EFFECT OF ANISOTROPY ON THRESHOLD PHENOMENA IN SUPERCONDUCTORS

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We consider the absorption of ultrasound and of electromagnetic waves in anisotropic superconductors at absolute zero at frequencies close to the threshold frequency ($\omega \sim 10^{11}$ to 10^{12} $\sec^{-1}$). We show that the threshold frequency depends on the direction of propagation of the waves. The absorption near threshold is in the acoustic case proportional to $(\omega - \Omega_0)^{1/2}$ (Ω_0 is the threshold frequency). There is an essential difference in the electromagnetic case for the absorption in Pippard, London, or intermediate superconductors. If one averages over the direction, one obtains the linear increase of the absorption near threshold, as predicted for the isotropic model by the BCS theory,[1,2] only for intermediate superconductors.

INTRODUCTION

USING the conservation laws for energy and momentum we get for the absorption of a phonon (or photon) combined with the dissociation of a Cooper pair

$$\Phi(\mathbf{p}, \mathbf{q}) \equiv \varepsilon_{\mathbf{p}} + \varepsilon_{\mathbf{p}-\mathbf{q}}$$

$$= \sqrt{\xi_{\mathbf{p}}^2 + \Delta_{\mathbf{p}}^2} + \sqrt{(\xi_{\mathbf{p}} - \mathbf{v}\mathbf{q})^2 + \Delta_{\mathbf{p}-\mathbf{q}}^2} = \omega \qquad (1)$$

(we use throughout the notation introduced earlier[3]). The threshold frequency is determined by the minimum value of $\Phi(\mathbf{p}, \mathbf{q})$ for fixed $\mathbf{q}$. In the isotropic case, $\Delta = \text{const}$ and the minimum which is equal to 2Δ is attained on the line $\theta = \pi/2$ (θ is the angle between $\mathbf{v}$ and $\mathbf{q}$). The absorption of ultrasound and electromagnetic waves near threshold in the isotropic model was considered in a number of papers[1,2,4,5] and their results lead to the following conclusions.

The absorption of ultrasound[5] near threshold increases from zero to a magnitude of the order of c/v, where c is the sound velocity (strictly speaking this increase occurs over an exponentially small region of frequencies).

The absorption of electromagnetic waves[1,2,4] near threshold increases linearly as $\sim \omega - 2\Delta$.

We shall show in the following that the anisotropy of a superconductor changes the frequency dependence of the absorption fundamentally. This is caused by two circumstances:

1) The minimum of the left-hand side of (1) occurs for anisotropic superconductors not on the line $\theta = \pi/2$ as in the isotropic case, but in separate points on the Fermi surface. Because of this there is near threshold a contribution to the absorption from an appreciably smaller part of the Fermi surface than in the isotropic case.

2) In the electromagnetic case the decrease in the region of allowed dissociation of a pair is accompanied by an appreciable increase in the probability for dissociation and this leads to an increase in the absorption in the region immediately adjoining the threshold as compared to the absorption in isotropic superconductors.

ABSORPTION OF ULTRASOUND

We use a diagram technique for our calculations. To find the absorption it is necessary to evaluate the imaginary part of the polarization operator $\Pi(\mathbf{q}, \omega)$. We have shown earlier[5] that one can perform the calculation in the weak coupling approximation and that the results remain qualitatively correct also in the strong coupling approximation, which in actual fact occurs in superconductors. It is therefore sufficient to take into account the diagrams of Fig. 1 to calculate Im Π. Apart from numerical factors we then get (see reference 3)

$$\operatorname{Im}\Pi(\mathbf{q}, \omega) \sim \int d^3 p \left[u^2 v'^2 + \frac{\Delta\Delta'}{4\varepsilon\varepsilon'}\right] g^2 g'^2 \delta(\varepsilon + \varepsilon' - \omega), \qquad (2)$$

where $g = g(\mathbf{p})$ is a dimensionless function which characterizes the anisotropy of the electron-phonon interaction while the rest of the notation is the

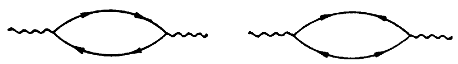

FIG. 1

same as the one used before.[3] The primed quantities are here functions of the argument $\mathbf{p}-\mathbf{q}$. Assuming $q \ll p_0$ we can in the following put $\Delta' \approx \Delta$, $g' \approx g$. It is, however, impossible to put ξ and ξ', and ϵ and ϵ' equal to one another as near threshold the quantities ξ, vq, and Δ may be of the same order of magnitude. We have

$$\xi' = \xi - vq\cos\theta, \qquad \varepsilon' = \sqrt{\xi'^2 + \Delta^2}. \tag{3}$$

Owing to the δ-function, the integration in (2) is performed over a thin layer of thickness $\sim\Delta$ near the Fermi surface. This enables us to reduce the integration over momentum space to an integration over ξ and over the unit sphere which is the stereographic projection of the Fermi surface (see reference 3).

We get thus

$$\operatorname{Im}\Pi(\mathbf{q},\omega) \sim \iint g^4 \frac{\sin\theta\, d\theta\, d\varphi}{Kv} \times \int_{-\infty}^{\infty} d\xi \left[\left(1+\frac{\xi}{\varepsilon}\right)\left(1-\frac{\xi'}{\varepsilon'}\right)+\frac{\Delta^2}{\varepsilon\varepsilon'}\right]\delta(\varepsilon+\varepsilon'-\omega) \tag{4}$$

(the direction of $\mathbf{q}$ is chosen as the polar axis); $K(\theta,\varphi)$ is the Gaussian curvature of the Fermi surface.

The roots of the equation $\epsilon + \epsilon' - \Delta = 0$ are determined by the equation

$$\xi_{1,2} = \frac{1}{2}\left\{vq\cos\theta \pm \omega\left[\frac{\omega^2 - v^2q^2\cos^2\theta - 4\Delta^2(\theta,\varphi)}{\omega^2 - v^2q^2\cos^2\theta}\right]^{1/2}\right\}. \tag{5}$$

For real phonons $(q = \omega/c)$ this equation can be transformed to

$$\xi_{1,2} = \tfrac{1}{2}\left[\omega v c^{-1}\cos\theta \pm \sqrt{\omega^2 - \Omega^2(\theta,\varphi)}\right], \tag{6}$$

where

$$\Omega^2(\theta,\varphi) = 4\Delta^2(\theta,\varphi)/(1 - v^2c^{-2}\cos^2\theta). \tag{7}$$

The quantities ξ_1 and ξ_2 obviously satisfy the relations

$$\xi_1' = -\xi_2, \qquad \xi_2' = -\xi_1, \qquad \varepsilon_1' = \varepsilon_2, \qquad \varepsilon_2' = \varepsilon_1, \tag{8}$$

the geometric meaning of which is clear from Fig. 2.

If we perform the integration over ξ and use (8) we get

$$\operatorname{Im}\Pi(\mathbf{q},\omega) \sim \iint \frac{\sin\theta\, d\theta\, d\varphi}{Kv} g^4 \frac{(\xi_1+\varepsilon_1)(\xi_2+\varepsilon_2)+\Delta^2}{\xi_1\varepsilon_2 - \xi_2\varepsilon_1}. \tag{9}$$

The domain of integration in (9) is limited by the

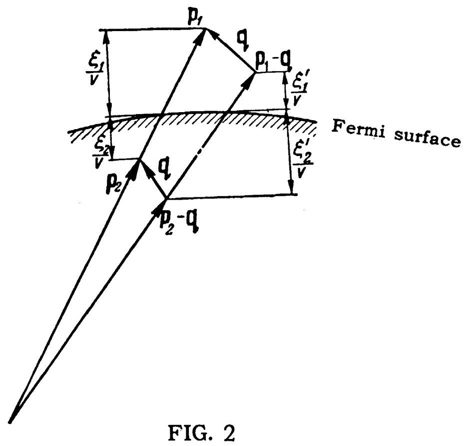

FIG. 2

conditions that the roots ξ_1 and ξ_2 be real

$$\omega^2 - \Omega^2(\theta,\varphi) \geqslant 0, \tag{10}$$

from which we get the inequality

$$|\cos\theta| \leqslant c\sqrt{\omega^2 - 4\Delta^2}/v\omega. \tag{11}$$

The threshold frequency Ω_0 is determined by the minimum value of $\Omega(\theta,\varphi)$. Since the ratio c/v ($c/v \sim 10^{-2}$ to 10^{-3}) is small, it follows from (11) that the minimum of $\Omega^2(\theta,\varphi)$ occurs near the line $\theta = \pi/2$. We expand Δ^2 in a power series in $x = \cos\theta$ and $\varphi - \varphi_0$, where φ_0 is the point where $\Delta^2(\pi/2,\varphi)$ is a minimum:

$$\Delta^2(x,\varphi) = \Delta_0^2\,[1 + ax + b(\varphi-\varphi_0)^2], \tag{12}$$

where $\Delta_0^2 = \min(\Delta^2(0,\varphi))$, and a and $b \sim 1$. Near this point we have

$$\Omega^2(x,\varphi) = 4\Delta_0^2\,[1 + ax + b(\varphi-\varphi_0)^2]/(1 - v_0^2c^{-2}x^2). \tag{13}$$

If we minimize this expression with respect to x we get at the minimum

$$x_0 = -\frac{a}{2}\left(\frac{c}{v_0}\right)^2, \qquad \Omega_0^2 = 4\Delta_0^2\left[1 - \frac{a^2}{4}\left(\frac{c}{v_0}\right)^2\right] \approx 4\Delta_0^2. \tag{14}$$

Near threshold we have

$$\omega^2 = \Omega_0^2(1+\alpha), \qquad \alpha \ll 1. \tag{15}$$

Using (10) and (13) we find that the domain of integration in the x,φ-plane is the interior of the ellipse

$$v_0^2 c^{-2}(x-x_0)^2 + b(\varphi-\varphi_0)^2 \leqslant \alpha. \tag{16}$$

Evaluating the integral in (9) up to terms of order $\sim\alpha^{1/2}$ we get

$$\operatorname{Im}\Pi = Ac\,(\omega - \Omega_0)^{1/2}/v_0, \qquad A \sim g_0^4/K_0 v_0 \Omega_0^{1/2}. \tag{17}$$

It was shown earlier[3] that the damping of ultrasound is determined by the equation

$$\alpha_s(\mathbf{n}) = \gamma(\mathbf{n})\,\omega\,\operatorname{Im}\Pi, \tag{18}$$

where $\gamma(\mathbf{n})$ is some experimentally immeasurable function of the direction. We have thus near threshold

$$\alpha_s(\mathbf{n}) = B(\mathbf{n})\, c\,(\omega - \Omega_0)^{1/2}/v_0, \qquad B(\mathbf{n}) = \gamma(\mathbf{n})\, A\Omega_0. \quad (19)$$

The minimum of the quantity $\Omega^2(\theta, \varphi)$ occurs in the isotropic case everywhere along the line $\theta = \pi/2$. The integration is in that case performed over a "ribbon" of width $\sim c\sqrt{\alpha}/v_0$ girdling the equator of the unit sphere, and it leads to the result obtained before.[5]

ABSORPTION OF ELECTROMAGNETIC WAVES

The absorption is in this case also determined by the imaginary part of the polarization tensor $\Pi_{\alpha\beta}$. If we restrict our considerations to the diagrams of Fig. 1 we have (apart from numerical factors)

$$\operatorname{Im}\Pi_{\alpha\beta} \sim \int d^3p\, v_\alpha v_\beta \left[u^2v'^2 + v^2u'^2 - \frac{\Delta\Delta'}{2\varepsilon\varepsilon'}\right]\delta(\varepsilon + \varepsilon' - \omega). \quad (20)$$

Changing to an integration over ξ and over the unit sphere, we find

$$\operatorname{Im}\Pi_{\alpha\beta} \sim \iint \frac{\sin\theta\, d\theta\, d\varphi}{Kv} v_\alpha v_\beta \int_{-\infty}^{\infty} d\xi\, \frac{\varepsilon\varepsilon' - \xi\xi' - \Delta^2}{\varepsilon\varepsilon'}\, \delta(\varepsilon + \varepsilon' - \omega). \quad (21)$$

In our problem the characteristic dimension is the penetration depth δ, which is appreciably less than the wavelength in vacuo $2\pi c/\omega \sim c/\Delta$; there is thus no dispersion relation of the form $\omega = cq$. The roots of the equation $\epsilon + \epsilon' - \omega = 0$ are thus determined by (5), and the condition that they be real assumes the form

$$\omega^2 - v^2q^2\cos^2\theta - 4\Delta^2(\theta, \varphi) \geqslant 0. \quad (22)$$

If we evaluate the integral over ξ we get

$$\operatorname{Im}\Pi_{\alpha\beta} \sim \iint \frac{\sin\theta\, d\theta\, d\varphi}{Kv} v_\alpha v_\beta \frac{\varepsilon_1\varepsilon_2 + \xi_1\xi_2 - \Delta^2}{\xi_1\varepsilon_2 - \xi_2\varepsilon_1}. \quad (23)$$

The magnitude of the integral in (23) depends essentially on whether we are considering a Pippard, London, or intermediate metal.

a) Pippard metal, $vq \gg \Delta$. The threshold frequency Ω_0 is determined by the minimum value of the quantity $\Omega^2(\theta, \varphi) = v^2q^2\cos^2\theta + 4\Delta^2(\theta, \varphi)$. Because of the condition $vq \gg \Delta$ the minimum of $\Omega^2(\theta, \varphi)$ occurs near the line $\theta = \pi/2$. Calculations similar to the ones for the acoustic case give

$$\Omega^2(x, \varphi) = 4\Delta_0^2\,[1 + a_1x + b_1(\varphi - \varphi_0)^2 + (vq/2\Delta_0)^2x^2],$$
$$x_0 = -\frac{a_1}{2}\left(\frac{2\Delta_0}{v_0q}\right)^2, \qquad \Omega_0^2 = 4\Delta_0^2\left[1 - \frac{a_1^2}{4}\left(\frac{2\Delta_0}{v_0q}\right)^2\right] \cong 4\Delta_0^2. \quad (24)$$

We find from condition (22) that the domain of integration is the interior of the ellipse

$$\left(\frac{v_0q}{2\Delta_0}\right)^2(x - x_0)^2 + b_1(\varphi - \varphi_0)^2 \leqslant \alpha. \quad (25)$$

The absorption is essentially determined by the transverse components of the tensor $\Pi_{\alpha\beta}$ which do not contain Fermi-velocity components v_α and v_β proportional to the small quantity $\cos\theta$.

If we evaluate the integral in (23) up to the lowest powers of α and Δ_0/v_0q we get

$$\operatorname{Im}\Pi_{\alpha\beta} \sim \frac{v_{\alpha 0}v_{\beta 0}}{K_0v_0}\frac{\Delta_0}{v_0q}\left[\alpha^{3/2} + 3a_1^2\left(\frac{\Delta_0}{v_0q}\right)^2\alpha^{1/2}\right], \quad (26)$$

where $v_{\alpha 0}$ and $v_{\beta 0} \sim v_0$. There are in the Pippard case thus two regions of frequencies near threshold:

1) $\alpha \gg (\Delta_0/v_0q)^2$. In that case we have

$$\operatorname{Im}\Pi_{\alpha\beta} = A_1(\Delta_0/v_0q)(\omega - \Omega_0)^{3/2}, \qquad A_1 \sim v_{\alpha 0}v_{\beta 0}/K_0v_0\Omega_0^{3/2}. \quad (27)$$

2) $\alpha \ll (\Delta_0/v_0q)^2$. Then

$$\operatorname{Im}\Pi_{\alpha\beta} = A_2(\Delta_0/v_0q)^3(\omega - \Omega_0)^{1/2}, \qquad A_2 = 3A_1a_1^2\Omega_0. \quad (28)$$

The minimum of the quantity Ω^2 occurs in the isotropic model everywhere along the line $\theta = \pi/2$. If the frequency is close to the threshold frequency integration leads in that case to the result of references 2 and 4:

$$\operatorname{Im}\Pi_{\alpha\beta} \sim \Delta(\omega - \Omega_0)/vq. \quad (29)$$

A comparison of Eqs. (27) and (28) with (29) shows that the anisotropy reduces the absorption in region 1), but enhances it in region 2) which is in the immediate vicinity of the threshold.

Experiments to measure the absorption of electromagnetic waves (see, for instance, reference 6) are usually performed on polycrystalline superconductors and at various angles of incidence. In that case one must average the result (26) over the various directions. The role of the threshold frequency for such experiments is played by twice the absolute minimum of the quantity Δ. An effective contribution to the averaged absorption is given near threshold only by those great circles on the stereographic projection of the Fermi surface which pass through a small ellipse (with semi-axes $\sim\sqrt{\alpha}$) which lies near the points where $\Delta(\theta, \varphi)$ is an absolute minimum. These circles occupy a band of width $\sim\sqrt{\alpha}$ on the sphere.

The averaging thus results in an additional small factor $\sqrt{\alpha}$ in Eq. (26). Moreover, in this case we must put in that equation $a_1 = 0$. The experimentally observed absorption must thus for such experiments be proportional to $\sim \alpha^2$.

b) London metal, $vq \ll \Delta$. The point where $\Omega^2(x, \varphi)$ is a minimum is in that case practically the same as the point where the energy gap $\Delta(x, \varphi)$ has its absolute minimum. Near threshold we have

$$\omega^2 - \Omega^2(x, \varphi) = 4\Delta_m^2 [\alpha - a_2(x - x_0)^2 - b_2(\varphi - \varphi_0)^2] \geqslant 0.$$

Evaluating the integral in (23) we get

$$\operatorname{Im} \Pi_{\alpha\beta} \sim \frac{v_\alpha v_\beta}{K_0 v_0} \left(\frac{v_0 q}{2\Delta_m}\right)^2 x_0^2 \alpha^{1/2}. \tag{30}$$

The averaging over directions introduces in this case no added small factors.

c) Intermediate metal, $vq \sim \Delta$. In this case, which is the most interesting from an experimental point of view, the quantity $\Omega^2(x, \varphi)$ has its minimum in some point x_1, φ_1 which neither lies on the line $x = 0$ nor is the same as the point where $\Delta(x, \varphi)$ is a minimum. The threshold frequency $\Omega_0 = \min \Omega(x, \varphi)$ has therefore no simple relation with the values of the energy gap.

The integration is in this case performed over the interior of the ellipse

$$a_3(x - x_1)^2 + b_3(\varphi - \varphi_1)^2 \leqslant \alpha. \tag{31}$$

The numerator of the integrand in (23) is not small (as far as the parameter α is concerned) while the denominator is of the order of $\sqrt{\alpha}$. The absorption changes thus in this case, as in the case of the London metal, near threshold proportional to $x_0^2 \alpha^{1/2}$ but the factor $(v_0 q/\Omega_0)^2$ which occurs in the expression for $\operatorname{Im} \Pi_{\alpha\beta}$ instead of the factor $(v_0 q/2\Delta_m)^2$ in (30) is not small.

Averaging over the directions gives for intermediate metals an additional small factor $\sqrt{\alpha}$. In fact, the absolute threshold frequency is in that case equal to $2\Delta_m$ and occurs for any directions of $\mathbf{q}$ which are perpendicular to the Fermi velocity in the point where the gap has its absolute minimum. The integration over directions is therefore performed over a narrow band of width $\sim\sqrt{\alpha}$ near the line $\theta_q = \pi/2$ and this leads to the factor $\sqrt{\alpha}$.

The absorption in polycrystalline samples of intermediate metals is thus proportional to $\omega - 2\Delta_m$. We note that the isotropic BCS model also leads to the same result (both for intermediate and for London metals). We emphasize once again that the agreement of qualitative results for the isotropic and for the averaged anisotropic model occurs only for the intermediate superconductors.

CONCLUSION

There are as yet no experiments on the absorption of ultrasound in the threshold region, since the corresponding frequencies cannot yet be attained. The only experiment we know of, in which the frequency dependence of the absorption of electromagnetic waves near threshold was measured at low temperatures, is described by Richards and Tinkham.[6] The total number of experimental points and the experimental accuracy are insufficient to compare theory and experiment. Moreover, the majority of the data obtained in that paper refer to intermediate metals for which the results of the isotropic model are the same as those for the averaged anisotropic model. The linear increase of the absorption with frequency is apparently verified experimentally.

The fact that the threshold frequencies for In, Hg, and Pb measured by the above mentioned authors[6] turned out to be larger than the ones predicted by the isotropic theory ($\Omega_0 = 3.5\,kT_c$) is very surprising. Indeed, in a real polycrystalline superconductor the threshold frequency is determined by the minimum value of the gap, and in the relation $\Delta(0) = 1.75\,kT_c$ of the isotropic model $\Delta(0)$ stands for some average over directions of the magnitude of the gap. Similar experiments in a region near the threshold with Pippard metals and also with single crystals are thus very desirable.

Experiments with single crystals are moreover important for the following reason. A measurement of the threshold frequency Ω_0 in Pippard metals allows us to find the minimum value of the energy gap on the great circle of the stereographic projection of the Fermi surface perpendicular to a given direction $\mathbf{n}$ ($\mathbf{n} = \mathbf{q}/q$). We have shown earlier[3] that one can obtain the same result by studying the absorption of ultrasound at frequencies below the threshold frequency at low temperatures ($T \ll \Delta$).

There are already experiments about the anisotropic absorption of ultrasound in that region.[7] It would be very useful to compare the experimental data obtained by Bezuglyĭ and co-workers[7] with electromagnetic measurements near threshold.

Pokrovskii and Toponogov[8] developed a method which allows us to construct the energy gap as a function of direction (everywhere, except for several "blank spots") once we know $\Delta_0 = \Delta_{min}(\mathbf{n})$. A study of threshold phenomena for the absorption of ultrasound and of electromagnetic waves gives thus still another method to construct the energy gap.

In conclusion the authors express their gratitude to L. P. Gor'kov for drawing their attention to the problem considered here.

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