A RELATIVISTIC FIELD-THEORY MODEL WITH AN EXACT SOLUTION

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Submitted to JETP editor January 30, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 205-208 (July, 1961)

We have considered a model which is the relativistic generalization of the Ruijgrok-Van Hove-Lee model, and which is free of the difficulties of that model.

1. For the study of the general problems of quantum field theory it is useful to have at one's disposal a relativistically-invariant model which can be solved explicitly. We consider one such model in the present paper. It will be used in the following to analyze the difficulties of a field theory with a point interaction, the singularities of a non-local theory, and so on.

We use as the basis from which to construct such a model the static Ruijgrok-Van Hove model [1] (in short: RVH) which is a generalization of the usual Lee model. As these two, the model proposed here describes the scalar interaction between scalar mesons (θ) and a nucleon field (V, N). The interaction Hamiltonian

$$H' = \sum_u \bar{\psi}_u (\sigma_+ \varphi_+ + \sigma_- \varphi_-) \psi_u \tag{1}$$

(the meaning of the index u is explained below) corresponds to the processes $V \rightleftarrows N + \theta$ and $N \rightleftarrows V + \theta$ with bare coupling constants g_V and g_N, respectively. In (1)

$$\psi = \begin{pmatrix} \psi_V \\ \psi_N \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & g_N \\ g_V & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & g_V \\ g_N & 0 \end{pmatrix},$$

and $\varphi_\pm$ is the creation and annihilation part of the field φ.

2. To construct a relativistic theory we must give up the static character of the nucleon field.* We shall describe the latter by the Lagrangian (see [2])

$$L_0 = \sum_u \bar{\psi}_u (i (u\nabla) - M) \psi_u. \tag{2}$$

Here and henceforth the summation is over all values of the four-vector u which satisfy the condition $u_0^2 - \mathbf{u}^2 = 1$, $u_0 > 0$. The vector u which has the meaning of the four-velocity of a particle is not at all connected with its momentum and is together with the momentum characteristic for its state.* The magnitude of u is not changed during the interaction process as the Hamiltonian (1) is diagonal.

The presence of a separate vector u enables us to consider the quantity ψ_u as a two-component spinor without coming into conflict with relativity or with the parity-conservation law. Indeed, a Lorentz transformation corresponding to a velocity $\delta\mathbf{v}$ gives

$$\psi'_u = (1 + \delta\mathbf{v}\mathbf{l})\, \psi_u,$$

where the vector $\mathbf{l} \sim [\boldsymbol{\sigma} \times \mathbf{u}]$ is a polar vector. We remind ourselves of the fact that in the usual theory $\mathbf{l}$ is proportional to the axial spin vector $\boldsymbol{\sigma}$ because there are not two vectors, and that we have thus a four-component spinor.

We make two remarks concerning the nucleon dispersion law

$$E = (\mathbf{u}\mathbf{p} + M)/u_0. \tag{3}$$

First of all, it follows from (3) that sometimes (for instance, when two nucleons with the same u are scattered) the energy and momentum conservation laws are not independent. It is thus necessary to check specially whether the vacuum and single-particle states are stable. From this point of view it is essential that there is no pair creation in our model and that the processes where a θ particle is emitted by a free nucleon are also forbidden. Indeed, the energy conservation law gives for those processes

$$\mathbf{u}\Delta\mathbf{p}/u_0 = \omega\,, \tag{4'}$$

where $\Delta\mathbf{p}$ is the change in the nucleon momentum. The momentum conservation law $\Delta\mathbf{p} = \mathbf{k}$, or

*The θ-N scattering cross section becomes then different from zero; it vanishes in the RVH model because recoil is neglected.

*The model considered here differs from the Bloch-Nordsieck model[2] in a number of important points; in particular, there is no connection between u and p and the problem is completely isotropic.

$$\mathbf{u}\Delta\mathbf{p}/u_0 = \mathbf{uk}/u_0 \qquad (4'')$$

is easily seen to be incompatible with (4′).

It is moreover clear from (3) that the energy of a state is not always positive and can change its sign (when $p^2 < 0$) when the frame of reference changes. On this point there is an essential difference with the usual model where always $p^2 > 0$ and where the sign of the energy is invariant. The natural relativistic generalization of the sign of the energy in the model considered is the sign of the quantity (up) which in the rest frame of the particle (when $\mathbf{p} = 0$) is the same as the sign of the energy and which is always positive. It is essential in this connection that just the quantity (up) and not p^2 determines the characteristics of the excited single-nucleon states (see Sec. 4 below).

To conclude this section we establish the connection between our model and the RVH and Lee models. For $g_N = 0$ we get the relativistic generalization of the Lee model. If we retain in the sums in (1) and (2) only the one term with $\mathbf{u} = 0$, $u_0 = 1$, we get the RVH model. Finally, if we put $g_N = 0$, we come back to the usual Lee model.

3. We turn now to a study of the proposed model. We start with the charge normalization. We note that it is performed exactly as in the RVH model[1] and is reduced to finding the exact Schrödinger wave function for the single-nucleon state. The only difference consists in that it is necessary to replace $\omega^{3/2}$ by $\omega^{1/2}(u_0\omega - \mathbf{u}\cdot\mathbf{k})$ in the corresponding integrals. We give here the final expression for the renormalized charges g_{0V} and g_{0N}:

$$g_{0V} = g_{0N} = (g_V g_N)^{1/2}. \qquad (5)$$

The renormalized charge is thus different from zero in the model considered, as in the RVH model. The vanishing of the charge in the Lee model—both the usual and the relativistic model—is simply connected with the fact that the process $N \rightleftarrows V + \theta$ is forbidden: we see from (5) that g_{0V} tends to zero as $g_N = 0$. This fact indicates once more that the situation in the Lee model is by no means indicative of similar difficulties in real field theories where there is complete symmetry between the emission and absorption of particles.*

4. We turn now to a study of the structure of the renormalized theory. Because of (5) we can at once put $g_V = g_N = g$, $\sigma_V = \sigma_N = g\tau_1$. We shall start from the functional equation for the fermion Green's function[4]

$$[i\,(u\nabla) - M + g\tau_1\Phi\,(x)]\,G_{uu'}\,(x, x'\,|\,\varphi) = \delta\,(x - x')\,\delta_{uu'},$$

$$\Phi\,(x) = \varphi\,(x) - i\int d\xi\Delta\,(x - \xi)\,\frac{\delta}{\delta\varphi\,(\xi)}\,.$$

*A subsequent paper[3] is devoted to this kind of problems.

The boson Green's function is in our model the same as the free function $\Delta(x - x')$.

One solves the equation given here easily as the dependence on the external field φ is at once split off as an exponential

$$\exp\left\{g\tau_1\int d\xi\varphi\,(\xi)\,[K\,(x' - \xi) - K\,(x - \xi)]\right\},$$

$$K\,(x) = (2\pi)^{-4}\int d^4k\,[(u\,k) + i\varepsilon]^{-1}\exp(-ikx). \qquad (6)$$

Performing in the usual way the mass renormalization and splitting off the Z factor, we get

$$G_c\,(p) = -\,i\,(2\pi)^{-4}\int_0^\infty d\xi\,\exp\{i\xi\,((up) - M_0) + \chi(\xi)\},$$

$$\chi\,(\xi) = ig^2\int_\xi^\infty d\xi'\,(\xi' - \xi)\,\Delta\,(\xi' u). \qquad (7)$$

This expression has no ghost singularities whatever.

The vertex part is immediately obtained from the relation

$$g\Gamma\,(x, x', \xi) = (\delta G\,(x, x')/\delta\varphi\,(\xi))_{\varphi=0}.$$

Using (6) we get the generalized Ward equation

$$(uk)\,\Gamma\,(p, k) = \tau_1\,\{G^{-1}\,(p) - G^{-1}\,(p - k)\}, \qquad (8)$$

through which the vertex part is completely determined. Using (7) and (8) we can also verify that the equality $g_0 = g$ holds and also find the asymptotic behavior

$$G\,(p) \sim p^{\alpha}/(up), \qquad \Gamma \sim \zeta^{-\alpha},$$

where $\alpha = g^2/4\pi^2$, $\zeta = \max(p, k)$. The decrease in the vertex part is in complete agreement with the well-known Lehmann-Symanzik-Zimmermann theorem.

5. The model constructed here satisfies thus the general requirements of field theory and the simultaneity condition.* One sees easily that the same properties are also possessed by a somewhat modified model with a complex charge

$$\sigma_V = g\tau_1, \qquad \sigma_N = g^*\tau_1.$$

All relations in the foregoing, except (5), remain valid if we perform the substitution

$$g^2 \to |g|^2. \qquad (9)$$

The charge renormalization is now of the form

$$g_0 = g, \qquad (g^*)_0 = g^*.$$

To verify the validity of what we just stated, it is sufficient to note that the contribution from every virtual line is, indeed, connected with the substitution (9).

*We note that the unitarity of the theory follows immediately from the Hermiticity of (1).

[1] T. W. Ruijgrok and L. Van Hove, Physica **22**, 880 (1956).

[2] N. N. Bogolyubov and D. V. Shirkov, Introduction into the Theory of Quantized Fields, Interscience, 1957, Sec. 41.

[3] D. A. Kirzhnits, JETP **41**, 417 (1961), Soviet Phys. JETP **14**, No. 2 (1962).

[4] S. F. Edwards and R. E. Peierls, Proc. Roy. Soc. (London) **A224**, 24 (1954).

Translated by D. ter Haar
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