RESONANCE DIFFRACTION OF WAVES IN LAMELLAR INHOMOGENEOUS MEDIA

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A peculiar type of diffraction effect due to resonance accumulation of waves inside a lamellar system is observed when the waves are incident at an oblique angle on a finite laminar system. The main properties of this effect are elucidated and a characteristic length is derived which defines the distance along the layers from the boundary for which the effect can be observed.

RESONANCE phenomena can arise in wave propagation in inhomogeneous lamellar media. This takes place if the waves encounter two or more low-transmission, non-absorbing layers en route; these play the role of barriers, in the region between which standing waves can form. At resonance, the standing-wave amplitude between the barriers increases many times in comparison with the incident wave amplitude. Simultaneously, the transmission of the barriers for the incident wave can increase sharply. The quantum-mechanical phenomenon of the analogous process in nonplanar systems is the resonance transmission of de Broglie waves through a system of two potential barriers (Ramsauer effect) and also the resonance penetration of waves through a barrier according to Breit-Wigner.

These resonance phenomena in lamellar media are frequently encountered in practice: in acoustics, in wave propagation in a plasma, particularly in the ionosphere, in optical systems of the type of interference filters, Fabry-Perot etalons, etc. Many researches have been devoted to the analysis of resonance effects of this type (see, for example, reference 1). As a rule, the calculations here are carried out for infinite systems. The basis for this is the fact that in practice the dimensions of the systems are many orders of magnitude greater than the wavelength. Therefore, it is assumed that the diffraction effects from the boundaries of real systems are always small corrections which can safely be neglected.

The latter is not true, generally speaking; that is, the fact that the system is very large in comparison with the wavelength is not sufficient that the asymptotic theory of an infinite system can be used as a valid first approximation. In particular, a peculiar resonance diffraction effect takes place for oblique incidence of waves on a lamellar system. This process can propagate along the stratified system for many orders of wavelengths from its boundary. In this connection, a characteristic length of resonance diffraction l_0 appears and can be many orders larger than the wavelength. Only when the dimensions of the system are large in comparison with the characteristic resonance diffraction length l_0, and not with the wavelength, can one use the asymptotic theory of an infinite system as a first approximation.

The lack of understanding of this process, and the invalid application of the asymptotic theory in a number of cases, have already led to a radical disparity between theory and experiment as, for example, in total reflection filters.[2]

Until recently, mistaken attempts have been made to attribute these disparities to all sorts of imperfections of the system, inasmuch as the phenomenon of resonance diffraction has never been described or calculated up to now. Recently, a calculation of electromagnetic wave transmission through finite lamellar dielectric systems was carried out by the author.[3] It was established that the diffraction effects begin to play an important role at resonance. It was shown in the same place, in particular, that just these effects produce a sharp decrease in the transmission of total reflection filters. A complete calculation of the transmission coefficient of the finite total internal reflection filter is in excellent agreement with experimental data. Thus, the special case of the resonance diffraction effect was considered for the first time.[3]

The purpose of the present research was to formulate the basic general laws of resonance diffraction. It seems to us that this is of interest, since resonance diffraction is a rather general phenomenon which can arise for waves any nature obliquely propagating through lamellar media. To reduce the amount of calculation, we considered the case of scalar wave propagation in the simplest semi-infinite resonant system. The results obtained can be written down in very simple form, which contains nothing that is specific to the nature of the waves and to the character of the actual resonant system. These results admit immediate generalization and make it possible to formulate the general laws of resonance diffraction.

We write down the scalar wave equation

$$\Delta\varphi - (n/c)^2 \ddot{\varphi} = 0. \tag{1}$$

Here $\varphi(\mathbf{r}, t)$ is the scalar wave function of the coordinates and time, n is a material constant of the medium (index of refraction), and c is the characteristic velocity. We consider the case of a harmonic time dependence $\varphi(\mathbf{r}, t) = \varphi(\mathbf{r}) e^{i\omega t}$. In this case, we have from (1)

$$\Delta\varphi(\mathbf{r}) + k^2\varphi(\mathbf{r}) = 0, \tag{2}$$

where $k^2 = (n\omega/c)^2$.

We assume that the plane z = 0 separates media with different material constants. We require that the following boundary conditions be satisfied at the separation boundary:

$$\partial\varphi_1/\partial t = \partial\varphi_2/\partial t, \qquad \partial\varphi_1/\partial z = \partial\varphi_2/\partial z. \tag{3}$$

The equations (1) — (3) just considered can be used for the description of various wave processes. In particular, they describe the propagation of longitudinal sound waves in an elastic medium of constant density. In this case, φ is the velocity potential of the medium and c/n is the velocity of the sound waves. The same equations describe electromagnetic wave propagation in a nonmagnetic dielectric in the case in which the electric vector lies in the plane of separation of the two dielectrics. Here φ is the only nonvanishing component of the vector potential of the wave, c is the velocity of light in vacuum, and n is the index of refraction of the medium.

We take the plane of incidence of the waves on the boundary separating the media as the xz plane. We assume that the layered system is semi-infinite and fills the half-space x > 0. Consequently, the desired solution of the wave equation in the region x < 0 must be identically equal to zero. It is obvious that individual plane monochromatic waves are insufficient for the solution of the given diffraction problem. Below, we shall give the approximate solutions of the wave equation (1) obtained previously,[3] with the help of which one can solve our boundary problem.

We assume that total internal reflection takes place in the incidence of waves from medium I, which is located in the region z < 0, on medium II, located in the region z > 0. In medium I we shall have a wave consisting of a combination of deformed incident and reflected plane homogeneous waves:

$$\varphi(\mathbf{r}, t) = \left\{A\left(x - \frac{k_x}{k_z} z\right)\exp[-i(k_x x + k_z z)] + B\left(x + \frac{k_x}{k_z} z\right)\exp[-i(k_x x - k_z z)]\right\} e^{i\omega t}. \tag{4}$$

In medium II, we shall have a wave consisting of a combination of deformed inhomogeneous waves which grow and decay along the z axis:

$$\varphi(\mathbf{r}, t) = \left\{F\left(x + i\frac{k_x}{q_z} z\right)\exp[-i(k_x x + iq_z z)] + G\left(x - i\frac{k_x}{q_z} z\right)\exp[-i(k_x x - iq_z z)]\right\} e^{i\omega t}. \tag{5}$$

Here A, B, F, G are certain complex functions of their arguments:

$$k_x = (n\omega/c)\sin\alpha, \qquad k_x^2 + k_z^2 = (n\omega/c)^2, \qquad k_x^2 - q_z^2 = (n_1\omega/c)^2, \tag{6}$$

where ω is the circular frequency of the waves, c the characteristic velocity, α the angle of incidence of the waves from medium I on the plane of separation z = 0, n and n_1 the index of refraction of medium I and medium II, respectively. It is assumed that the angle of incidence is sufficiently large that the condition for total reflection is satisfied; i.e., $\sin\alpha \geq n_1/n$.

Generally speaking, Eqs. (4) and (5) are not exact solutions of the wave equation (1) for arbitrary form of the amplitude functions A, B, F, and G. They are exact solutions of (1) in the special case when these amplitudes are constant or are linear functions of their arguments. However, (4) and (5) are approximate solutions of (1), with accuracy up to first derivatives for arbitrary amplitudes A, B, F, and G, if these amplitudes are sufficiently slowly changing functions of their arguments. The condition "sufficiently slowly" in the given case means the smallness of the change of amplitudes over a distance of the order of a wavelength: $|\partial A/\partial x| \ll |k_x A|$, $|\partial A/\partial z| \ll |k_z A|$, etc. In practice, for the resonance diffraction phenomenon of interest to us, this condition of slowness of change of the amplitudes is always satisfied by a wide margin, and the Eqs. (4) and (5) can be taken practically as exact solutions. We shall call the waves (4) and (5) "orthogonal," which reflects the fact that the planes where the phases and amplitudes are constant, are mutually orthogonal.

Substituting (4) and (5) in (3), we get a differential relation connecting the amplitudes of the waves

(4) and (5) in the plane of separation z = 0. These equations can be solved with accuracy up to the first derivatives; one obtains the following direct and inverse systems of differential equations for the determination of A and B in terms of F and G, and vice versa:

$$A(x) = \tfrac{1}{2}\left(1 + i\frac{q_z}{k_z}\right)F(x) + \tfrac{1}{2}\left(1 - i\frac{q_z}{k_z}\right)G(x) - \tfrac{1}{2}\frac{k_x}{q_z k_z}\left(1 + \frac{q_z^2}{k_z^2}\right)[F'(x) - G'(x)],$$

$$B(x) = \tfrac{1}{2}\left(1 - i\frac{q_z}{k_z}\right)F(x) + \tfrac{1}{2}\left(1 + i\frac{q_z}{k_z}\right)G(x) + \tfrac{1}{2}\frac{k_x}{q_z k_z}\left(1 + \frac{q_z^2}{k_z^2}\right)[F'(x) - G'(x)]; \qquad (7)$$

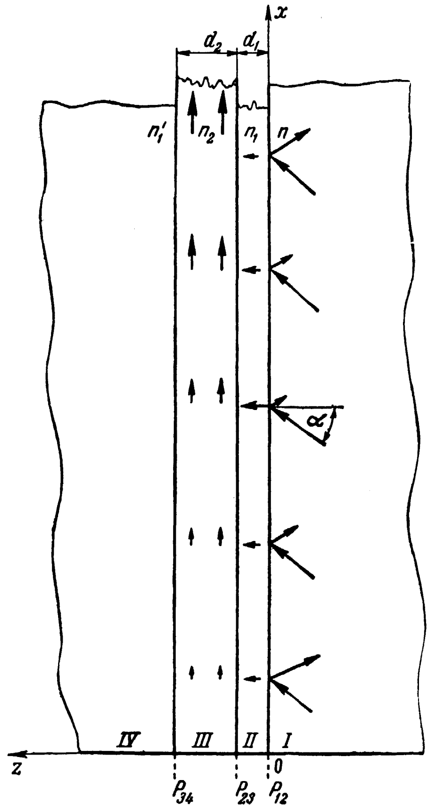
$$F(x) = \tfrac{1}{2}\left(1 - i\frac{k_z}{q_z}\right)A(x) + \tfrac{1}{2}\left(1 + i\frac{k_z}{q_z}\right)B(x) - \tfrac{1}{2}\frac{k_x}{q_z k_z}\left(1 + \frac{k_z^2}{q_z^2}\right)[A'(x) - B'(x)],$$

$$G(x) = \tfrac{1}{2}\left(1 + i\frac{k_z}{q_z}\right)A(x) + \tfrac{1}{2}\left(1 - i\frac{k_z}{q_z}\right)B(x) + \tfrac{1}{2}\frac{k_x}{q_z k_z}\left(1 + \frac{k_z^2}{q_z^2}\right)[A'(x) - B'(x)]. \qquad (8)$$

We now proceed to consideration of resonant systems. The simplest resonant system consists of a single barrier located in front of an impenetrable wall. We shall call such a system a resonant condenser. Such a semi-infinite resonant condenser is shown in cross section in the drawing, with a totally reflecting sheet bounded by the region x > 0. Here the boundary medium I and the layer-resonator III possess large indices of refraction n and n_2, respectively. The totally reflecting layer II and the boundary medium IV possess small indices of refraction n_1 and n_1', respectively. It is assumed that the waves are incident from medium I on the separation boundary at such large angles of incidence α that complete internal reflection takes place in the planes P_{12} and P_{34}; i.e., the following conditions hold:

$$\sin\alpha \geqslant n_1/n \ \text{ and } \ \sin\alpha \geqslant n_1'/n.$$

The appearance of waves along the totally reflecting layer II is due to the smallness of its thickness d_1. In this case, the appearance of energy inside the resonator III has a resonance character and for certain fixed conditions depends on the thickness d_2 of the resonator. To be precise, upon satisfaction of definite resonance conditions, which are determined from the Eqs. (9) set forth below, the amplitude of the wave inside resonator III increases sharply in the case of an unbounded system. In the presence of a boundary,



this resonance accumulation of waves will take place gradually in a certain portion adjacent to the boundary; that is, a special resonant diffraction layer appears with its own characteristic length. It is just this layer that we wish to discuss.

We note the curious fact* that in the case of oblique propagation of electromagnetic waves polarized in the plane of incidence in an unbounded lamellar plasma with continuously changing dielectric properties, the effect of leakage and resonant accumulation of energy takes place in the region behind the totally reflecting boundary, similar to what happens in the resonant condenser considered by us, in which the medium has properties which undergo discrete changes.[4] If we take into account the presence of the boundary in this case, then a characteristic length will obviously exist which is analogous to that obtained in the present research, over the extent of which flow of energy takes place into the region behind the barrier.

In the calculation of the semi-infinite resonant condenser shown in the drawing, one should determine eight complex amplitude functions, two for each of the media from I to IV. These amplitudes are connected by six matching conditions of the form (7) or (8) on the three boundary surfaces

*I am most grateful to V. L. Ginzburg who turned my attention to this point.

P_{12}, P_{23}, and P_{34}. Moreover, it is clear from physical considerations that the amplitude of the wave in medium IV which arises on the side $z > 0$ must be equal to zero, i.e., $F_{P_{34}}(x) \equiv 0$., since it would otherwise increase without limit as $z \to \infty$. In order that the problem be completely determined, one must assume one of the remaining amplitudes as given. We shall assume that the amplitude of the wave incident from I on P_{12} is given, and shall assume that an ordinary plane wave with constant amplitude A_1 is incident from medium I. Thus there remain six unknown complex amplitude functions, connected by the set of six linear differential equations of first order.

By making use of the approximate method developed earlier,[3] these equations can be integrated with accuracy up to first derivatives of slowly changing amplitudes. As a result, we obtain all six desired amplitude functions, expressed in terms of the amplitude A_1 of the wave incident on the system. These amplitudes depend in resonant fashion on the conditions in the resonator. The conditions of resonance are periodically repeated with change of thickness of the resonator d_2, and have the approximate form

$$\operatorname{tg}(k_z' d_2)_{\text{рез}} = -k_z'(q_z + q_z')/(q_z q_z' - k_z'^2), \quad (9)^*$$

where, by analogy with (6),

$$(k_x^2 + k_z'^2) = (n_2\omega/c)^2, \qquad (k_x^2 - q_z^2) = (n_1'\omega/c)^2. \quad (10)$$

For exact resonance, the amplitudes of the waves inside resonator III in the region $x > 0$ are equal to

$$|A_{P_{23}}(x)| = |B_{P_{23}}(x)| = \left[\sqrt{(q_z^2 + k_z^2)(q_z^2 + k_z'^2)}/2q_z k_z'\right] \times \exp(q_z d_1)(1 - e^{-x/l_0})|A_{\mathrm{I}}|, \quad (11)$$

while the amplitude of the wave reflected from the leading plane of separation is equal to

$$|B_{P_{12}}(x)| = |1 - 2e^{-x/l_0}| \cdot |A_{\mathrm{I}}|, \quad (12)$$

where the characteristic length of resonant diffraction enters:

$$l_0 = (k_x/k_z)\left[(q_z^2 + k_z^2)(q_z^2 + k_z'^2)/(2q_z k_z')^2\right] \exp(2q_z d_1) \times (d_2 + 1/q_z + 1/q_z'). \quad (13)$$

For an infinite system in the region $x \gg l_0$ we have $e^{-x/l_0} \approx 0$; therefore, we find from (11) and (12):

$$|A_{P_{23}}(\infty)|^2/|A_{\mathrm{I}}|^2 = \left[(q_z^2 + k_z^2)(q_z^2 + k_z'^2)/(2q_z k_z')^2\right] \exp(2q_z d_1). \quad (14)$$

From (12) we have $|B_{P_{12}}(\infty)| = |A_{\mathrm{I}}|$; that is, the square of the modulus of the amplitude of the wave inside the resonator increases by a factor of $\exp(2q_z d_1)$ in comparison with the incident wave, while the amplitude of the reflected wave is equal to the amplitude of the incident wave.

The situation is different in the region $0 \le x \lesssim l_0$. It is just in this region that the diffraction phenomena play a role. As is seen from (11), the amplitudes of the waves inside the resonator in this region increase monotonically from zero, gradually approaching the resonance value for an infinite system. On the other hand, the amplitude of the reflected wave, which at $x = 0$ is equal to the amplitude of the incident wave, gradually decreases, vanishing at $x = l_0 \ln 2$, and then increases again, approaching the amplitude of the incident wave.

The characteristic resonant diffraction length l_0, which is determined for exact resonance by the expression (13), can be written in the following simple form with the help of (6) and (14):

$$l_0 = \left[|A_{P_{23}}(\infty)|^2/|A_{\mathrm{I}}|^2\right](d_2 + 1/q_z + 1/q_z') \operatorname{tg}\alpha, \quad (15)$$

that is, this length is equal to the effective thickness of the resonator, multiplied by the coefficient of amplification of the square of the amplitude of the wave inside the resonator and by the tangent of the angle of incidence. The effective thickness of the resonator is equal to its geometric thickness plus the effective depth of penetration of the waves in the totally reflecting medium adjoining it, which plays the role of the barrier.

In conclusion, we emphasize that the results obtained determined the general phenomenon of resonance diffraction, which is characteristic for any wave processes in lamellar media. The essence of this phenomenon consists of the fact that, in the oblique incidence of waves, there is always a certain region abutting the boundary of the system, in which a gradual leakage of the waves inside the resonator takes place. In this region, the amplitude of the waves inside the resonator gradually increases, as the distance from the boundary increases as $1 - e^{-x/l_0}$, from zero up to the resonant value of the amplitude of the wave inside an unbounded resonant system. The dimensions of the region in which these phenomena take place are determined by the characteristic length of resonance diffraction, which is in turn determined by Eq. (15).

*tg = tan.

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