac{(n\,|\,G_{x'}(\mathbf{r})\,|\,0)\,(0\,|\,G_{y'}(\mathbf{r}')\,|\,n)}{E_n - E_0 + \hbar\omega + i\varepsilon_n(-\omega)}\right], \quad (3)$$

$$\mathrm{div}_{\mathbf{r}} K^{\perp}_{y'}(\mathbf{r}, \mathbf{r}') = 0, \quad \mathrm{curl}_{\mathbf{r}} K^{\parallel}_{y'}(\mathbf{r}, \mathbf{r}') = 0. \quad (4)$$

Here E_0 is the energy of the ground state, E_n is the energy of the excited state; ϵ_n is connected with the lifetime of the state in the usual fashion, and $G(\mathbf{r})$ is the operator of the electric moment per unit volume.

In the present work, we shall apply these results to the case of exciton states of the crystal. This case was considered by Pekar,[2,3] who started from the differential equation derived by him for the polarization and with additional boundary conditions which require the vanishing of the exciton part of the polarization on the surface of the crystal. However, these conditions, even if one uses the wave functions of the exciton states chosen by Pekar, are approximate in the first place, and in the second place, are applicable only in the case of an isolated exciton band, transition to which is permitted in the dipole approximation.

The analysis in this paper is based on the integral-differential equation (1), which automatically takes into account the additional conditions and requires only the choice of the exciton wave functions (Pekar's functions were used for this purpose). This makes it possible to obtain general results which include Pekar's results as a special case and which show where the approximation in the latter lies.

1. WAVES IN AN INFINITE CRYSTAL

The Hamiltonian of the infinite crystal commutes with the translation operators $\hat{T}_{\mathbf{n}}$ ($\mathbf{n}$ is the lattice vector). Therefore, one can use the eigenfunctions of $\hat{T}_{\mathbf{n}}$ in the calculation of matrix elements entering into (3); the corresponding eigenvalues are $e^{i\mathbf{k}\cdot\mathbf{n}}$, where $\mathbf{k}$ is the quasi-momentum of the exciton ($-\pi \le \mathbf{k}\cdot\mathbf{a}_i < \pi$), and $\mathbf{a}_i$ ($i = 1, 2, 3$) is the basic lattice vector. It follows from translational symmetry that the quantity

$$\mathbf{f}(\mathbf{r}, \mathbf{k}, \alpha) = \exp(-i\mathbf{k}\mathbf{r})\,(0\,|\,\mathbf{G}(\mathbf{r})\,|\,\mathbf{k}\alpha),$$

(α are the other variables that enumerate the states along with $\mathbf{k}$) has the same period as the lattice, and can be expanded in the vectors of the reciprocal lattice $\mathbf{b}$:

$$\mathbf{f}(\mathbf{r}, \mathbf{k}, \alpha) = \sum_{\mathbf{b}} \mathbf{g}(\mathbf{b}, \mathbf{k}, \alpha)\, e^{i2\pi\mathbf{b}\mathbf{r}}.$$

Transforming to the variables $\mathbf{k}$, α in (3), we obtain

$$K_{x'y'}(\mathbf{r}, \mathbf{r}') = K^{+}_{x'y'}(\mathbf{r}, \mathbf{r}') + K^{-}_{x'y'}(\mathbf{r}, \mathbf{r}'), \quad (5)$$

$$K^{\pm}_{x'y'}(\mathbf{r}, \mathbf{r}') = \frac{L^3}{(2\pi)^3 \hbar\omega} \sum_{\alpha}{}' \int \Gamma^{\pm}(\mathbf{k}, \alpha) \times \sum_{\mathbf{bb}'} g^{\pm}_{x'y'}(\mathbf{b}, \mathbf{b}', \mathbf{k}, \mathbf{k}, \alpha) \exp [\mp i\mathbf{k}(\mathbf{r} - \mathbf{r}') \mp i2\pi(\mathbf{br} - \mathbf{b}'\mathbf{r}')]\, d\mathbf{k};$$

$$\Gamma^{\pm}(\mathbf{k}, \alpha) = \mp \frac{E(\mathbf{k}, \alpha) - E_0}{E(\mathbf{k}, \alpha) \pm i\varepsilon(\mathbf{k}, \alpha, \mp\omega) - E_0 \pm \hbar\omega},$$
$$g^{+}_{x'y'}(\mathbf{b}, \mathbf{b}', \mathbf{k}, \mathbf{k}', \alpha) = g^{*}_{x'}(\mathbf{b}, \mathbf{k}, \alpha)\, g_{y'}(\mathbf{b}', \mathbf{k}', \alpha), \quad g^{-} = g^{+*}, \tag{6}$$

$L \to \infty$ are the dimensions of the fundamental region.

We shall assume that the macroscopic field $\mathcal{E}^{\perp}(\mathbf{r})$ entering in (1) is a smooth function of its coordinates. Then, in the integration over $\mathbf{r}'$, the main contribution is made to the part due to K^+ by the terms with $\mathbf{b}' = 0$ and with small k, since K^+ does not contain any singularities. When k is small, the matrix element $(0|\mathbf{G}(\mathbf{r})|\mathbf{k}\cdot\alpha)$ is a smooth function of the coordinates: when $\mathbf{k} = 0$, it has the same period as the lattice, as a consequence of which, being a macroscopic quantity, it is constant. Therefore, $\mathbf{g}(\mathbf{b}, \mathbf{k}, \alpha)$ is small for $\mathbf{b} \neq 0$ and small k. The circumstances mentioned lead to the result that the part of the polarization which is due to K^+ is locally associated with the field.

Following Pekar,[2] we limit ourselves to consideration of states for which $\mathbf{k}$ and only $\mathbf{k}$ is a continuous quantum number. In this case α is discrete and one can separate components in K^- for which $E(0, \alpha)$ differs appreciably from $E(\omega) = E_0 + \hbar\omega$. Like K^+, they yield the local part of the polarization. Taking it into account that $f^*(z^*)$ is the analytic continuation in the complex plane of the function $f^*(z)$, and that the functions $\mathbf{k}$ entering into (6) may not be analytic at the point $\mathbf{k} = 0$, we can write (1) in the form

$$\Delta\mathcal{E}^{\perp}_{x'}(\mathbf{r}) + \gamma\beta^{\perp}_{x'y'}\mathcal{E}^{\perp}_{y'}(\mathbf{r}) + 4\pi\gamma\int \chi^{\perp}_{x'y'}(\mathbf{r}, \mathbf{r}')\,\mathcal{E}^{\perp}_{y'}(\mathbf{r}')\, d\mathbf{r}' = 0, \tag{7}$$

$$\beta^{\perp}_{x'y'} = \delta_{x'y'} + \frac{4\pi L^3}{\hbar\omega}\Big[\sum_{\alpha}\Gamma^{+}\Big(-ai\frac{\partial}{\partial\mathbf{r}}, \alpha\Big) \times g^{+\perp}_{x'y'}\Big(0, 0, \Big(-ai\frac{\partial}{\partial\mathbf{r}}\Big)^{*} - ai\frac{\partial}{\partial\mathbf{r}}, \alpha\Big) + \sum_{\alpha}{}'\Gamma^{-}\Big(-ai\frac{\partial}{\partial\mathbf{r}}, \alpha\Big) \times g^{-\perp}_{x'y'}\Big(0, 0, -ai\frac{\partial}{\partial\mathbf{r}}, \Big(-ai\frac{\partial}{\partial\mathbf{r}}\Big)^{*}, a\Big)\Big]_{a=+0}. \tag{8}$$

The prime on the summation sign in (8) means that the components containing the singularity are omitted; $\chi^{\perp}$ is obtained from K^-, which appears in (6), by replacement of g^- by $g^{-\perp}$ and $\sum_{\alpha}$ by $\sum_{\alpha}{}'' = \sum_{\alpha} - \sum_{\alpha}{}'$; $g^{\pm\perp}_{x'y'}$ is obtained from $g^{\pm}_{x'y'}$ by replacing $g_{x'}$ by the projection $g^{\perp}_{x'}$ of this quantity, on the plane perpendicular to $\mathbf{k} + 2\pi\mathbf{b}$.

We seek a solution of (7) in the form

$$\mathcal{E}_{\mathbf{n}}(\mathbf{r}) = \sum_{s} \mathbf{c}_s \exp(i\mathbf{n}\varkappa_s\mathbf{r}), \qquad n^2 = 1, \tag{9}$$

$$\mathbf{nc}_s = 0 \tag{10}$$

subject to a condition that guarantees the smoothness of the field

$$|\varkappa_s| \ll \pi/d \tag{11}$$

(d is the lattice constant), which should be satisfied for those s for which c_s is large. Computing the integral over $\mathbf{k}$ in (7) by a method similar to that used in the Appendix, we find that κ_s and c_s are determined from

$$-\varkappa^2 c_{x'} + \gamma\beta^{\perp}_{\mathbf{n}x'y'}c_{y'} + \frac{4\pi\gamma L^3}{\hbar\omega}\sum_{\alpha}{}''\Gamma^{-}_{\mathbf{n}}(\varkappa, \alpha)\, g^{-\perp}_{\mathbf{n}x'y'}(0, 0, \varkappa, \varkappa^*, \alpha)\, c_{y'} = 0 \tag{12}$$

and (10), where the subscripts of g and Γ indicate the direction of $\mathbf{k}$, and $\beta^{\perp}_{\mathbf{n}x'y'}$ is obtained from (8) by setting $a = 0$ and writing Γ and g with the index $\mathbf{n}$. From (12) one can obtain (in the corresponding case and approximation) the results of Pekar,[2] which refer to a transverse field in an infinite crystal. $\mathcal{E}^{\parallel}(\mathbf{r})$ can be found with the help of (2) and (9).

2. WAVES IN A THICK PLATE

We consider the normal incidence of a wave on a plane-parallel plate of a cubic crystal. As wave functions of the exciton state, we choose the functions[2]

$$\Phi_{\alpha\mathbf{k}} = 2^{-1/2}(\Psi_{\alpha\mathbf{k}} - \Psi_{\alpha\widetilde{\mathbf{k}}}),$$

where $\Psi_{\alpha\mathbf{k}}$ is the wave function of the infinite crystal,

$$\widetilde{\mathbf{k}}\mathbf{a}_{1,2} = \mathbf{k}\mathbf{a}_{1,2} \equiv k_{1,2}, \qquad -\widetilde{\mathbf{k}}\mathbf{a}_3 = \mathbf{k}\mathbf{a}_3 \equiv k_3 = \pi\nu/(N+1),$$
$$\nu = 1, \ldots, N,$$

$\mathbf{a}_{1,2}$ are parallel to the surfaces of the plate, N is the number of elementary cells that make up the thickness of the plate l.

We direct the z axis normal to the plate so that $z = 0$ and l on the surfaces. Now the polarizability kernel is obtained in the form (5), where

$$K^{\pm}_{x'y'}(\mathbf{r}, \mathbf{r}')$$

$$= \frac{L^2}{(2\pi)^2 2\hbar\omega} \sum_{\alpha} \sum_{k_3=-k_N}^{k_N} \int \Gamma^{\pm}(\mathbf{k}, \alpha) \sum_{\mathbf{bb}'} \exp\{\mp i [k_3 b_{3z} z + k_x (x - x') + k_y (y - y') + (k_1 b_{1z} + k_2 b_{2z})(z - z') + 2\pi (\mathbf{br} - \mathbf{b}'\mathbf{r}')]\} \times [g^{\pm}_{x'y'}(\mathbf{b}, \mathbf{b}', \mathbf{k}, \mathbf{k}, \alpha) \exp(\pm i k_3 b_{3z} z') - g^{\pm}_{x'y'}(\mathbf{b}, \mathbf{b}', \mathbf{k}, \tilde{\mathbf{k}}, \alpha) \exp(\mp i k_3 b_{3z} z')]\, dk_x\, dk_y, \quad (13)$$

where $k_N = \pi N/(N+1)$, $\mathbf{b}_i$ $(i = 1, 2, 3)$ is the fundamental vector of the reciprocal lattice.

As above, we can separate in (13) the components that give the local contribution to the polarization. We then arrive at Eq. (7), in which one must now integrate over the volume $L^2 l$, $\beta^{\perp}$ is obtained from (8) by the replacement of L^3 by $L^2 l$, and $\chi^{\perp}$ from the K^- appearing in (13) by replacement of g^- by $g^{-\perp}$ and $\sum_{\alpha}$ by $\sum_{\alpha}''$.

It is not difficult to prove that a solution of (7) which does not depend on x and y, exists with the given kernel. We further assume that one can separate excitons with transverse and longitudinal polarizations in a cubic crystal at small k. Finally, the tensors entering into (7) in the case of a cubic crystal reduce to scalars; therefore, one can consider a field with fixed polarization. Let $\mathscr{E}(z)$ be directed along the y axis. Omitting the corresponding signs, we obtain

$$\frac{d^2\mathscr{E}(z)}{dz^2} + \eta\, \mathscr{E}(z) + \frac{\rho}{2l} \sum_{\alpha}'' \sum_{bb'} \int_0^l \chi(z, z', b, b', \alpha)\, \mathscr{E}(z')\, dz' = 0. \quad (14)$$

Here the sum over α takes into account only the transverse excitons, $b = nb_{3Z}$ (n an integer), $\eta = \gamma\beta$, $\rho = 4\pi\gamma L^2 l/\hbar\omega$,

$$\chi(z, z', b, b', \alpha) = \exp[i2\pi(bz - b'z')] \sum_{k=-k_N}^{k_N} \Gamma_0^-(k, \alpha)\, g_0(b, k, \alpha) \times \exp(ikz)\, [g_0^*(b', k, \alpha) \exp(-ikz') - g_0^*(b', -k, \alpha) \times \exp(ikz')], \quad (15)$$

where

$$f_0(b, k) = f(n\mathbf{b}_3, \mathbf{k})|_{k_x, y=0}, \qquad f = \Gamma, g.$$

As is shown in the Appendix, the solution of (14) has the form

$$\mathscr{E}(z) = \sum_s c_s \exp(i\varkappa_s z), \quad (16)$$

where κ_S are the roots of the equation

$$-\varkappa^2 + \eta + \rho \sum_{\alpha}'' g_0^-(0, 0, \varkappa, \varkappa^*, \alpha)\, \Gamma_0^-(\varkappa, \alpha) = 0, \quad (17)$$

and it is assumed that (11) is satisfied for all s for which c_S is large; the quantities c_S satisfy the equations

$$\sum_s c_s \sum_b [k^+_{\alpha q}\, r_s^+(b, k^+_{\alpha q}, \alpha) + (\varkappa_s - 2\pi b)\, r_s^-(b, k^+_{\alpha q}, \alpha)] = 0,$$

$$\sum_s c_s \exp(i\varkappa_s l) \sum_b \exp(-2\pi b l)\, [k^-_{\alpha q}\, r_s^+(b, k^-_{\alpha q}, \alpha) + (\varkappa_s - 2\pi b)\, r_s^-(b, k^-_{\alpha q}, \alpha)] = 0,$$

$$r_s^{\pm}(b, k, \alpha) = [g_0^*(b, k^*, \alpha) \pm g_0^*(b, -k^*, \alpha)]\, [(\varkappa_s - 2\pi b)^2 - k^2]^{-1}, \quad (18)$$

where $k^{\pm}_{\alpha q}$ are the roots of the equation

$$E_0(k, \alpha) - i\varepsilon_0(k, \alpha, \omega) - E(\omega) = 0, \qquad q = 1, \ldots, m_\alpha, \quad (19)$$

and the sign on $k_{\alpha q}$ denotes the sign of the imaginary part.

The number of components in (16) exceeds the number of equations in (18) by two, so that there are exactly two independent quantities among the c_S. The results that have been given are valid in the case of a sufficiently thick plate. From (16), the Maxwell boundary conditions, and the additional conditions (18), one can find the waves reflected from and emerging from the plate. The corresponding formulas are very cumbersome in the general case and we shall not write them out.

3. EXCITONS OF ZERO AND FIRST ORDER

We consider the simplest case in which the sum (17) contains only one component, and $E(\omega) \approx E_0(0, \alpha)$. Under these conditions, (11) is satisfied for all s and we can therefore set $\kappa = 0$ in the numerator of Γ in (17), and expand $E_0(\kappa, \alpha)$ in the denominator and in (19) in powers of κ, restricting ourselves to the quadratic term. Moreover, we can neglect the dependence of ϵ_0 on κ and ω. In this case (19) yields two roots $k^- = -k^+$, which are small, and which makes it possible to neglect components in (18) with $b \neq 0$. Omitting α, we get from (17) — (19)

$$-\varkappa^2 + \eta + \rho g_0^-(0, 0, \varkappa, \varkappa^*)\, [E_0(0) - E_0] \times (\xi\varkappa^2 - i\varepsilon - \zeta)^{-1} = 0; \quad (20)$$

$$\sum_s c_s [k^+ r^+(0, k^+) + \varkappa_s r^-(0, k^+)] = 0,$$

$$\sum_s c_s \exp(i\varkappa_s l)\, [k^- r^+(0, k^-) + \varkappa_s r^-(0, k^-)] = 0; \quad (21)$$

$$\xi (k^{\pm})^2 - i\varepsilon - \zeta = 0, \qquad (22)$$

where $\zeta = E(\omega) - E_0(0)$, $\xi = \hbar^2/2m^*$, m^* is the effective mass of the exciton.

We consider two cases, in which

$$g_0 \equiv g_0(0, 0) \neq 0$$

and

$$g_0 = 0, \qquad g_0' \equiv (dg_0(0, k)/dk)_0 \neq 0.$$

In these cases, we can speak of excitons of zero and first order, respectively (or of dipole and quadrupole excitons). Excitons of zero order are found in anthracene, and of first order in copper oxide (references to the corresponding experiments are given in the paper of Pekar et al.[4]). For excitons of zero order, we can regard $g_0(0, k) = g_0$ in (20) and (21); here, (20) – (22) are identical with the corresponding results of Pekar.[2,3] Thus the results derived are approximately valid for frequencies arbitrarily close to the resonant frequency point $\mathbf{k} = 0$.

We now consider excitons of first order, assuming $g_0(0, k) = g_0'k$ for them. Here the solutions of (20) are $\pm\kappa_j$, $j = 1, 2$,

$$\kappa_j^2 = \{\xi\eta + \zeta + i\varepsilon + \alpha + (-1)^j [(\xi\eta + \zeta + i\varepsilon + \alpha)^2 - 4\xi\eta(\zeta + i\varepsilon)]^{1/2}\}/2\xi,$$

where $\alpha = \rho |g_0'| [E_0(0) - E_0]$, which are identical with the corresponding results of Pekar et al.[4] It is now convenient to write (16) in the form

$$\mathscr{E}(z) = \sum_{j=1,2} [c_j^+ \exp(i\kappa_j z) + c_j^- \exp(-i\kappa_j z)].$$

In corresponding fashion, (21) yields

$$\sum_{j=1,2} \gamma_j (c_j^+ - c_j^-) = 0,$$

$$\sum_{j=1,2} \gamma_j [c_j^+ \exp(i\kappa_j l) - c_j^- \exp(-i\kappa_j l)] = 0,$$

$$\gamma_j = \kappa_j [\kappa_j^2 - (k^+)^2]^{-1}. \qquad (23)$$

Let the incident, reflected, and transmitted waves have respectively the forms

$$\mathscr{E}_0 \exp(ik_0 z), \qquad R \exp(-ik_0 z), \qquad B \exp(ik_0 z).$$

From the Maxwell boundary conditions and the additional conditions (23), we find*

$$R = A(u^2 - v^2 - w^2), \qquad B = A\, i2uw \exp(-ik_0 l);$$

$$A = \mathscr{E}_0 (u^2 - v^2 + w^2 + i2vw)^{-1},$$

$$u = \gamma_1 / \sin\kappa_2 l - \gamma_2 / \sin\kappa_1 l,$$

$$v = \gamma_1 \operatorname{ctg}\kappa_2 l - \gamma_2 \operatorname{ctg}\kappa_1 l, \qquad w = (\gamma_1\kappa_2 - \gamma_2\kappa_1)/k_0.$$

*ctg = cot.

In the case of a semi-infinite crystal, we then obtain

$$B_\infty = 0, \qquad R_\infty = \mathscr{E}_0 (\gamma_1 - \gamma_2 - w)(\gamma_1 - \gamma_2 + w)^{-1},$$

which gives the generalization of the corresponding Fresnel formula with account of spatial dispersion.

We note that in the case of excitons of first order one should have, for greater consistency, considered higher approximations in the material equations.[5]

4. WAVES IN A PLATE OF ARBITRARY THICKNESS

As is seen from the Appendix, the solution of Eq. (14) used above is inappropriate for sufficiently small l. Here we obtain a solution appropriate for arbitrary l, limiting ourselves to the case in which the sum over α in (14) contains only one component (an isolated exciton band) and $E(\omega) \approx E_0(0)$. In this case the terms of the sum over k in (15) vanish, for smooth functions $\mathscr{E}(z)$, with increase in $|k|$ so that for all z the principal role in the last component of (14) is played by terms with small $|k|$, as a consequence of which one can discard terms with b and $b' \neq 0$ and keep the first non-vanishing term in the expansion of $g_0(0, k)$ in powers of k. We note that for excitons of zero order this agrees in accuracy with the approximate theory of Pekar.[2]

In the case of excitons of zero order, we obtain in this fashion

$$\frac{d^2\mathscr{E}'(t)}{dt^2} + \eta'\mathscr{E}'(t) + \int_0^\pi \sum_{n=1}^\infty f_0'(n) \sin nt \sin nt' \mathscr{E}'(t')\, dt' = 0,$$

$$t = (\pi/l) z, \qquad \mathscr{E}'(t) = \mathscr{E}(z),$$

$$\eta' = (l/\pi)^2 \eta, \qquad f'(n) = (l/\pi)^3 f(k_n),$$

$$f_0(k_n) = \begin{cases} (2\rho/l) |g_0|^2 \Gamma_0^-(k_n), & n \leqslant N \\ 0, & n > N. \end{cases} \qquad (24)$$

We now seek the solution of (24) in the form

$$\mathscr{E}'(t) = \sum_{n=1}^\infty a_n \sin nt, \qquad 0 < t < \pi.$$

Taking it into account that

$$\frac{d\mathscr{E}'}{dt} = \frac{c_0}{2} + \sum_{n=1}^\infty [na_n - d_0 + (-1)^n (c_0 + d_0)] \cos nt,$$

$$c_0 = (2/\pi)[\mathscr{E}'(\pi) - \mathscr{E}'(0)], \qquad d_0 = (2/\pi)\mathscr{E}'(0), \qquad (25)$$

we can differentiate termwise the cosine series[6] and obtain

$$a_n = n[d_0 - (-1)^n (c_0 + d_0)][n^2 - \eta' - (\pi/2) f_0'(n)]^{-1}.$$

Further, denoting

$$\sigma_p = \sum_{n=1}^{\infty} (-1)^{pn} \left[\eta' + \frac{\pi}{2} f_0'(n)\right]\left[n^2 - \eta' - \frac{\pi}{2} f_0'(n)\right]^{-1},$$

we obtain

$$d\mathscr{E}'/dt|_0 = c_0/2 + d_0\sigma_2 - (c_0 + d_0)\sigma_1,$$
$$d\mathscr{E}'/dt|_\pi = c_0/2 + d_0\sigma_1 - (c_0 + d_0)\sigma_2. \quad (26)$$

(25), (26) express $\mathscr{E}'$ and $d\mathscr{E}'/dt$ on the surface of the plate in terms of the constants c_0 and d_0, which play the role of arbitrary constants.

With the help of the Maxwell boundary conditions, we find the amplitudes of the reflected and transmitted waves:

$$R_0 = A_0 [k_0^2 l^2 - 4(\sigma_2 - \sigma_1)(1 - \sigma_1 - \sigma_2)],$$

$$B_0 = A_0 i 2k_0 l (1 - 2\sigma_1) \exp(-ik_0 l);$$

$$A_0 = \mathscr{E}_0 [k_0^2 l^2 + 4(\sigma_2 - \sigma_1)(1 - \sigma_1 - \sigma_2) + 2ik_0 l (1 - 2\sigma_2)]^{-1}.$$

In the case of excitons of first order, (14) and (15) yield the following expression in the approximation under consideration:

$$\frac{d^2\mathscr{E}'(t)}{dt^2} + \eta'\mathscr{E}'(t) + \int_0^{\pi} \sum_{n=1}^{\infty} f_1'(n) \cos nt \cos nt' \mathscr{E}'(t')\, dt' = 0,$$

where $f_1(k_n)$ is obtained from $f_0(k_n)$ by replacing $|g_0|^2$ by $|g_0'|^2 k_n^2$. Working in a fashion similar to the above, we obtain the amplitudes of the reflected and transmitted waves

$$R_1 = A_1 [4k_0^2 l^2 (\sigma_1' - \sigma_2')(1/\eta' - \sigma_1' - \sigma_2') + \pi^4],$$

$$B_1 = A_1 i \cdot 2k_0 l \pi^2 (1/\eta' - 2\sigma_1') \exp(-ik_0 l);$$

$$A_1 = \mathscr{E}_0 [4k_0^2 l^2 (\sigma_1' - \sigma_2')(1/\eta' - \sigma_1' - \sigma_2') - \pi^4 - i2k_0 l \pi^2 (1/\eta' - 2\sigma_2')]^{-1},$$

$$\sigma_p' = \sum_{n=1}^{\infty} (-1)^{pn} \left[n^2 - \eta' - \frac{\pi}{2} f_1'(n)\right]^{-1}.$$

5. ABSORPTION

In the presence of spatial dispersion, there is meaning to considering only absorption throughout the entire volume:

$$\overline{W} = i\omega \int \mathscr{E}_y(\mathbf{r})\, P_y^*(\mathbf{r})\, d\mathbf{r} + \text{compl. conj.}$$

where the bar denotes averaging over the period, while $\mathscr{E}_y(\mathbf{r})$ and $P_y(\mathbf{r})$ are the amplitudes of the electric field intensity and polarization. We resolve $\mathscr{E}_y(\mathbf{r})$ into a solenoidal part and an irrotational part, in accord with which we obtain

$$\overline{W} = W^{\perp} + W^{\parallel},$$

$$W^{\perp,\parallel} = i\omega \int \mathscr{E}_y^{\perp,\parallel}(\mathbf{r})\, P_y^*(\mathbf{r})\, d\mathbf{r} + \text{compl. conj.} \quad (27)$$

Further, substituting (2), in the expression

$$P_{x'}(\mathbf{r}) = \int K_{x'y'}(\mathbf{r}, \mathbf{r}')\, \mathscr{E}_{y'}^{\perp}(\mathbf{r}')\, d\mathbf{r}' \quad (28)$$

obtained in reference 1, and (3) in (24), and taking (4) into account, we find that $W^{\parallel} = 0$. Finally, by introducing (28) and the relation $K_{x'y'} = Q_{x'y'} + R_{x'y'}$ which figures in reference 1, ($Q_{x'y'}$, $R_{x'y'}$ are the Hermitian and anti-Hermitian parts of the kernel) into (27) we obtain

$$\overline{W} = -2i\omega \int R_{x'y'}(\mathbf{r}, \mathbf{r}')\, \mathscr{E}_{x'}^{\perp *}(\mathbf{r})\, \mathscr{E}_{y'}^{\perp}(\mathbf{r}')\, d\mathbf{r}\, d\mathbf{r}'.$$

Thus the absorption for a given field is determined by the anti-Hermitian part of the polarizability kernel, just as the absorption is determined by the anti-Hermitian part of the polarizability tensor in the absence of spatial dispersion.

It is seen from (3) that $R_{x'y'} \sim \epsilon$ when $\hbar\omega \neq E_n - E_0$. Therefore, upon satisfaction of the inequality mentioned, the absorption must tend to zero if $\epsilon \to 0$. It is not difficult to prove that the expressions obtained above for R, B, R_0, B_0 and R_1, B_1 satisfy this condition.

In conclusion we note that in this research, just as in reference 1, we did not make any distinction between the average and effective values of the field. Therefore, the results obtained apply only to the case in which these values are identical. The general case needs special consideration.

APPENDIX

We consider the integral which appears in (14):

$$I(z, b, b', \alpha) = \int_0^l \chi(z, z', b, b', \alpha)\, \mathscr{E}(z')\, dz',$$

where $\mathscr{E}(z)$ is given by Eq. (16). In view of the fact that (14) is a differential equation, it suffices in the case of large l to know I for z not too close to the boundary values. For such z the principal contribution to I is made in the summation over k in (15) by the immediate vicinity of the points $\pm \mathrm{Re}\,\kappa_s$ and $\mathrm{Re}\,k_{\alpha q}$. Let us consider the contribution made by the points $\pm \mathrm{Re}\,\kappa_s$. In view of (11), only the terms with b and b' = 0 are important:

$$I_s(z, 0, 0, \alpha) = c_s \Gamma_0^- (-i\partial/\partial z, \alpha)\, g_0(0, -i\partial/\partial z, \alpha) \times [g_0^*(0, (-i\partial/\partial z)^*, \alpha)\, I_s^-(z) - g_0^*(0, (i\partial/\partial z)^*, \alpha)\, I_s^+(z)],$$

$$I_s^{\pm}(z) = \int_0^l \sum_k^{(s)} \exp[ik(z \pm z') + i\kappa_s z']\, dz',$$

where the summation is carried out over the values $k = (\pi/l)\nu$ which lie close to the points $\pm \mathrm{Re}\,\kappa_s$. Inasmuch as the vicinities of the points mentioned

make the principal contribution, we can extend the summation over all k. Then

$$I_s(z, 0, 0, \alpha) = 2lc_s g_0^-(0, 0, \varkappa_s, \varkappa_s^*, \alpha)\,\Gamma_0^-(\varkappa_s, \alpha)\exp(i\varkappa_s z). \tag{A.1}$$

We now consider the contribution of the points $k_{\alpha q}$:

$$I_\alpha = \sum_q \sum_s c_s \sum_k{}^{(q)} \exp[i(2\pi b + k)z]\,\Gamma_0^-(k, \alpha)\, g_0(b, k, \alpha) \times \left\{ g_0^*(b', k, \alpha)\frac{\exp[i(\varkappa_s - k - 2\pi b')l] - 1}{i(\varkappa_s - k - 2\pi b')} - g_0^*(b', -k, \alpha)\frac{\exp[i(\varkappa_s + k - 2\pi b')l] - 1}{i(\varkappa_s + k - 2\pi b')} \right\},$$

where the summation over k includes the vicinity of the points $k_{\alpha q}$. We transform from summation over k to integration; this is possible in the case of sufficiently large l. Here, we must replace e^{ikl} in the second term in curly brackets by the equivalent expression e^{-ikl} to obtain a smoother function of k. Furthermore, we can close the contour of integration in the lower half-plane in the interval containing $e^{ik(z-l)}$, and in the upper half-plane in the interval containing e^{ikz}. Then, with the aid of residues, we find

$$I_\alpha = -2l\sum_q \sum_s c_s e^{i2\pi bz}\{\exp[ik_{\alpha q}^-(z - l) + i(\varkappa_s - 2\pi b')l] \times p_s(b, b', k_{\alpha q}^-, \alpha) + \exp(ik_{\alpha q}^+ z)\, p_s(b, b', k_{\alpha q}^+, \alpha)\},$$

$$p_s(b, b', k, \alpha) = \left[\frac{g_0^-(b, b', k, k^*, \alpha)}{\varkappa_s - k - 2\pi b'} - \frac{g_0^-(b, b', k, -k^*, \alpha)}{\varkappa_s + k - 2\pi b'}\right]\frac{E_0(k, \alpha) - E_0}{dE_0(k, \alpha)/dk}. \tag{A.2}$$

Taking into account (A.1), (A.2), (16) and (14), it is easy to obtain (17), (18). We write (17) in the form

$$\varphi(\varkappa) + \psi(\varkappa) = 0;$$

$$\varphi(\varkappa) = (-\varkappa^2 + \eta)\prod_\alpha{}'' [E_0(\varkappa, \alpha) - i\varepsilon_0(\varkappa, \alpha, \omega) - E(\omega)],$$

$$\psi(\varkappa) = \rho\sum_\alpha{}'' g_0^-(0, 0, \varkappa, \varkappa^*, \alpha) \times \prod_{\alpha' \neq \alpha} [E_0(\varkappa, \alpha') - i\varepsilon_0(\varkappa, \alpha', \omega) - E(\omega)].$$

$\varphi(\kappa)$ and $\psi(\kappa)$ are analytic functions of κ. The number of components in (16), N_1, is equal to the number of zeros of $\varphi(\kappa) + \psi(\kappa)$ lying close to the origin of the coordinates, in accord with (11). The number of equations in (18) is $N_2 = \sum_\alpha{}'' m_\alpha$. We shall show that $N_1 = N_2 + 2$. We draw the closed contour C in the plane passing through the points $-\pi/d$, $+\pi/d$ so that for them $|\kappa| \geq \pi/d$ and $\varphi(\kappa)$, $\psi(\kappa)$ do not vanish on them. Inasmuch as $|\kappa|$ is large on C, then $|\varphi(\kappa)| > |\psi(\kappa)| > 0$ on C. Then, by Rouche's theorem,[7] the number of zeros of $\varphi(\kappa) + \psi(\kappa)$ inside C is identical with the number of zeros of $\varphi(\kappa)$, which is obviously equal to $N_2 + 2$.

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