INTEGRAL EQUATIONS FOR ππ-SCATTERING AND PROBLEMS RELATED TO CONVERGENCE OF THE AMPLITUDE EXPANSION

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Convergence of the expansions of the cosine dependence of the amplitude, employed in the deduction of the integral equations from the Mandelstam representation, is investigated in the case of ππ scattering. A system of equations for low energies in presented, in which rapid convergence of the expansion of the real part of the amplitude can be attained by a conformal mapping of the cosine plane. Since any power of the function employed contains an infinite number of partial waves, this approach should be especially convenient in cases when high number waves may be important.

1. QUESTIONS OF CONVERGENCE

MANY authors have recently investigated the derivation of a system of equations for elastic pion-pion scattering.[1-3] A general feature of these investigations is that the singularities of the scattering amplitude are determined by means of the Mandelstam two-dimensional integral representation,[4-6] which discloses explicitly the analytic properties of the amplitude and leads to various one-dimensional dispersion relations.

Chew and Mandelstam[1] obtained dispersion relations for the partial waves. The imaginary part of the amplitude in the nonphysical region is obtained by analytic continuation from the physical region of the crossing reactions by expansion in Legendre polynomials.

However, this continuation leads to principal difficulties[1,3,7] because the Legendre series does not converge in the region of the spectral functions and because, furthermore, it converges very slowly in a sufficiently large region near the boundary of the spectral function, so that high-order waves cannot be neglected.

Hsien, Ho, and Zoellner[3] have proposed a different approach, getting around these difficulties. They use dispersion relations for the forward (or backward) scattering only. The path of integration does not cross the regions of the spectral functions. The path of the left-hand integral coincides with the boundary of the physical region of the second (or third) reactions, so that no analytic continuation is necessary. Only integrals over the positive energies at $\cos^2\theta = 1$ remain after the crossing transformation.

To obtain expressions for the partial amplitudes, Hsien et al use, along with the dispersion relation for A, also its derivative* with respect to t (see also [8]). If the isotopic spin I is 0 or 2, only even waves are present:

$$A_l^{0,2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n A^{0,2}}{(\partial \cos^2\theta)^n} \right)_{\cos^2\theta=1} c_{ln} \qquad (l \text{ even}), \tag{1}$$

where

$$c_{ln} = \int_0^1 (\cos^2\theta - 1)^n P_l(\cos\theta)\, d\cos\theta,$$

$$A^{0,2}(\nu, -\cos\theta) = A^{0,2}(\nu, \cos\theta).$$

Only the S wave is taken into account in [3] and the first two terms are retained on the right side of expansion (1). These expressions are substituted into the unitarity condition only in the given approximation. Analogously, only the P wave is taken into account in the case of odd l.

The series (1) converges not only for all energies $\nu < 3$, where scattering only is possible, but also up to $\nu = 4.8$. Naturally, the rate of convergence of the series depends in this case on the distance between the nearest singularity and the point $\cos^2\theta = 1$. This can be seen from the fact that unitarity necessitates knowledge of the amplitude in

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*$t = -2\nu(1 - \cos\theta)$ is the square of the momentum transfer, and $s = 4(\nu + 1)$ is the square of the total energy; both are given in units of μ^2 in the c.m.s. of the first reaction. We also use the symbol $\nu = q^2/\mu^2$.[1]

the entire physical region $0 \le \cos^2\theta \le 1$. Since we continue the amplitude into the entire physical region using its values at the point $\cos^2\theta = 1$ only, the amplitude A can be represented only in the form of a series in powers of $\cos^2\theta - 1$, which has its own radius of convergence. With increasing ν, the radius decreases and the convergence becomes worse;[9] starting with $\nu_{max} = 4.8$, the series in the right half of (1) diverges. From this point of view, we can say that the accuracy of the approximation made in [3] is small in the region $\nu > \nu_{max}/2$.

The system of integral equations for the amplitude of the $\pi\pi$ scattering proposed in the present article differs from the system of Hsien et al[3] in two main points. First, to obtain a better approximation of the amplitude we use not the expansion in powers of $\cos^2\theta - 1$, but the expansion

$$A^{0,2}(\nu, \cos^2\theta) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\partial^n A}{\partial w^n}\right)_{w=0} w^n(\cos^2\theta).$$

$$w\,|_{\cos^2\theta=1} = 0.$$

Here $w(\cos^2\theta)$ is chosen to make the expansion converge as fast as possible. In other words, the function $w(\cos^2\theta)$ is a conformal mapping (see [9]) of the complex plane $\cos^2\theta$, with the aid of which ν_{max} can be shifted as far as desired towards higher energies. We can therefore conclude that our equations take into account the region $\nu > 2$ more accurately, because the corresponding expression for the partial wave

$$A_l^{0,2} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\partial^n A}{\partial w^n}\right)_{w=0}\int_0^1 w^n(\cos^2\theta)\,P_l(\cos\theta)\,d\cos\theta \qquad (2)$$

(l even) converges for all energies. This may prove important in the case when the expected resonance occurs at $\nu > 2$.

Second, our expansion in powers of $w(\cos^2\theta)$, contains an infinite number of waves even in the first approximation, so that the contributions from the higher waves can be estimated. In order not to lose this advantage, we use everywhere expansion in powers of w only; the unitarity condition, in particular, is also expanded in w^n, since a transition to partial waves would lead to the problem of rearrangement of two infinite series, and thereby to loss of accuracy.

As shown earlier[9] the series in w^n converges most rapidly when $w(\cos^2\theta)$ maps the cut $\cos^2\theta$ plane on a unit circle. The cuts themselves are mapped on the boundary of this circle. Consequently, the expansion in powers of w converges in all the cut plane $\cos^2\theta$, particularly in the physical region $0 \le \cos^2\theta \le 1$. This optimal value of w is

$$w_M(\cos^2\theta, \nu) = 1 + 2\sqrt{\tau^2 - 1}\,(\sqrt{\tau^2 - 1} - \sqrt{\tau^2 - \cos^2\theta})\,/\,(1 - \cos^2\theta), \qquad (3)$$

where $\tau(\nu)$ is the cosine of the nearest singularity.

Since the function w_M is rather complicated, we sometimes use for preliminary estimates and calculations that do not influence the final result the simpler function

$$w_P(\cos^2\theta, \nu) = (1 - \cos^2\theta)/(\alpha^2 - \cos^2\theta),$$
$$\alpha^2 = 2\tau^2 - 1, \qquad (3')$$

which maps the left half plane $\mathrm{Re}(\cos^2\theta) < \tau^2$ on the unit circle.

2. UNITARITY CONDITION

As can be seen from (3) and (3′), the functions w_M and w_P contain contributions from all the partial waves. We therefore write the unitarity conditions not for the partial waves, but directly for the amplitude $A(\nu, \cos\theta)$:

$$\mathrm{Im}\,A(\nu, \cos\theta) = \frac{1}{4\pi}\sqrt{\frac{\nu}{\nu+1}}\int_0^{2\pi}\int_{-1}^{+1} A^*(\nu, \cos\theta_1)\,A(\nu, \cos\theta_2)\,d\cos\theta_1 d\varphi, \qquad (4)$$

$$\cos\theta_2 = \cos\theta\cos\theta_1 - \sin\theta\sin\theta_1\cos\varphi.$$

The amplitude $A(\nu, \cos\theta)$ is expanded in powers of w:

$$A^{0,2}(\nu, \cos\theta) = A_0^{0,2}(\nu) + A_1^{0,2}(\nu)\,w + A_2^{0,2}(\nu)\,w^2 + \ldots,$$

$$A^1(\nu, \cos\theta) = \cos\theta\,(A_0^1(\nu) + A_1^1(\nu)\,w + A_2^1(\nu)\,w^2 + \ldots). \qquad (5)$$

Differentiating (4) with respect to $t = -2\nu(1 - \cos\theta)$ and putting $t = 0$, we obtain*

$$\mathrm{Im}\,A^I(\nu, \cos\theta = 1) = \frac{1}{4\pi}\sqrt{\frac{\nu}{\nu+1}}\sum_{m,n=0}^{\infty} A_m^{I*}(\nu)\,A_n^I(\nu)\,K_{m,n}^I(\nu),$$

$$\left(\frac{\partial^i\,\mathrm{Im}\,A^I}{\partial t^i}\right)_{t=0} = \frac{1}{4\pi(2\nu)^i}\sqrt{\frac{\nu}{\nu+1}}\sum_{m,n=0}^{\infty} A_m^{I*}(\nu)\,A_n^I(\nu)\,K_{m,n}^{I(i)}(\nu), \qquad (6)$$

where

$$K_{m,n}^{I(i)} = \frac{\partial^i}{\partial\cos\theta^i}\int_0^{2\pi}\int_{-1}^{+1} w^m(\cos\theta_1)\,w^n(\cos\theta_2)\,d\cos\theta_1 d\varphi\,\Big|_{\cos\theta=1} \quad \text{for } I = 0, 2,$$

$$K_{m,n}^{1(i)} = \frac{\partial^i}{\partial\cos\theta^i}\int_0^{2\pi}\int_{-1}^{+1}\cos\theta_1 w^m(\cos\theta_1)\cos\theta_2 w^n(\cos\theta_2)\,d\cos\theta_1\,d\varphi\,\Big|_{\cos\theta=1} \quad \text{for } I = 1. \qquad (7)$$

*The dispersion integrals are written for t = const, and not for $\cos\theta$ = const. When t = 0 both formulations are equivalent, but the condition t = const is convenient because differentiation of the dispersion integrals with respect to t corresponds to differentiation with respect to θ with a single subtraction.

In the present article we take into account in (6) only i = 0 and 1. Since w_M is a rather complicated function, it is advantageous to evaluate the integrals $K_{m,n}^{I(i)}$, defined by (7), with the aid of electronic computers. In the cases when explicit closed expressions are necessary for the integrals $K_{m,n}^{I(i)}$ we must forego the optimum w_M and substitute w_P into (7). We then obtain for I = 0 or 2:

$$K_{00}^{I} = 4\pi,$$

$$K_{10}^{I} = K_{01}^{I} = 4\pi - 2\pi\alpha^{-1}(\alpha^2 - 1)\ln[(\alpha \mp 1)/(\alpha - 1)],$$

$$K_{11}^{I} = 2\pi\alpha^{-2}(3\alpha^2 - 1) - \pi\alpha^{-3}(3\alpha^2 + 1)$$
$$\times (\alpha^2 - 1)\ln[(\alpha + 1)/(\alpha - 1)];$$

$$K_{00}^{I'} = K_{10}^{I'} = K_{01}^{I'} = 0,$$

$$K_{11}^{I'} = \tfrac{1}{6}\pi\alpha^{-4}(3\alpha^4 - 2\alpha^2 + 3)$$
$$- \tfrac{1}{4}\pi\alpha^{-5}(\alpha^2 + 1)(\alpha^2 - 1)^2\ln[(\alpha + 1)/(\alpha - 1)] \qquad (8a)$$

and for I = 1

$$K_{00}^{1} = 4\pi/3,$$

$$K_{01}^{1} = K_{10}^{1} = 4\pi/3 + 4\pi(\alpha^2 - 1)$$
$$- 2\pi(\alpha^2 - 1)\alpha\ln[(\alpha + 1)/(\alpha - 1)],$$

$$K_{11}^{1} = 4\pi/3 + 10\pi(\alpha^2 - 1)$$
$$- \pi\alpha^{-1}(\alpha^2 - 1)(5\alpha^2 - 1)\ln[(\alpha + 1)/(\alpha - 1)];$$

$$K_{00}^{1'} = K_{00}^{1}, \qquad K_{10}^{1'} = K_{01}^{1'} = K_{01}^{1},$$

$$K_{11}^{1'} = 2\pi + \tfrac{1}{2}\pi\alpha^{-2}(\alpha^2 - 1)(15\alpha^2 + 1)$$
$$- \tfrac{1}{4}\pi\alpha^{-3}(\alpha^2 - 1)(15\alpha^4 + 1)\ln[(\alpha + 1)/\alpha - 1)]. \qquad (8b)$$

3. INTEGRAL EQUATIONS

We start from the dispersion relations for constant t:

$$A^I(\nu, t) = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu' - \nu}\,\mathrm{Im}\,A^I(\nu', t)$$
$$+ \sum_{J=1}^{2}\frac{1}{2}\tilde{\alpha}_{IJ}\frac{1}{\pi}\int_0^\infty \frac{d\nu'}{1 + \nu + \nu' + t/4}\,\mathrm{Im}\,A^J(\nu', t), \qquad (9)$$

where $\tilde{\alpha}_{IJ}/2$ denotes the matrix of the crossing transformation in isotopic space:

$$\tilde{\alpha}_{IJ} = \begin{pmatrix} 2/3 & -2 & 10/3 \\ -2/3 & 1 & 5/3 \\ 2/3 & 1 & 1/3 \end{pmatrix}. \qquad (10)$$

$\tilde{\alpha}_{IJ}$ is connected with α_{IJ} of [1] by the relations $\tilde{\alpha}_{IJ} = (-1)^{I+J}\alpha_{IJ}$.

Restricting ourselves to two terms in the expansion (5), we must differentiate (9) with respect to t and set t = 0. Using (6), we obtain integral equations (without subtraction) for $A_0^I(\nu)$ and $A_1^I(\nu)$:

$$A_0^I(\nu) = \frac{1}{4\pi^2}\int_0^\infty \frac{d\nu'}{\nu' - \nu}\sqrt{\frac{\nu'}{\nu' + 1}}\sum_{m,n=0}^{1} A_m^{I*}A_n^I K_{mn}^I$$
$$+ \sum_{J=0}^{2}\tilde{\alpha}_{IJ}\frac{1}{8\pi^2}\int_0^\infty \frac{d\nu'}{1 + \nu + \nu'}\sqrt{\frac{\nu'}{\nu' + 1}}\sum_{m,n=0}^{1} A_m^{J*}A_n^J K_{mn}^J, \qquad (11a)$$

$$\left(\frac{\partial w}{\partial\cos\theta}\right)_{\cos\theta=1} A_1^I(\nu) = -A_0^1(\nu)\,\delta_{1,I}$$
$$+ \frac{\nu}{4\pi^2}\int_0^\infty \frac{d\nu'}{\nu' - \nu}\frac{1}{\sqrt{\nu'(\nu' + 1)}}\sum_{m,n=0}^{1} A_m^{I*}A_n^I K_{mn}^{I'}$$
$$+ \sum_{J=0}^{2}\tilde{\alpha}_{IJ}\frac{\nu}{8\pi^2}\int_0^\infty \frac{d\nu'}{1 + \nu + \nu'}\frac{1}{\sqrt{\nu'(\nu' + 1)}}\sum_{m,n=0}^{1} A_m^{J*}A_n^J K_{mn}^{J'}$$
$$- \sum_{J=0}^{2}\tilde{\alpha}_{IJ}\frac{\nu}{16\pi^2}\int_0^\infty \frac{d\nu'}{(1 + \nu + \nu')^2}\sqrt{\frac{\nu'}{\nu' + 1}}\sum_{m,n=0}^{1} A_m^{J*}A_n^J K_{mn}^J. \qquad (11b)$$

The factor $(\partial w/\partial\cos\theta)|_{\cos\theta=1}$ is equal to $-1/2(\tau^2 - 1)$ when $w \equiv w_M$ and to $-1/(\tau^2 - 1)$ when $w \equiv w_P$.

As shown in Sec. 2, the subtraction has already been carried out in (11b). It is therefore sufficient to subtract only in (11a). Unlike [1] and, [3] where the subtraction is made at the points $s = \bar{s} = t = 4/3$ and $t = 0$, $s = \bar{s} = 2$, respectively, we choose for the subtraction point the threshold of the first reaction $s = 4$, $\bar{s} = t = 0$.

We introduce the following notation

$$A^0(\nu = 0, t = 0) = a^0,$$
$$A^2(\nu = 0, t = 0) = a^2 \quad (A^1(\nu = 0, t = 0) = 0).$$

Both scattering lengths a^0 and a^2 are related by

$$a^0 = \frac{5}{2}a^2 + \frac{1}{8\pi^2}\int_0^\infty \frac{d\nu'}{\sqrt{\nu'}(\nu' + 1)^{3/2}}\left(2\sum_{m,n=0}^{1} A_m^{0*}A_n^0 K_{mn}^0\right.$$
$$\left. + 3\sum_{m,n=0}^{1} A_m^{1*}A_n^1 K_{mn}^1 - 5\sum_{m,n=0}^{1} A_m^{2*}A_n^2 K_{mn}^2\right). \qquad (12)$$

After subtraction, (11a) becomes

$$A_0^I(\nu) = a^I + \frac{\nu}{4\pi^2}\int_0^\infty \frac{d\nu'}{\nu' - \nu}\frac{1}{\sqrt{\nu'(\nu' + 1)}}\sum_{m,n=0}^{1} A_m^{I*}A_n^I K_{mn}^I$$
$$- \sum_{J=0}^{2}\tilde{\alpha}_{IJ}\frac{\nu}{8\pi^2}\int_0^\infty \frac{d\nu'}{1 + \nu + \nu'}\frac{\sqrt{\nu'}}{(\nu' + 1)^{3/2}}\sum_{m,n=0}^{1} A_m^{J*}A_n^J K_{mn}^J. \qquad (11a')$$

4. ESTIMATE OF ACCURACY OF THE UNITARITY CONDITION

The rate of convergence of the expansion depends on the position of the nearest singularity of the amplitude A, i.e., on the distance from the line t = 4, which is the first of the lines t = const intersecting (asymptotically) the regions of the spectral functions A_{13} and A_{23}. (In the complex-cosine plane this corresponds to a cut beginning with $\tau_\infty = 1 + 2/\nu$.) However, according to the main premises of the theory we can expect the influence of the far regions of the spectral functions to be negligibly small at low energies. We can therefore assume for estimating purposes that the cut in t begins with the line $t = 44/7$, which crosses the boundary $s = 16t/(t-4)$ of the spectral function at the point $\nu = 10$ ($s = 44$). In the cosine plane, this corresponds to a branch point at $\tau_{10} = 1 + 22/7\nu$.

We give below estimates made at the threshold of the first inelastic process, i.e., $\nu = 3$, where τ_∞ and τ_{10} are respectively equal to $5/3$ and 2.047. To estimate the upper limit of the errors, we choose as the "amplitude" the function

$$A^l = (\tau + \cos\theta)^{-1} + (-1)^l(\tau - \cos\theta)^{-1},$$

the singularities of which are concentrated in the very start of the former cut. For a comparison of the convergence of the series in powers of w and $\cos^2\theta - 1$ we refer the reader to Table 1 of [9]. where several partial waves are calculated for $I = 0$ and 2 in both approximations.

The errors in the unitarity conditions are caused by the fact that we restrict ourselves to the constant linear terms in the expansion of the amplitude in w^n. For w we choose w_p [see (3′)] with $\tau = 2.047$. Accordingly, the integrals K_{mn} are determined by the formulas (8).

In the case of even isotopic spin I, the errors in the function $\sum_{m,n=0}^{1} A_m^{I*}A_n^I K_{mn}^I$ and of its derivative, compared with their exact expressions (summation from 0 to ∞), are respectively -4.08 and $+33.2$ percent. Although the second of these errors seems large, the total error in (11) is small. This is brought about by the fact that (11b) contains along with $\Sigma A_m^{I*}A_n^I K_{mn}^{I'}$ also terms with $\Sigma A_m^{I*}A_n^I K_{mn}^I$ which are of much greater order of magnitude (when $I = 0$ or 2). Therefore the error due to the derivative is only 0.67 percent.

In the case when $I = 1$, both terms are of the same order of magnitude, but the errors of both are very small (-1.74 and -1.046 percent, respectively).

5. CONCLUSION

The analyticity assumptions implied in the Mandelstam representation, together with the unitarity property of the S matrix, serve as a basis for derivation of integral equations for A.

Naturally, such equations cannot be solved without making certain approximations such as the two-particle approximation in the unitarity condition, or the account of only several terms of the expansion of the amplitude in powers of the scattering-angle cosine.

The Mandelstam representation is frequently used to obtain dispersion relations in one variable only (for example, the energy), and these are simpler in form than the two-dimensional relations. The dependence of A on another variable (the momentum transfer or the cosine) is represented in series form. A series in Legendre polynomials can be used in principle, but it diverges in a large part of the nonphysical region. This circumstance was taken into account in [3], but the approach proposed there calls for knowledge of one or several derivatives ($\partial^n A/\partial\cos\theta^n$) for $\cos\theta = \pm 1$, in terms of which the partial waves are expressed. In other words, it is necessary to use in addition to the Legendre series the Taylor series, which in turn has its own convergence region.

In the present article we have, in accordance with [3], also expanded the dependence of A on the cosine in the vicinity of $\cos\theta = \pm 1$, but in powers of a definite function which has singularities precisely where A has them. This causes, first, the amplitude to be expanded in a power series that converges in the most rapid manner (see [9], Appendix 1). The errors due to the inclusion of the first two terms only, estimated in Sec. 4, confirm this result. Second, we have attained, albeit partly, symmetry in the analysis of the energy and of the momentum transfer.

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Note added in proof (June 16, 1961): Wolf and Zoellner advised us that they obtain good agreement with experiments on τ decay, choosing $a^0 = 0.3$ and $a^2 = 0.2$. Their article will be published in JETP.

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