SOVIET PHYSICS JETP VOLUME 14, NUMBER 1 JANUARY, 1962

THE SHELL MODEL AND THE SHIFT OF SINGLE-PARTICLE LEVELS IN NUCLEI OF THE "CORE + NUCLEON" TYPE, DUE TO ADDITION OF A PAIR OF NUCLEONS

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Submitted to JETP editor, January 11, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 171-182 (July, 1961)

The shift of single-particle levels in nuclei of the "core + nucleon" type, caused by the addition of a pair of particles (nucleons or holes) of the same type to the nucleus, is investigated within the framework of the shell model. The shifts are estimated quantitatively by perturbation-theory calculation of the change in level spacing due to unequal shifting of the levels when an additional pair is added to the nucleus. Two mechanisms of level shift are examined, one where the shift is due to a change in the core parameters (isotopic and isotonic shifts) and where the shift results from direct interaction between the pair and odd particle, the core remaining unperturbed (in this case the pair enters the nucleus as a system which is autonomous and independent of the core and in which the particles are paired with respect to the angular momentum J = 0). An estimate of the isotopic and isotonic shifts for a number of typical nuclei indicates that the first mechanism does not correspond to the experiments. On the other hand, a calculation of the relative level shifts based on the second mechanism, performed for a large number of nuclei of the "core + nucleon" type for which experimental level schemes are available, leads to results which are in good agreement with the experimental data. It is shown that there is a competition between the closely-spaced levels on which the pair is located.

INVESTIGATIONS (particularly quantitative) of nuclear configurations within the framework of the shell model entail considerable difficulties, because the nuclear levels are not identified to the same degree of unambiguity in all nuclei. The clearest picture of nuclear configurations is observed in odd nuclei of the "core + nucleon" type, in which both groups of nucleons (after subtracting the odd nucleon) fill closed shells or subshells and form an inert "core." Single-particle energy levels in these nuclei arise as real excited states of the odd nucleon, which is situated in a certain averaged field of the core. The sequence of levels in nuclei of the "core + nucleon" type is satisfactorily described by a Mayer scheme[1] intermediate between the energy schemes corresponding to idealized potentials (oscillator and well).

As shown by the presently known experimental data on nuclear level schemes,[2] the sequence of excited levels, established for odd nuclei of the "core + nucleon" type, can be regarded as valid also for other odd nuclei, which contain one or several added pairs of like nucleons in an even or odd group of like particles. On going to these nuclei, however, i.e., on adding, for example, one pair of nucleons to the initial "core + nucleon" nucleus, the levels of the latter are subjected to unequal shifts and the distances between the corresponding levels change (to the extent that certain level pairs cross).

A level shift of this kind can be attributed either to a change in the core parameters (depth or radius of the well) when a pair of like nucleons is added (isotopic and isotonic shifts), or to direct interaction (in the form of a perturbation) of a pair with the odd nucleon producing the levels; the core remains unperturbed and the pair forms a certain autonomous system in the nucleus, wherein the nucleons are paired with respect to the angular momentum (J = 0).

The purpose of the present article is to establish the character and the main cause of this shift of single-particle levels, and to derive quantitative estimates for the changes in level spacing in nuclei of the "core + nucleon" type when a pair of like nucleons (neutrons or protons) is added.* It is expected that the relative level shift in the two nuclei ("core + nucleon" and "core + nucleon + pair") will indeed be determined by the interaction between the nucleons of the additional pair with the core and the unpaired nucleon, and that a

*This analysis is also valid for nuclei of the "core + hole" type, and for the case when a pair of nucleons is removed (and not added), i.e., when a pair of "holes" is produced. This is implied throughout, and occasionally the single term "particle" will be used.

quantitative analysis of this effect is possible within the framework of the shell model through the use of perturbation theory (in spite of the fact that the shell model does not predict with satisfactory accuracy the level spacings themselves).

1. ISOTOPIC AND ISOTONIC SHIFTS

We examine the level shift of the unpaired nucleon in a "core + nucleon" nucleus, a shift due to the change in the average potential of the core when a pair of neutrons or a pair of protons is added to the nucleus.

Within the framework of the model, the average potential of the core, which characterizes the self-consistent field produced by the nucleons in the closed neutron and proton shells, can be regarded as independent (except for very light nuclei). Consequently, the total self-consistent field is made up of two independent self-consistent fields, the neutron and proton fields, and the average potential of the core can be expressed as a sum of two potentials with corresponding statistical weights, proportional to the number of neutrons and protons in the nucleus. If the unpaired nucleon is a neutron, the average potential of the core can be assumed to have the form[3]

$$V^{(n)} = \frac{N}{A} V_{nn} + \frac{Z}{A} V_{np} = \frac{N_0}{A} V_{nn} + \frac{Z_0}{A} V_{np} + \frac{N - N_0}{A} (V_{nn} - V_{np}), \quad (1)$$

where V_{nn} characterizes the interaction of the neutron with the neutron medium, V_{np} the interaction with the proton medium, and N_0 and Z_0 are the numbers of neutrons and protons corresponding to the stability of the nucleus A. Analogously, for the unpaired proton, the average potential of the core is assumed to be

$$V^{(p)} = \frac{Z}{A} V_{pp} + \frac{N}{A} V_{pn} = \frac{Z_0}{A} V_{pp} + \frac{N_0}{A} V_{pn} - \frac{N - N_0}{A} (V_{pp} - V_{pn}), \quad (2)$$

where V_{pp} and V_{pn} determine the interaction of the proton with the proton and neutron media.

For stable nuclei, the potentials (1) and (2) consist of only the first two terms, which characterize the potential of the stable core V_0. After subtracting the Coulomb interaction, V_0 is identical for the unpaired neutron and proton ($V_0^{(n)} \approx V_0^{(p)} \approx V_0$). On the other hand, in view of the charge independence of the nuclear forces, it is natural to assume $V_{pp} \approx V_{nn}$ and $V_{np} \approx V_{pn}$. As a result we obtain a two-parameter average core potential for the external unpaired neutron and proton*

$$V^{(n)}(Z, N) = V_0 + \frac{N - N_0}{A} V',$$

$$V^{(p)}(Z, N) = V_0 + \frac{Z - Z_0}{A} V' + V_{\text{Coul}}, \quad (3)$$

where

$$Z - Z_0 = -(N - N_0), \qquad V' \approx V_{nn} - V_{np} \approx V_{pp} - V_{pn}$$

and V_{Coul} is the Coulomb energy of the unpaired proton.

In the calculation of the relative shifts of the nuclear levels, potentials (3) were regarded as ordinary rectangular potential wells (without account of the smearing of the potential on the boundary) that depend on the neutron-proton distribution in the nucleus A(Z, N).

We chose for the potentials (3) the parameters $V_0 = 45$ Mev and $V' = -89$ Mev (the nuclear radius is $R = 1.4\,A^{1/3}$ Fermi units). This choice is not an essential limitation, since it can be shown that the results of the calculations vary very little with small changes in the parameters of the potentials in (3), and that any combination of parameters in this range can be used in practice.

The distance between the levels nlj (E_{nlj}) and $n'l'j'$ ($E_{n'l'j'}$) in the nucleus A(Z, N) is given by the following well-known expression:

$$\delta E_{n'l'j',\, nlj}(Z, N) = E_{n'l'j'}(Z, N) - E_{nlj}(Z, N) = 10.5 \cdot A^{-2/3} \{[x_{n'l'j'}(Z, N)]^2 - [x_{nlj}(Z, N)]^2\} \text{ Mev}; \quad (4)$$

where here x_{nlj} are the roots of the transcendental equation obtained from the condition that the logarithmic derivatives of the external and internal wave functions of the unpaired nucleon be equal on the boundary of the potential well (3). On going from nuclei A(Z, N) of the "core + particle" type to nuclei $A \pm 2\,(Z, N \pm 2)$ and $A \pm 2\,(Z \pm 2, N)$ (i.e., on "adding" or "removing" a pair of like nucleons), the energy levels of the unpaired particle E_{nlj} experience an unequal shift ΔE_{nlj}, as a result of which the distances between levels (4) change. Depending on the type of additional pair of nucleons, the relative level shifts due to the change in the core potential can be classified either as isotopic ($\Delta Z = 0$, $\Delta N = \pm 2$)

$$\Delta_{(\Delta N)}(\delta E_{n'l'j',\, nlj}) = \delta E_{n'l'j',\, nlj}(Z, N \pm 2) - \delta E_{n'l'j',\, nlj}(Z, N) \quad (5)$$

or isotonic ($\Delta N = 0$, $\Delta Z = \pm 2$)

*A potential of this kind, but with smeared edge (with account of the diffuse boundary of the nucleus) was investigated by Sliv and Volchok.[4]

$$\Delta_{(\Delta Z)}\,(\delta E_{n'l'j',\,nlj}) = \delta E_{n'l'j',\,nlj}\,(Z\pm 2,\,N) - \delta E_{n'l'j',\,nlj}\,(Z,\,N). \qquad (5')$$

Equations (4), (5), and (5′) were used to calculate the change in distances between different levels for the transitions $A \rightarrow A \pm 2$ in a series of typical nuclei with known experimental level schemes. The results of the calculations, together with the experimental values of the change in distance between corresponding levels, are listed in Tables I and II.

As can be seen from these tables, the nuclei considered were those in which the numbers of neutrons N and protons Z corresponded to the filled shells (subshells) in the Mayer scheme. The numbers (+1) or (−1) in the columns marked N and Z indicate the type of odd nucleon (or hole) whose levels are shifted by a change in the potential of a core made up of a combination of nucleons (Z, N) from the closed shells (subshells). In many cases the problem is not single valued: the energy spectrum of the nucleus can be simultaneously regarded as pertaining either to a nucleon or to a hole, but with different core potentials. For example, for the nucleus with 39 protons, we can consider, along with the variant having an unpaired proton Z = 38 (+1), an alternate variant with an unpaired hole Z = 40 (−1) (the filled subshells for Z = 38 and Z = 40 are $1f_{5/2}$ and $2p_{1/2}$ respectively). The core potentials corresponding to these two cases will be different, owing to the dependence of the potential V(Z, N) on Z. Duplicate calculations were carried out for these nuclei (the results of these calculations were not listed in Tables I and II because they did not improve the agreement with experiment).

From a comparison of the theoretical change in level spacing with the experimental data in Tables I and II it follows quite obviously that the proposed mechanism, where the levels are shifted by a change in the core parameters (the isotopic and isotonic potential shifts), does not correspond to reality at all (the discrepancies are both in sign and in order of magnitude); this was expected from preliminary estimates of the effect. In any case, this mechanism could not be the only cause of level shift of an odd particle when a pair is added to a "core + particle" nucleus. Characteristically, the determined isotopic and isotonic relative level shifts are as a rule one order of magnitude smaller (in absolute value) than the corresponding experimental values; this demonstrates that the proposed level-shift mechanism is patently insufficient to explain the real level shifts.

2. PERTURBATION OF SINGLE-PARTICLE LEVELS BY A NUCLEON PAIR ADDED TO THE CORE

We shall assume that when two like nucleons (or holes) are added to a nucleus A of the "core + nucleon" (or "core + hole") type, the added

Table I. Isotopic shift ($\Delta Z = 0$, $\Delta N = \pm 2$)

Level difference $n'l'j' - nlj$	Z	N	ΔN	$\Delta(\delta E)_{theor}$, kev	$\Delta(\delta E)_{exp}$, kev
$1h_{11/2} - 2d_{3/2}$	58	78(+1)	+2	+51	+485
$1g_{9/2} - 2p_{1/2}$	50(−1)	64	+2	−6	+58
$1g_{9/2} - 2p_{1/2}$	50(−1)	66	+2	+1	+23
$1g_{9/2} - 2p_{1/2}$	38(+1)	50	−2	+103	−532
$1g_{9/2} - 2p_{1/2}$	38(+1)	50	+2	+31	−356
$1g_{9/2} - 2p_{1/2}$	50	40(+1)	+2	+76	−286
$1h_{11/2} - 3s_{1/2}$	50	66(+1)	+2	+72	−231
$1f_{5/2} - 2p_{3/2}$	38(−1)	50	−2	+18	−550
$2d_{3/2} - 3s_{1/2}$	50	66(+1)	+2	+4	−137
$2p_{3/2} - 1f_{7/2}$	20	20(+1)	+2	−346	−1359
$2p_{3/2} - 1f_{7/2}$	20(+1)	28	−2	−55	−2590

Table II. Isotonic shifts ($\Delta N = 0$, $\Delta Z = \pm 2$)

Level difference $n'l'j' - nlj$	N	Z	ΔZ	$\Delta(\delta E)_{theor}$, kev	$\Delta(\delta E)_{exp}$, kev
$1h_{11/2} - 2d_{3/2}$	82(−1)	58	−2	+ 9	− 79
$1h_{11/2} - 2d_{3/2}$	78(+1)	58	−2	− 10	+ 13
$1g_{9/2} - 2p_{1/2}$	50(−1)	38	−2	+ 12	+ 83
$1g_{9/2} - 2p_{1/2}$	50(−1)	38	+2	+ 10	− 200
$1g_{9/2} - 2p_{1/2}$	56	40(+1)	+2	− 33	+ 608
$1g_{9/2} - 2p_{1/2}$	50(−1)	40	+2	− 40	− 70
$1g_{9/2} - 2p_{1/2}$	50	38(+1)	+2	+150	−1017
$1g_{9/2} - 2p_{1/2}$	40(+1)	50	+2	− 46	+ 75

pair is not incorporated in the core, but enters autonomously into one of the shells outside the core, in a state "paired" with respect to the angular momentum ($J = 0$). This added pair interacts with the unpaired nucleon, perturbs the states of the latter, and thereby produces unequal shifts of the various single-particle excitation levels of the nucleus. The unequal shifts change the relative distances between the levels.

The relative shift of the single-particle levels nlj and $n'l'j'$, for two corresponding nuclear configurations A and $A \pm 2$, is determined by the expression

$$\Delta_{(\pm 2)}(\delta E^{(A)}_{n'l'j',\, nlj}) = \Delta_{(\pm 2)} E^{(A)}_{n'l'j'} - \Delta_{(\pm 2)} E^{(A)}_{nlj}, \qquad (6)$$

where $\Delta_{(\pm 2)} E^{(A)}_{nlj}$ and $\Delta_{(\pm 2)} E^{(A)}_{n'l'j'}$ are individual shifts of the levels nlj and $n'l'j'$, analogous to isotopic and isotonic shifts. These shifts, however, are no longer due to a change in the core potential upon "addition" of the pair, but to the direct interaction between the pair and the unpaired nucleon V_{1P}. The perturbing potential V_{1P} was chosen in the form of a δ-function two-particle potential with all types of interactions between nucleons (with exception of tensor forces)

$$V_{1P} = 4\pi \sum_{i=1}^{2} \delta(\rho - r_i)\{a(\tau\tau_i)(\sigma\sigma_i) + c + d(\sigma\sigma_i) + e(\tau\tau_i)\} \qquad (7)$$

Within the framework of the single-particle model, the individual level shift $\Delta_{(\pm 2)} E^{(A)}_{nlj}$ was considered, in accordance with perturbation theory, as a first-approximation correction to the eigenvalue of the energy of the unpaired particle in the nucleus, without account of the correlation interaction* between the particles within the pair occupying the level $n_0 l_0 j_0$:

$$\Delta_{(\pm 2)} E_{nlj;\, n_0 l_0 (j_0) J=0} = \langle V_{1P} \rangle = (\Psi_{nlj;\, n_0 l_0 (j_0) J=0} \,|\, \hat{V}_{1P} \,|\, \Psi_{nlj;\, n_0 l_0 (j_0) J=0}). \qquad (8)$$

We chose for the core potential the two idealized potentials of the shell model—a rectangular well and oscillator potential. In the analysis of the nucleon (or hole) distribution over the levels, however, we used the real level sequence, which is intermediate between the levels corresponding to the two idealized potentials. A characteristic feature of our calculations was that in the case of the rectangular well we used the core potential $V(Z, N)$ defined in (3) and dependent of the number of neutrons N and protons Z in the nucleus (not only on the atomic number A), with a well radius $R = 1.4\, A^{1/3}$ Fermi units. The oscillator potential considered was standard (with the usual dependence on A but not on N or Z), and consequently isotopic and isotonic effects of the core potential were not taken into account.

Substitution of the antisymmetrized wave function of the nucleus into (A) leads to the following final formula for the level shifts (for either the rectangular-well or for the oscillator potential):

$$\Delta_{(\pm 2)} E^{(A)}_{nlj;\, n_0 l_0 (j_0) J=0} = 2(2j+1)\{f + gB(l, l_0; j, j_0)\} D_{nl,\, n_0 l_0}, \qquad (9)$$

where

$$B(l, l_0; j, j_0) = \frac{2}{(2j+1)(2j_0+1)} \sum_L K(l, l_0, L) \frac{(l+L-l_0)!\,(l+l_0-L)!\,(L+l_0-l)!}{(l+l_0+L+1)!} \times \left\{\left(\frac{l+l_0+L}{2}\right)! \Big/ \left(\frac{l+l_0+L}{2} - l\right)! \left(\frac{l+l_0+L}{2} - l_0\right)! \left(\frac{l+l_0+L}{2} - L\right)!\right\}^2,$$

$$K(l, l_0, L) = \begin{cases} (L + l_0 - l + 1)(L - l_0 + l), & j = l - \frac{1}{2}, \quad j_0 = l_0 + \frac{1}{2} \\ (L + l_0 + l + 2)(-L + l_0 + l + 1), & j = l + \frac{1}{2}, \quad j_0 = l_0 + \frac{1}{2} \\ (L + l_0 + l + 1)(-L + l_0 + l), & j = l - \frac{1}{2}, \quad j_0 = l_0 - \frac{1}{2} \\ (L + l_0 - l)(L - l_0 + l + 1), & j = l + \frac{1}{2}, \quad j_0 = l_0 - \frac{1}{2} \end{cases}$$

$$|l - l_0| \leqslant L \leqslant l + l_0, \qquad l + l_0 + L - \text{even},$$

$$D_{nl,\, n_0 l_0} = \int |R_{nl}(\rho)|^2 |R_{n_0 l_0}(\rho)|^2 \rho^2 d\rho. \qquad (10)$$

The constants f and g (9) are combinations of the constants a, c, d, and e of the perturbing interaction potential (7), and depend on whether unlike particles [n(pp) or p(nn)] or like particles [n(nn) or p(pp)] enter into the interacting "particle–pair" system: in the case of a combination of unlike particles

$$f_u = c - d - e + a, \qquad g_u = d - a; \qquad (11)$$

and in the case of combination of like particles

*The correlation interaction could change the wave function of the nucleus, and this would lead to some new theoretical values of the relative level shifts. We assume this effect too small in the first approximation.

$$f_1 = c - d + e - a, \quad g_1 = d + a. \qquad (11')$$

The radial integrals D_{nl,n_0l_0} can be calculated only by numerical means. Calculations for an oscillator potential were made by Zeldes,[5] who tabulated all the main radial integrals. For a rectangular well, the tabulation of the integrals becomes impossible, in view of the fact that the integrals are not standard and in view of the explicit dependence on the well radius R and of the isotopic or isotonic properties of the potential V(Z, N), so that an individual calculation is necessary for each specific case.

An analysis was made of practically all nuclei A of the "core + particle" type, and of the corresponding nuclei $A \pm 2$ of the "core + particle + pair" type, for which either the experimental level schemes (or at least one pair of corresponding levels with an established relative shift) were known. The core nuclei chosen had closed shells or subshells, the Mayer scheme serving as a criterion. The constants f and g were chosen from a comparison of the calculated relative level shifts with the experimental values, made simultaneously for all investigated typical nuclei. Nuclei in which the level shift is due to a combination of interacting unlike particles (the constants f_u, g_u) were distinguished from those with a combination of interacting like particles (the constants f_l, g_l).

The constants were chosen such as to obtain satisfactory relative level shift values for the greatest possible number of nuclei. For this purpose, the system of equations in the form $\alpha f + \beta g = \Delta(\delta E)_{exp}$, obtained for the entire group of investigated nuclei with the aid of formulas (6) and (9) for the ground levels (according to the Mayer scheme) and near-lying levels of the pair, was reduced to a system of equations in the form $\alpha' f/g + \beta' = 1/g$. This system was solved by using a wide range of trial values of f/g, in such a way that a constant (1/g) common to all nuclei was obtained from the left half of the corresponding equation for at least one level of each nucleus. We chose in this fashion the following values of the constants for the two basic potentials and for the two types of interacting "particle-pair" combinations:

$f_{u(well)} = +1$, $g_{u(well)} = -0.1$; $f_{u(osc)} = +0.317$, $g_{u(osc)} = -0.033$ and $f_{l(well)} = +0.5$, $g_{l(well)} = -2.5$; $f_{l(osc)} = +2.5$, $g_{l(osc)} = -1.25$,

With these constants the theoretical values of the relative nuclear-level shifts deviated least from the corresponding experimental values. The ratio of the constants f_u/g_u and f_l/g_l was compared with the results of certain energy calculations based on the shell model (in particular, with Flowers' results,[6] where the best values of these constants were determined from a theoretical analysis of the energy scheme of the O^{16} nucleus), and also with the corresponding analysis of experiments on np scattering at high energies. It turned out that when an odd neutron interacts with a pair of protons or an odd proton interacts with a pair of neutrons, the estimated ratio of the constants is $f_u/g_u \sim (+10)/(-1)$ for either the oscillator or the rectangular-well potential. This agrees both with Flowers' data and with the corresponding results of the analysis of np-scattering experiments. To the contrary, when an odd nucleon interacts with a pair of like particles [n(nn) or p(pp)], the ratio f_l/g_l for the oscillator potential differs from that for the rectangular well, namely

$$f_{l(osc)}/g_{l(osc)} \sim (+2)/(-1), \quad f_{l(well)}/g_{l(well)} \sim (+1)/(-5).$$

Whereas in the case of an oscillator potential the ratio of the constants (~ -2) coincides with the np-scattering data (but not with the data obtained by energy calculations based on the shell model), in the case of the rectangular well the ratio obtained for the constants $(-1/5)$ has never been encountered before, and agrees neither with Flowers' shell calculations $(-4/5)$ nor with the results of the analysis of the np-scattering experiments (-2).

The results of the calculations and their comparison with the experimental data are summarized in Tables III, IIIa (combinations of unlike particles) IV, and IVa (combinations of like particles).* All possible pairs of nuclear levels (of typical nuclei), for which a relative shift has been established experimentally, are considered. Omissions from the tables denote that the corresponding theoretical values greatly deviate from the experimental result; the relative shifts in parentheses are those for which calculations yield only qualitatively correct results. For the levels that the pair added to the nucleus can occupy, Tables IIIa and IVa indicate two or three alternative versions, the first of which (underlined) is the ground level (above the core) within the framework of the Mayer scheme. The latter is the result of the approach used in this investigation, wherein the data on the relative level shifts are distributed among Tables III, IV, IIIa, and IVa.

In each calculation of the relative level shifts of an odd particle, we assumed first that the pair

*$\bar{n}$ and $\bar{p}$ denote in the tables holes in neutron and proton shells respectively.

Table III. Relative shift of single-particle levels $\Delta(\delta E)$ (combinations of unlike interacting particles)

${}_{Z}X_{N}^{A} \to {}_{Z'}X_{N'}^{A\pm 2}$	Level difference $n'l'j' - nlj$	$\Delta(\delta E)_{exp}$, kev	Combination of interacting particles	Level of pair $n_0 l_0 j_0$	$\Delta(\delta E)_{well}$, kev	$\Delta(\delta E)_{osc}$, kev
${}_{41}Nb_{50}^{91} \to {}_{41}Nb_{52}^{93}$	$1g_{9/2} - 2p_{1/2}$	+75	$p\,(nn)$	$2d_{5/2}$	+77	(+90)
${}_{49}In_{64}^{113} \to {}_{49}In_{66}^{115}$	$1g_{9/2} - 2p_{1/2}$	+58	$\bar{p}\,(nn)$	$3s_{1/2}$	+61	+60
${}_{39}Y_{50}^{89} \to {}_{39}Y_{48}^{87}$	$1g_{9/2} - 2p_{1/2}$	−532	$\bar{p}\,(\bar{n}\bar{n})$	$1g_{9/2}$	−536	
${}_{38}Sr_{49}^{87} \to {}_{40}Zr_{49}^{89}$	$1g_{9/2} - 2p_{1/2}$	−200	$\bar{n}(pp)$	$2p_{1/2}$	−182	
${}_{28}Ni_{33}^{61} \to {}_{26}Fe_{33}^{59}$	$1f_{5/2} - 2p_{3/2}$	−261	$n\,(\bar{p}\bar{p})$	$1f_{7/2}$	−260	
${}_{21}Sc_{28}^{49} \to {}_{21}Sc_{26}^{47}$	$1f_{5/2} - 2p_{3/2}$	−140	$p\,(\bar{n}\bar{n})$	$1f_{7/2}$	(−268)	−133

Table IIIa. Relative shift of single-particle levels $\Delta(\delta E)$ (combinations of unlike interacting particles). Competition of pair levels

${}_{Z}X_{N}^{A} \to {}_{Z'}X_{N'}^{A\pm 2}$	Level difference n'l'j'−nlj	$\Delta(\delta E)_{exp}$, kev	Combination of interacting particles	Level of pair $n_0 l_0 j_0$	$\Delta(\delta E)_{well}$, kev	$\Delta(\delta E)_{osc}$, kev
${}_{38}Sr_{49}^{87} \to {}_{36}Kr_{49}^{85}$	$1g_{9/2} - 2p_{1/2}$	+83	$\bar{n}(\bar{p}\bar{p})$	$1f_{5/2}$ / $2p_{1/2}$	+78	(+92)
				$2p_{3/2}$		+82
${}_{40}Zr_{49}^{89} \to {}_{42}Mo_{49}^{91}$	$1g_{9/2} - 2p_{1/2}$	−70	$\bar{n}(pp)$	$1g_{9/2}$ / $2p_{1/2}$	−69	(−94)
${}_{39}Y_{50}^{89} \to {}_{39}Y_{52}^{91}$	$1g_{9/2} - 2p_{1/2}$	−356	$\bar{p}(nn)$	$2d_{5/2}$ / $1g_{7/2}$	−378	
${}_{49}In_{66}^{115} \to {}_{49}In_{68}^{117}$	$1g_{9/2} - 2p_{1/2}$	+23	$\bar{p}(nn)$	$1h_{11/2}$ / $3s_{1/2}$	+21,5	(+60)
${}_{58}Ce_{81}^{139} \to {}_{56}Ba_{81}^{137}$	$1h_{11/2} - 2d_{3/2}$	−79	$\bar{n}(\bar{p}\bar{p})$	$1g_{7/2}$ / $2d_{5/2}$	−79	
${}_{58}Ce_{79}^{137} \to {}_{56}Ba_{79}^{135}$	$1h_{11/2} - 2d_{3/2}$	+13	$n(\bar{p}\bar{p})$	$1g_{7/2}$ / $3s_{1/2}$	+15	
${}_{50}Sn_{65}^{115} \to {}_{48}Cd_{65}^{113}$	$1h_{11/2} - 3s_{1/2}$	−247	$\bar{n}(\bar{p}\bar{p})$	$1g_{9/2}$ / $2p_{1/2}$	−240	−241
${}_{37}Rb_{50}^{87} \to {}_{37}Rb_{48}^{85}$	$1f_{5/2} - 2p_{3/2}$	−550	$\bar{p}(\bar{n}\bar{n})$	$1g_{9/2}$ / $2p_{1/2}$	−540	

Table IV. Relative shift of single-particle levels $\Delta(\delta E)$ (combinations of like interacting particles)

${}_{Z}X_{N}^{A} \to {}_{Z'}X_{N'}^{A\pm 2}$	Level difference n'l'j'−nlj	$\Delta(\delta E)_{exp}$, kev	Combination of interacting particles	Level of pair $n_0 l_0 j_0$	$\Delta(\delta E)_{well}$, kev	$\Delta(\delta E)_{osc}$, kev
${}_{38}Sr_{49}^{87} \to {}_{38}Sr_{47}^{85}$	$1g_{9/2} - 2p_{1/2}$	+155	$\bar{n}(\bar{n}\bar{n})$	$1g_{9/2}$	+163	
${}_{41}Nb_{50}^{91} \to {}_{43}Tc_{50}^{93}$	$1g_{9/2} - 2p_{1/2}$	−286	$p(pp)$	$1g_{9/2}$	−260	
${}_{39}Y_{50}^{89} \to {}_{41}Nb_{50}^{91}$	$1g_{9/2} - 2p_{1/2}$	−1017	$\bar{p}(pp)$	$2p_{1/2}$	(−800)	−1089
${}_{50}Sn_{67}^{117} \to {}_{50}Sn_{69}^{119}$	$1h_{11/2} - 2d_{3/2}$	−94	$n(nn)$	$1h_{11/2}$	−100	
${}_{58}Ce_{79}^{137} \to {}_{58}Ce_{81}^{139}$	$1h_{11/2} - 2d_{3/2}$	+485	$n(nn)$	$2d_{3/2}$	+470	(+660)
${}_{50}Sn_{67}^{117} \to {}_{50}Sn_{69}^{119}$	$1h_{11/2} - 3s_{1/2}$	−231	$n(nn)$	$1h_{11/2}$	−220	

Table IVa. Relative shift of single-particle levels $\Delta(\delta E)$ (combinations of like interacting particles). Competition of pair levels

${}_{Z}X^{A}_{N} \to {}_{Z'}X^{A\pm 2}_{N'}$	Level difference $n'l'j' - nlj$	$\Delta(\delta E)_{exp}$, kev	Combination of interacting particles	Level of pair $n_0 l_0 j_0$	$\Delta(\delta E)_{well}$, kev	$\Delta(\delta E)_{osc}$, kev
${}_{41}Nb^{97}_{56} \to {}_{43}Tc^{99}_{56}$	$1g_{9/2} - 2p_{1/2}$	+608	$p\,(pp)$	$1g_{9/2}$ $2p_{1/2}$	+608	+633
${}_{50}Sn^{115}_{65} \to {}_{50}Sn^{117}_{67}$	$1h_{11/2} - 3s_{1/2}$	—192	$\bar{n}\,(nn)$	$3s_{1/2}$ $1g_{7/2}$	—290	
${}_{20}Ca^{41}_{21} \to {}_{20}Ca^{43}_{23}$	$2p_{3/2} - 1f_{7/2}$	—1359	$n\,(nn)$	$1f_{7/2}$ $2p_{3/2}$ $1f_{5/2}$	—1369	—1313
${}_{21}Sc^{49}_{28} \to {}_{23}V^{51}_{28}$	$2p_{3/2} - 1f_{7/2}$	—2142	$p\,(pp)$	$1f_{7/2}$ $2p_{3/2}$ $1f_{5/2}$	(∼—1000)	(—1273) (—1562)
${}_{21}Sc^{49}_{28} \to {}_{23}V^{51}_{28}$	$1f_{5/2} - 2p_{3/2}$	—1575	$p\,(pp)$	$1f_{7/2}$ $2p_{3/2}$	(∼—1000)	(—1036)

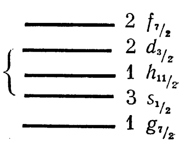
added to the nucleus is located in the lowest level above the core (shell or subshell). The favorable cases where the theoretical shifts agreed with the corresponding experimental values are gathered in Tables III and IV. However, along with this ground level of the pair (according to the Mayer scheme), we considered simultaneously a group (3 or 4) of neighboring alternative occupation levels for the pair, and the relative level shifts were determined on the basis of the method developed above (simultaneous solution of a system of equations for all groups of nuclei), with allowance for all possible locations of the pair. Quantitative estimates have shown that when the relative shifts agree with the experimental values the pair is not always at the ground level (above the core) called for by the scheme employed.

In many cases, singled out in Tables IIIa and IVa, the theoretical and experimental values agree if the pair is at some level neighboring with the ground level. These neighboring levels, which the pair entering the nucleus can occupy instead of the ground level indicated by the Mayer scheme, will be called competing levels. It is characteristic that the competing levels encountered in all the foregoing cases are precisely those closest to each other in the Mayer scheme, which can, generally speaking, overlap when the pair is added to the nucleus, i.e., the sequence of two or three neighboring levels can change. This result is all the more worthy of attention, for along with the pair levels that undoubtedly compete with the ground levels, we considered also two or three (and in some cases even more) other levels, which were not close from the point of view of the Mayer scheme. In none of these nuclei, however, did the calculations indicate a possibility that the pair can occupy these levels.

By way of illustration let us describe several characteristic cases of level competition in several nuclei.

For the case of the indium isotopes ${}_{49}In^{115}_{66} \to {}_{49}In^{117}_{68}$ ($\Delta N = +2$, Table IIIa), the relative shift of the levels $1g_{9/2}$ and $2p_{1/2}$ is characterized, for different pair locations, by the following values (for a rectangular well):

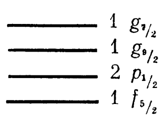
Pair level	$\Delta(\delta E)_{theor}$, kev
$1\,h_{11/2}$	+ 262
$3\,s_{1/2}$	+ 21.5
$2\,d_{3/2}$	+ 104
	$\Delta(\delta E)_{exp} = +\,23$ kev



Of the three competing levels, satisfactory results are obtained not with the first level above the core, $1h_{11/2}$, but with the lower level $3s_{1/2}$, (according to the adopted shell scheme) in the direct vicinity of the ground level $1h_{11/2}$.

In the case of a transition from ${}_{41}Nb^{97}_{56}$ to ${}_{43}Tc^{99}_{56}$ ($\Delta Z = +2$, Table IVa), the relative shift of the same levels is (for a rectangular well):

Pair level	$\Delta(\delta E)_{theor}$, kev
$1\,g_{9/2}$	— 162
$2\,p_{1/2}$	+ 608
	$\Delta(\delta E)_{exp} = +\,608$ kev



In this case, the competing near-lying levels are $1g_{9/2}$ (ground) and $2p_{1/2}$. Calculations of the relative level shifts, carried out for other possible occupations of non-competing remote levels by the pair, always lead, in the case of the investigated nuclei, to values that deviate from experiment. For example, in the case of the transition ${}_{41}Nb^{97}_{56} \rightarrow {}_{43}Tc^{99}_{56}$, the results are clearly unsatisfactory for all possible non-competing pair levels: $1f_{5/2}$, $1g_{7/2}$, $2d_{3/2}$, etc; the situation for the transition ${}_{49}In^{115}_{66} \rightarrow {}_{49}In^{117}_{68}$ is analogous.

In general, for each type of nucleon there exist, as it turned out, definite groups of closely-spaced levels. These compete, in particular, when the number of nucleons of the nucleus is changed. In our case considered the competition between levels manifests itself under level shifts caused by adding an additional pair to a nucleus of the "core + particle" type. On the basis of the analysis of Tables IIIa and IVa, we can single out the following groups of competing levels:

a) for deuterons

$$(1f_{7/2},\ 2p_{3/2},\ 1f_{5/2}),\ (2p_{1/2},\ 1g_{9/2}),\ (2d_{5/2},\ 1g_{7/2},\ 3s_{1/2});$$

b) for protons

$$(1f_{7/2},\ 2p_{3/2},\ 1f_{5/2}),\ (2p_{1/2},\ 1g_{9/2}),\ (1g_{7/2},\ 2d_{5/2}),\ (1h_{11/2},\ 2d_{3/2},\ 3s_{1/2}).$$

We note that in many cases the type of potential (well, oscillator) decides which of the competing levels will predominate when the pair is added to the nucleus (see next to the last line in Table IIIa and last three lines in Table IVa).

3. DISCUSSION OF RESULTS

The theoretical analysis of the relative shifts of single-particle levels in nuclei of the "core + particle" type, when a pair of like particles (nucleons or holes) is added to the nucleus, shows unequivocably that the level shift is due to direct interaction (regarded as a small perturbation) between the pair and the odd particle. As can be seen from Tables III and IV, this mechanism explains satisfactorily the relative level shifts in typical nuclei, if account is taken of the isotopic and isotonic properties of the nuclear potential. In the case of a rectangular well, the agreement with relative level shifts is found to be very good, with the exception of the series of light nuclei for which, as expected, better results are obtained with the oscillator potential (in spite of the neglect of its isotopic and isotonic properties). For the oscillator potential, the results of the calculations frequently disagree with the experiment, this being a natural consequence of neglecting the dependence of the potential on N and Z. The good tabulated results, obtained without account of the correlation interactions inside the pair, offer some evidence that the correlation effects may affect insignificantly the energy values under consideration.

As can be seen from the calculations of the relative level shifts of specific typical nuclei, the number of constants necessary for a satisfactory explanation of these shifts does not exceed the number of constants (f and g) involved in the theoretical formulas (particularly in the case of a rectangular potential well), i.e., the same constants f and g are valid for all the nuclei. The latter circumstance is, in the case of a rectangular potential well, the result of an account of the dependence of the nuclear radius and the depth of the well on the neutrons N and protons Z in the nucleus A(Z, N). As was noted by Nemirovskii,[3] the dependence of the radius of the nucleus and the depth of the well on N and Z can be reduced only to the dependence on N and Z established above [formula (3)] for the depth of the well of the core V(Z, N), with the nuclear radius having the usual form $R = 1.4 \times A^{1/3}$ Fermi units.

In the case of an oscillator potential, the choice of the constants f and g is less successful, but it is much better than that obtained by Zeldes,[5] who used from three to five parameters for each pair of levels, and who did not distinguish between the principal effects of interactions of the odd particle in the nucleus and the secondary effects (in particular, the effect of pair correlation). The result of the oscillator potential can be improved appreciably by taking into account the isotopic and isotonic properties of this potential, i.e., its dependence on N and Z. Critical remarks of this kind were advanced by Peierls (in connection with Zeldes' paper[7]), who noted that the excessive number of parameters used to explain the relative level shifts of even-odd nuclei is the result of failure to account for the dependence of radius of the nucleus (or, what is the same, of the depth of the well) on the number of neutrons and protons in the nucleus.

Peierls' remark does not apply to our work if a rectangular well is used. The choice of just a rectangular potential well as an exact potential dependent on N and Z is connected with the character of the available experimental data: most of the typical nuclei with known experimental schemes can be described by a rectangular well better than by an oscillator potential (the percentage of light nuclei is insignificant).

The competition between pair levels, established through the use of the approach employed here, needs additional verification by other means, although the effect itself was already discussed in earlier papers on this problem.[5,7] The verification becomes particularly essential if the correlation interactions in the pair are taken into account, particularly their dependence on the position of the pair in the nucleus (as is well known, the pairing energy of nucleons is different at different levels).

In conclusion we note that, within the framework of the shell model, the addition to a "core + particle" nucleus of not one but two, three, or more pairs (all within a single shell) should, if pairing is taken into account, lead to relative level shifts $\Delta(\delta E)$ that vary linearly with the number of pairs "introduced" into the nucleus. This effect was theoretically indicated by Zeldes,[5] and noted in Shpinel's[8] analysis of the experimental level schemes. However, the linear dependence of $\Delta(\delta E)$ on ΔN or ΔZ holds only for a definite number (2 or 3) of supplementary pairs in a given shell, beyond which one cannot regard a new pair entering into the shell as being independent of the remaining pairs; consequently the linear dependence is violated and the correlation of the pair must be taken into account. Nonetheless, the linearity observed when at least the first two pairs are added is evidence that these pairs are autonomous in the nucleus, and thereby demonstrates the possible insignificance of correlation effects in the interacting "particle-pair" system.

I express my deep gratitude to K. A. Ter-Martirosyan for numerous discussions and continuous interest in the work, and also to N. I. Gaĭdukov for help with the numerical calculations.

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Translated by J. G. Adashko
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