A GAUGE INVARIANT FORMULATION OF NEUTRAL VECTOR FIELD THEORY

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It is shown that the theory of a neutral vector field with nonzero rest mass admits of a gauge invariant formulation without the introduction of auxiliary fields. In this theory the gauge invariance has a trivial physical meaning: the quanta of zero spin, described by the four-vector A_μ, do not interact with anything. Only the quanta with spin 1 take part in interactions.

1. INTRODUCTION

THERE is a widespread belief[1-10] that the theory of the neutral vector field $A_\mu(x)$ with nonzero rest mass, unlike electrodynamic theory, cannot be formulated in a gauge invariant way without the introduction of auxiliary fields. This is regarded as a serious obstacle to attempts which have been made rather frequently in recent times to carry through an analogy between the baryon charge and the electric charge or between the hyperon charge by the introduction of corresponding vector fields (Lee and Yang, Sakurai, and others[5-7,9,10]).

In the usual formulation of the neutral vector field theory one uses the equation (cf., e.g., [11,12])

$$\Box A_\mu - \partial^2 A_\nu/\partial x_\mu \partial x_\nu - m^2 A_\mu = -j_\mu, \tag{1}$$

which is equivalent to the equation

$$(\Box - m^2) A_\mu = -j_\mu \tag{2}$$

with the supplementary condition

$$\partial A_\mu/\partial x_\mu = 0. \tag{3}$$

Neither Eq. (1), nor Eq. (2) with supplementary condition (3), is gauge invariant.

The need for the supplementary condition (3) is motivated by the wish to exclude the spin 0 and assure that the energy is positive definite.

We note that at the price of introducing, in addition to the four-vector $A_\mu(x)$, an auxiliary scalar field $B(x)$ Stueckelberg[3] succeeded in constructing a gauge invariant formalism for the vector field with the supplementary condition.[1,4,8,11] In such a theory, however, the meaning of the gauge invariance is to a large extent obscured.

It will be shown below that the theory of a neutral vector field with nonzero rest mass admits of a gauge invariant formulation without the introduction of any auxiliary fields. For this we must entirely renounce the supplementary condition. In the theory to be considered $A_\mu(x)$ obeys only the gauge-invariant equation (2) (Sec. 2). It turns out that only the part of $A_\mu(x)$ with the spin 0 is subject to the gauge transformation. Here the physical meaning of the gauge invariance is that the vector-field quanta with spin 0 have no interaction with other fields or with each other (Sec. 3). Therefore the use of a supplementary condition to exclude the spin 0 is superfluous. It is also not needed to assure positive definiteness of the energy (Sec. 3). This theory is completely equivalent to the usual theory of the neutral vector field with nonzero mass based on Eq. (1) or on Eqs. (2) and (3) (Sec. 4).

It can be said that in the present case the gauge invariance plays the part that is usually played by supplementary conditions in the theory of higher spins. Unlike the usual supplementary conditions it does not exclude the quanta of the undesired spin, but only "renders them harmless."

2. BASIC EQUATIONS

As is common practice (cf., e.g., [12]) we take the Lagrangian density which describes the neutral vector field A_μ in interaction with a spinor field* ψ in the form

$$L(x) = -\frac{1}{2}\frac{\partial A_\nu}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\mu} - \frac{m^2}{2}A_\mu A_\mu + j_\mu A_\mu - \bar{\psi}\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + M\right)\psi$$

$$j_\mu = ig\bar{\psi}\gamma_\mu\psi. \tag{4}$$

The Lagrangian and the equations of motion that follow from it,

*In analogous ways one can write the interaction for cases involving several fields, and also for couplings of the type of an anomalous magnetic moment.

$$(\Box - m^2) A_\mu = -j_\mu, \tag{5}$$

$$(\gamma_\mu \partial / \partial x_\mu + M) \psi = ig\gamma_\mu \psi A_\mu, \tag{6}$$

and also the equal-time ($x_0 = y_0$) commutation relations, which are analogous to those for electrodynamics,*

$$\{\psi(x), \psi(y)\} = 0, \qquad \{\psi(x), \overline{\psi}(y)\} = \gamma_4 \delta(\mathbf{x} - \mathbf{y}),$$

$$[A_\mu(x), A_\nu(y)] = 0, \qquad \left[A_\mu(x), \frac{\partial A_\nu}{\partial y_4}(y)\right] = \delta_{\mu\nu}\delta(\mathbf{x} - \mathbf{y}),$$

$$[\psi(x), A_\nu(y)] = 0, \qquad \left[\psi(x), \frac{\partial A_\nu}{\partial y_4}(y)\right] = 0, \tag{7}$$

are invariant† under the gauge transformations

$$\psi'(x) = \exp[ig\Lambda(x)]\,\psi(x), \tag{8}$$

$$A'_\mu(x) = A_\mu(x) + \partial\Lambda(x)/\partial x_\mu \tag{9}$$

with arbitrary $\Lambda(x)$ satisfying the equation

$$\frac{\partial}{\partial x_\mu}(\Box - m^2)\Lambda = 0. \tag{10}$$

Gauge invariance thus exists for $m \neq 0$ in just the same way as in the case in which one sets $m = 0$ in Eqs. (4), (5), and (10) (quantum electrodynamics).

We note that unlike the treatment in [5-7,9], where the function $\Lambda(x)$ was assumed completely arbitrary, in the transformations (8) and (9) considered here $\Lambda(x)$ is restricted by the condition (10). This same restriction also holds in quantum electrodynamics, unlike the classical theory.

From the gauge invariance of the Lagrangian or of the equations there follows the conservation law:‡

$$\frac{\partial}{\partial x_\mu}\left(-j_\mu\Lambda - \frac{\partial A_\nu}{\partial x_\mu}\frac{\partial\Lambda}{\partial x_\nu} + A_\nu\frac{\partial^2\Lambda}{\partial x_\mu \partial x_\nu}\right) = 0. \tag{11}$$

This equation must hold for any $\Lambda(x)$ that satisfies Eq. (10). In particular, for $\Lambda = \text{const}$ we get

$$\partial j_\mu / \partial x_\mu = 0 \tag{12}$$

the conservation law for the current.

Differentiating Eq. (5) and taking Eq. (12) into account, we get the equation

$$(\Box - m^2)\,\partial A_\mu / \partial x_\mu = 0, \tag{13}$$

which we shall need soon.

*The equal-time commutations can be taken in this form, since no supplementary condition is imposed on the field operators A_μ.

†The Lagrangian density (4) is invariant apart from a divergence, which of course affects no results (cf. Appendix 1).

‡For a derivation of this conservation law and a discussion of the associated operator transforming the state vectors see Appendix 1.

3. THE ABSENCE OF INTERACTIONS WITH THE QUANTA OF SPIN 0

One uses a four-vector $A_\mu(x)$ with a view to the description of particles of spin 1. Within the framework of the homogeneous Lorentz group there are no quantities that describe only particles of spin 1. In accordance with this, along with quanta of spin 1, $A_\mu(x)$ also describes quanta of spin 0. Namely, A_μ can be divided into two parts:

$$A_\mu = \left(A_\mu - \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}\right) + \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}, \tag{14}$$

where the first part describes quanta of spin 1 and the second quanta of zero spin. One can verify this by applying the invariant operator of the square of the spin[13-16] for the field $A_\mu(x)$ (cf. Appendix 2):

$$(\Gamma^2)_{\mu\nu} = 2(\delta_{\mu\nu}\Box - \partial^2/\partial x_\mu \partial x_\nu) \tag{15}$$

which has the eigenvalues $s(s+1)\Box$ for spin s. Namely, using Eq. (13), we can easily verify that

$$(\Gamma^2)_{\mu\nu}\left(A_\nu - \frac{1}{m^2}\frac{\partial}{\partial x_\nu}\frac{\partial A_\lambda}{\partial x_\lambda}\right) = 2\Box\left(A_\mu - \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\lambda}{\partial x_\lambda}\right), \tag{16}$$

$$(\Gamma^2)_{\mu\nu}\frac{1}{m^2}\frac{\partial}{\partial x_\nu}\frac{\partial A_\lambda}{\partial x_\lambda} = 0. \tag{17}$$

Naturally these same quanta of zero spin are also described by the scalar $\partial A_\mu / \partial x_\mu$ itself.

It must be emphasized that under the condition (1) the gauge transformation (9) changes only the part of A_μ with the spin 0:

$$\frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A'_\nu}{\partial x_\nu} = \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu} + \frac{\partial\Lambda}{\partial x_\mu}, \tag{18}$$

while the part with the spin 1 remains unchanged:

$$A'_\mu - \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A'_\nu}{\partial x_\nu} = A_\mu - \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}. \tag{19}$$

If we want to consider only the quanta with the spin 1, then it would seem to be necessary to take steps to exclude the quanta with zero spin. Our assertion is that in the case of vector quanta of nonzero rest mass no special steps of this sort (for example, imposition of a supplementary condition) are needed. It already follows from the gauge invariant field equations (5) and (6) themselves that the quanta with spin 0 do not interact with other fields nor with each other: the part of A_μ that describes them obeys the free-particle equation

$$(\Box - m^2)\frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu} = 0 \tag{20}$$

[a trivial consequence of Eq. (13)]. Consequently, the equation with interaction, Eq. (5), is obeyed by just the part of A_μ with spin 1:

$$(\square - m^2)\left(A_\mu - \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}\right) = -j_\mu. \qquad (21)$$

Thus the quanta with spin 0 do not affect the physics; if any set of such quanta is present in the initial state, precisely this same set is present in the final state. Otherwise the matrix element of the S matrix is equal to zero.

This assertion can be formulated in the language of conservation laws. From the free-particle equation (13) there follow conservation laws for the total four-vector momentum $P_\mu^{(0)}$ and angular momentum $M_{\mu\nu}^{(0)}$ of the scalar field $m^{-1}\partial A_\mu/\partial x_\mu$, and moreover, of the number of quanta with each value of the momentum. Thus the S matrix is diagonal in the quantum numbers of the quanta with spin 0.

The imposition of a supplementary condition of the form

$$\partial A_\mu/\partial x_\mu = 0 \qquad (22)$$

or*

$$\left(\frac{\partial A_\mu}{\partial x_\mu}\right)_- \Phi = 0 \qquad (23)$$

for the exclusion of the spin 0 is superfluous, since it affects only the part of $A_\mu(x)$ that describes the free quanta of spin 0, which do not interact with anything.

As still another, and sometimes the main, argument[1,2] in favor of the necessity of imposing the condition (22), the point has been made that otherwise the operator P_0 for the total energy is not positive-definite. But the energy operator $P_0^{(0)}$ of the always free quanta of spin zero is conserved. Therefore as the physical energy operator we may take the operator†

$$P_0^{\text{phys}} = P_0 - P_0^{(0)}. \qquad (24)$$

This operator is conserved and gauge invariant, and its spectrum is positive definite. Also it is really permissible not to make this subtraction. Then the energy would be reckoned not from zero, but from some gauge-dependent negative level. Thus also from this point of view there is no necessity of the supplementary condition (22).

4. EQUIVALENCE TO THE USUAL THEORY WITH THE SUPPLEMENTARY CONDITION

Although the scalar field $m^{-1}\partial A_\mu/\partial x_\mu$ has turned out to be free, it has not yet been eliminated from the Dirac equation (6). Let us now break up the operator $\psi(x)$ into factors depending on the gauge and independent of the gauge:

$$\psi(x) = \exp\left[ig\frac{1}{m^2}\frac{\partial A_\nu}{\partial x_\nu}\right]\varphi(x),$$

$$\varphi(x) \equiv \exp\left[-ig\frac{1}{m^2}\frac{\partial A_\nu}{\partial x_\nu}\right]\psi(x), \qquad (25)$$

where φ is the part of ψ that does not depend on the gauge.* In the variables φ and apart from a four-dimensional divergence the Lagrangian density (4) can be rewritten in the form

$$L(x) = -\frac{1}{4}F_{\mu\nu}^{(1)}F_{\mu\nu}^{(1)} - \frac{m^2}{2}A_\mu^{(1)}A_\mu^{(1)} + \frac{1}{2}\left(\frac{\partial}{\partial x_\mu}\frac{1}{m}\frac{\partial A_\nu}{\partial x_\nu}\right)^2 + \frac{m^2}{2}\left(\frac{1}{m}\frac{\partial A_\nu}{\partial x_\nu}\right)^2 + \frac{ig}{2}\bar\varphi\gamma_\mu(\varphi A_\mu^{(1)} + A_\mu^{(1)}\varphi) - \bar\varphi\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + M\right)\varphi, \qquad (26)$$

where

$$F_{\mu\nu}^{(1)} \equiv \frac{\partial A_\nu^{(1)}}{\partial x_\mu} - \frac{\partial A_\mu^{(1)}}{\partial x_\nu}, \qquad A_\mu^{(1)} \equiv A_\mu - \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}. \qquad (27)$$

This Lagrangian corresponds to the usual theory of the interaction of a spinor field φ and a vector field $A_\mu^{(1)}$ with the supplementary condition[17]

$$\partial A_\mu^{(1)}/\partial x_\mu = 0 \qquad (28)$$

and, in addition, describes a free scalar field $m^{-1}\partial A_\nu/\partial x_\nu$, which now does not appear in the Dirac equation.

The commutation relations for the fields φ and $A_\mu^{(1)}$ are the same as in the usual theory with the supplementary condition.† At the same time the interacting fields φ and $A_\mu^{(1)}$ commute with the scalar field $m^{-1}\partial A_\nu/\partial x_\nu$ and with all of its derivatives. Consequently, the scalar field $m^{-1}\partial A_\nu/\partial x_\nu$ is completely dynamically independent. Thus the theory under discussion is entirely equivalent to the usual theory of the neutral vector field with the supplementary condition. The equivalence of the two theories can also be established by a different method, by means of the unitary transformation of Dyson,[18,11] which also "turns off" the vector interaction of the scalar field.

5. CONCLUDING REMARKS

Let us make one remark about the mass renormalization. A gauge transformation affects

*In the theory under consideration this condition fixes the gauge and means that in the physical states Φ there are no quanta with spin 0.

†For the concrete form of the operator P_0^{phys} for a free field A_μ see Appendix 3.

*Apart from a constant phase factor, under the transformations (8)–(10) we have

$$\varphi' = \exp\left[-\frac{ig}{m^2}(\square - m^2)\Lambda\right]\varphi = \exp\left[-\frac{ig}{m^2}\cdot\text{const}\right]\varphi.$$

†These relations can be obtained from the commutation relations (7) and the equations of motion (5) and (6).

only the part of A_ν that describes the quanta of spin zero. Consequently, it is only for these quanta that we can expect that the mass will not have to be renormalized. And indeed this is so, because they do not interact with anything. As for the quanta with spin 1, their mass is naturally renormalized in the process of the interaction. The equation (5), with the mass renormalization of the quanta of spin 1 taken into account, can be written, for example, in the form

$$(\Box - m^2)A_\mu = -j_\mu - \delta m^2\left(A_\mu - \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}\right). \quad (29)$$

Finally we point out that in this theory of a vector field A_ν with nonzero mass the Green's functions obey the same laws of gauge transformation as in electrodynamics.[19-23] These laws connect averages over the same state of products of operators in two different gauges. This state is the vacuum in both gauges for the quanta with spin 1, but contains no quanta with spin 0 from the point of view of only one gauge (Appendix 1). The Ward identity follows from these laws, as in electrodynamics.

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APPENDIX 1

THE CONSERVATION LAW THAT FOLLOWS FROM GAUGE INVARIANCE

After a gauge transformation (8) and (9), with Eq. (10) taken into account, the Lagrangian density (4) differs from its original value by a divergence. For an infinitesimal transformation we have

$$\delta L(x) = -\frac{\partial}{\partial x_\mu}\left(A_\nu \frac{\partial^2 \Lambda}{\partial x_\mu \partial x_\nu}\right). \quad (1.1)$$

Calculating the variation of $L(x)$ in the usual way, using the Euler equations, we arrive at the conservation law (11), which can also be written in the equivalent form

$$\frac{\partial}{\partial x_\mu}\left[\left(\frac{\partial}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}\right)\Lambda - \frac{\partial A_\nu}{\partial x_\nu}\frac{\partial \Lambda}{\partial x_\mu} + A_\mu(\Box - m^2)\Lambda\right]$$

$$\equiv \frac{\partial}{\partial x_\mu}K_\mu = 0. \quad (1.2)$$

In Eq. (11) one can substitute any function $\Lambda(x)$, limited only by Eq. (10), and therefore Eq. (11) represents a continuum of conservation laws, corresponding to the continuum of the parameters of the gauge group. It is easy to verify that Eq. (11) is equivalent to Eq. (5).

We can now write the gauge transformation of the eigenvectors, without changing the eigenvalues, and of the field operators, in the form

$$\Psi' = \exp\left(\int d\mathbf{x} K_4\right)\Psi \equiv U\Psi, \qquad \psi'(x) = U\psi(x)U^{-1},$$
$$A'_\mu(x) = UA_\mu(x)U^{-1} \quad (1.3)$$

[the last two relations are a different way of writing Eqs. (8) and (9)].

From the points of view of different gauges the same state has different sets of the noninteracting gauge of spin 0. For example, let us define the vacuum in two different gauges by the relations

$$a^+_\mu p_\mu \Psi_0 = 0, \qquad a'^+_\mu p_\mu \Psi'_0 = 0, \quad (1.4)$$

where $a^+_\mu p_\mu$ are operators for annihilation of quanta of spin 0, corresponding to $\partial A_\mu/\partial x_\mu$ (see Appendix 3). If $ia_\mu p_\mu$, λ and $-ia_\mu p_\mu$, λ^+ are the amplitudes of the positive and negative frequency parts of the three-dimensional Fourier expansions of $\partial A_\mu/\partial x_\mu$ and $\Lambda(x)$, which hold by Eq. (13) and for the special case of a $\Lambda(x)$ which satisfies the same equation, then

$$\Psi'_0 = U\Psi_0 \equiv \exp\left\{-i\int d\mathbf{p}\,[a_\mu p_\mu \lambda^+(\mathbf{p}) + a^+_\mu p_\mu \lambda(\mathbf{p})]\right\}\Psi_0$$
$$= \exp\left\{-\frac{m^2}{2}\int d\mathbf{p}\lambda(\mathbf{p})\,\lambda^+(\mathbf{p}) - i\int d\mathbf{p}\lambda^+(\mathbf{p})\,p_\mu a_\mu\right\}\Psi_0. \quad (1.5)$$

The last expression gives the expansion of the vacuum of the new gauge, Ψ'_0, in terms of the states of the old gauge, which are produced by the creation operators $p_\mu a_\mu$ from the old vacuum Ψ_0.

Thus the vacuum from the point of view of one gauge is not the vacuum from the point of view of another gauge. This is understandable, since the conditions (1.4), and also the more general condition (23), are not gauge invariant, since they are conditions that establish and fix a gauge.

APPENDIX 2

THE OPERATOR FOR THE SQUARE OF THE SPIN FOR THE FIELD $A_\mu(x)$

The general definition of the operator for the square of the spin (one of the invariants of the inhomogeneous Lorentz group) for arbitrary many-component functions transforming according to representations of the inhomogeneous Lorentz group has been given and discussed in a number of papers.[13-16] This definition is [16]:

$$\Gamma^2 = -p^2_\lambda \tfrac{1}{2} m_{\rho\sigma} m_{\rho\sigma} + m_{\lambda\rho} m_{\lambda\sigma} p_\rho p_\sigma, \quad (2.1)$$

where

$$p_\lambda = \frac{1}{i}\frac{\partial}{\partial x_\lambda}, \qquad m_{\rho\sigma} = \frac{1}{i}\left(x_\rho \frac{\partial}{\partial x_\sigma} - x_\sigma \frac{\partial}{\partial x_\rho}\right) + s_{\rho\sigma} \qquad (2.2)$$

are the infinitesimal operators of displacements and four-dimensional rotations for the given function. The matrices $s_{\rho\sigma}$ are the infinitesimal operators for rotations of the components of the function.

Since for a vector function $A_\mu(x)$

$$(s_{\rho\sigma})_{\mu\nu} = i\,(\delta_{\rho\mu}\delta_{\sigma\nu} - \delta_{\rho\nu}\delta_{\sigma\mu}), \qquad (2.3)$$

substitution of Eqs. (2.2) and (2.3) in Eq. (2.1) at once gives the expression (15).

The law of transformation of the vector function $A_\mu(x)$, and thus also the infinitesimal operators p_λ and $m_{\rho\sigma}$ and the operator (2.1) or (15) for the square of the spin constructed from them, do not depend on whether $A_\mu(x)$ is subjected to any equations or not. Whereas, however, in the case of equations without interaction A_μ can be divided into independent parts with spins 0 and 1, in the case of an interaction this might not be so.

In our case, as we have seen (Sec. 3), even in the presence of the interaction $A_\mu(x)$ breaks up into dynamically independent parts with spins 0 and 1.

APPENDIX 3

THE NORMAL PRODUCT OF OPERATORS OF THE VECTOR FIELD AND THE DEFINITION OF THE PHYSICAL ENERGY OPERATOR IN THE FREE CASE

In the free case it is convenient to define all operators in the form of normal products, i.e., so that all creation operators shall stand to the left of the annihilation operators.

For example, it is in this sense that we would like to understand the operator for the four-momentum

$$P_\mu = \int d\mathbf{p}\, p_\mu :a_\nu^+ a_\nu: \qquad (p_0 = \sqrt{\mathbf{p}^2 + m^2}), \qquad (3.1)$$

as we indicate by the colons.

The operators a_ν and a_ν^+ obey the commutation relations

$$[a_\mu(\mathbf{p}_1),\, a_\nu^+(\mathbf{p}_2)] = \delta_{\mu\nu}\delta(\mathbf{p}_1 - \mathbf{p}_2). \qquad (3.2)$$

It is clear from these relations that the annihilation operators are a_m and a_0^+, and the creation operators are a_m^+ and a_0. But writing the normal product in Eq. (3.1) in the form

$$:a_\nu^+ a_\nu: = a_m^+ a_m - a_0 a_0^+ \qquad (3.3)$$

is not permissible, because of its lack of covariance. We can give a covariant definition if we break up the operators a_μ and a_μ^+ into parts with the spins 0 and 1:

	Annihilation operators	Creation operators
$s = 1$	$a_\mu + m^{-2} a_\nu p_\nu p_\mu$	$a_\mu^+ + m^{-2} a_\nu^+ p_\nu p_\mu$
$s = 0$	$-m^{-2} a_\nu^+ p_\nu p_\mu$	$-m^{-2} a_\nu p_\nu p_\mu$

(here $p^2 = -m^2$). One can establish the spins of these parts by means of the invariant operator for the square of the spin, Eq. (15) (cf. Appendix 2), written in the momentum representation. Their meanings as annihilation and creation operators follow from Eq. (3.2).

The ways of writing normal products are not, for example,

$$:\left(a_\mu^+ + \frac{a^+p}{m^2}p_\mu\right)\left(a_\nu + \frac{ap}{m^2}p_\nu\right): = :\left(a_\nu + \frac{ap}{m^2}p_\nu\right)\left(a_\mu^+ + \frac{a^+p}{m^2}p_\mu\right):$$
$$= \left(a_\mu^+ + \frac{a^+p}{m^2}p_\mu\right)\left(a_\nu + \frac{ap}{m^2}p_\nu\right), \qquad (3.4)$$

$$:\frac{a^+p}{m^2}p_\mu \frac{ap}{m^2}p_\nu: = :\frac{ap}{m^2}p_\nu \frac{a^+p}{m^2}p_\mu: = \frac{ap}{m^2}p_\nu \frac{a^+p}{m^2}p_\mu. \qquad (3.5)$$

In mixed products with $s = 1$ and $s = 0$ the order is immaterial because of the commutation.

The normal product in Eq. (3.1) can now be expanded as

$$P_\mu = \int d\mathbf{p}\, p_\mu \left\{\left[a_\nu^+ a_\nu + \frac{(a^+p)(ap)}{m^2}\right] - \frac{(ap)(a^+p)}{m^2}\right\}. \qquad (3.6)$$

The first term in the curly brackets is the positive definite operator for the number of quanta of spin 1, and the second is the negative definite operator for the number of quanta of spin 0. Accordingly the energy of the quanta of spin 1 is positive definite, and that of the quanta of spin 0 is negative definite.

If we subtract from P_μ the energy-momentum four-vector of the quanta of spin 0, we get the energy momentum operator of the quanta of spin 1 (the physical quanta), which in the free case is invariant under gauge transformations ($a_\mu \to a_\mu + i\lambda(\mathbf{p})p_\mu$)

$$P_\mu^{\text{phys}} = \int d\mathbf{p}\, p_\mu [a_\nu^+ a_\nu + (a^+p)(ap)/m^2] \qquad (3.7)$$

and has the energy positive definite.

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