EVOLUTIONALITY CONDITIONS OF STATIONARY FLOWS

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Application of the evolutionality conditions to flows in nozzles and to transonic flows shows that a continuous transition from subsonic to supersonic velocity is possible, whereas transition from supersonic to subsonic velocity is accompanied by the formation of shock waves. Application of the evolutionality conditions to an oblique shock wave attached to a wedge indicates that the flow behind the shock wave is subsonic. The evolutionality conditions are also applied to exothermal and endothermal discontinuities.

1. INTRODUCTION

ANALYSIS of magnetohydrodynamic shock waves has shown that not all shock waves on which the conservation laws are satisfied and the entropy increases can be realized in practice. For magnetohydrodynamic shock waves to exist it is necessary, in addition, that the evolutionality conditions be satisfied,[1] i.e., that the number of outgoing waves be equal to the number of independent boundary conditions on the discontinuity surface.[2,3]

An account of the evolutionality conditions is essential also in the investigation of the possibility of existence of several gasdynamic flows in the absence of a magnetic field. The present article is devoted to this subject.

We investigate the evolutionality conditions of continuous flows and of moving and attached discontinuities. An account of these conditions enables us to conclude that a continuous transition from supersonic to subsonic flow is impossible. In particular, shock waves must occur in the reversed Laval nozzle (which converts supersonic flow into subsonic). Similar shock waves must also occur in transonic flow about a bounded body. We shall advance later on certain arguments in favor of concluding that an attached oblique shock wave is a weak one.

2. CONTINUOUS ONE-DIMENSIONAL FLOWS

The meaning of the evolutionality condition is best seen by analyzing the flow of gas in a nozzle.

Let the stationary values of the velocity, pressure, and entropy experience infintesimal perturbations δv_x, δv_y, δp, and δs (the x axis is directed along the nozzle). These perturbations can be resolved into four terms, each propagating along one of the characteristics. The slopes of these characteristics in the (x, t) plane are determined by the following relations (we confine ourselves to the one-dimensional theory):

$$dx/dt = v, \quad (1)$$
$$dx/dt = v+c, \quad (2)$$
$$dx/dt = v-c. \quad (3)$$

The characteristic (1) is dual and serves the perturbations of both the entropy and the velocity curl. The perturbations of the Riemann invariants propagate along characteristic (2) and (3). The characteristics (1) and (2) are always directed downstream. The characteristic (3) is directed downstream in the case of supersonic velocity of the medium and upstream in the case of subsonic velocity. If the flow is everywhere subsonic or everywhere supersonic, each of the perturbations propagates downstream or upstream along one of the characteristics. Such a flow is evolutional.

In passing through the speed of sound, the two cases shown in Fig. 1 are possible. Figure 1a corresponds to the ordinary Laval nozzle in which

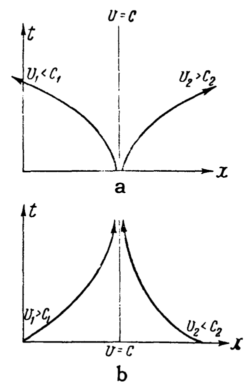


FIG. 1

subsonic flow becomes supersonic. Figure 1b corresponds to the reversed Laval nozzle, in which supersonic flow is transformed into subsonic.

In the former case (Fig. 1a) two characteristics of type (3) go from the sound line $v = c$. These characteristics carry the same perturbation to the entrance and to the exit from the nozzle. Such a flow is also evolutional.

In the second case (Fig. 1b), the different perturbations occurring at the input and at the output of the nozzle are carried to the sound line. Since these perturbations are independent, a discontinuity is produced on the sound line,* in other words, this flow is non-evolutional.

3. TRANSONIC FLOW

Let a stationary bounded body be placed in a subsonic gas stream. Since the velocity of the gas on the surface of the body is greater than the velocity at infinity, the gas will reach the velocity of sound for a certain critical Mach number at infinity. Starting with this value of the Mach number, the appearance of shock waves is possible. We want to know whether shock waves appear when the velocity of the incoming gas is increased, or whether a continuous flow about a finite body is possible such that the velocity of the gas is subsonic at infinity and bounded regions of supersonic flow exist on the surface of the body. Such continuous solutions do exist formally, but it will be shown presently that they are not evolutional and therefore cannot be realized.†

To prove the non-evolutionality of the transonic flow we note that the flow is one-dimensional near the body, where the deduction made above, concerning flows in nozzles, remains valid; viz., a continuous transition from subsonic to supersonic flow is evolutional and can be realized, but a contnous transition from supersonic to subsonic flow is not evolutional, i.e., not realizable.

The reason for the impossibility of having a continuous transonic flow about a body is that infinitesimal perturbations produce a radical change in the entire flow pattern. The impossibility of transonic flow about a body under an infinitesimal perturbation of the stationary state was demonstrated in a different manner by several writers.[6-13]

*That discontinuity is formed on the sound line in the reversed Laval nozzle was demonstrated by Kantrowitz,[4] who investigated the deformation of the profile of the perturbation wave; see also the article by Meyer.[5]

†Experiments show that shock waves are produced in the transition from supersonic to subsonic velocity.

4. MOVING DISCONTINUITIES

Let us write out the most general form of the conservation laws that hold on the discontinuity surface: the conservation of mass

$$\{\rho v_n\} = 0, \tag{4}$$

the conservation of momentum

$$\{p + \rho v_n^2\} = 0, \tag{5}$$

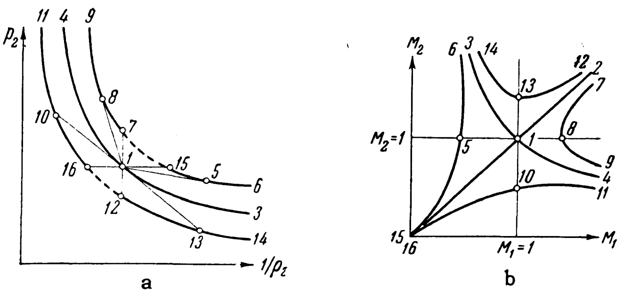
$$\{v_\tau\} = 0 \tag{6}$$

and the conservation of energy

$$\{w + v_n^2/2\} = \Delta E, \tag{7}$$

where w is the heat function, v_n the normal component of the gas velocity, v_τ the tangential component, and ΔE the change in energy due to dissociation, ionization, chemical reactions, or phase transitions, radiation* or absorption of photons; the subscripts 1 and 2 pertain to the regions in front and behind the discontinuity, respectively; the symbol $\{\ \}$ denotes the difference between the values of the corresponding quantities behind and in front of the discontinuity.

Let us examine now the form assumed by the shock adiabat (connection between the pressure p_2 behind the shock wave and the specific volume $1/\rho_2$). To find the equation of the shock adiabat we must eliminate v_{1n} and v_{2n} from the conservation laws (4), (5), and (7). A plot of the shock adiabat is shown in Fig. 2a, where the point 1 denotes the initial state (p_1; $1/\rho_1$). The line 4-1-3 denotes the shock adiabat without change of energy ($\Delta E = 0$), the line 9-8-7-15-5-6 is the shock adiabat with release of energy ($\Delta E > 0$), and finally the line 11-10-16-12-13-14 shows the shock adiabat with absorption of energy ($\Delta E < 0$). On the segments 7-15 and 16-12, bounded by the vertical line 7-1-12 and the horizontal line 16-1-15, the mass flux den-



a b

FIG. 2

*We have in mind discontinuities in which the energy radiated is not compensated by absorption in the adjacent layers of the matter. Otherwise we must put $\Delta E = 0$, and the presence of radiation affects only the form of the equation of state.

sity $\rho_1 v_{1n}$ becomes imaginary, and these segments cannot be realized.

To determine the evolutionality conditions, it is more convenient to use the shock adiabat in the (M_1, M_2) plane (M_1 and M_2 are the Mach numbers ahead of and behind the discontinuity, respectively). Such an adiabat is shown in Fig. 2b, where the numbers denote the same points as in Fig. 2a. In Fig. 2b the points 15 and 16 coincide with the origin. At 8 and 5 we have $M_2 = 1$ and the lines 1-8 and 1-5 are tangent to the shock adiabat 9-7-15-6 (see Fig. 2a). $M_1 = 1$ at 10 and 13, where the tangent 10-1-13 to the shock adiabat 4-1-3 crosses the shock adiabat 11-16-12-14.

Let us proceed to determine the evolutionality conditions of the discontinuities. We consider the shock waves first.

The speed of propagation of a shock wave depends on its amplitude. Therefore the evolutionality conditions of the shock wave have the form[14] $M_1 > 1$ and $M_2 < 1$. Such waves can propagate either without change of energy (segments 4-1 of Figs. 2a and 2b), with release of energy (segments 9-8), or with absorption of energy (segments 10-11). Corresponding to segment 9-8 is an overcompressed detonation wave; such a wave is a combustion wave in which the medium is heated in the shock wave and point 8 corresponds to the Chapman-Jouguet detonation. Segment 10-11 corresponds to a shock wave in which the heating of the medium is accompanied by absorption of energy resulting from dissociation or ionization. This section corresponds also to a shock wave in a fog, accompanied by evaporation.

Let us proceed now to examine discontinuities with propagation speed independent of the amplitude. In this case the evolutionality conditions have the form[14] $M_1 < 1$, $M_2 < 1$, or $M_1 > 1$, $M_2 > 1$. Such waves can propagate without change of energy only in the trivial case when there is no discontinuity, $M_1 = M_2$ (the line 0-1-2 of Fig. 2b). If energy is released, these waves correspond to segments 15-5 and 7-8 of Figs. 2a and 2b. Segment 15-5 corresponds to the ordinary combustion waves; the segments 7-8 corresponds either to a supersonic combustion wave (such combustion takes place in thermonuclear reactions, when the medium is heated by radiant heat conduction, and also in a supersonic stream) or to a discontinuity with recombination. Segments 15-5 and 7-8 correspond also to condensation jumps.* Discontinuities corresponding to segments 16-10 and 12-13 apparently cannot be realized.

Let us consider, finally, the discontinuities that convert subsonic gas flow into supersonic ($M_1 < 1$, $M_2 > 1$). Such continuities correspond to the segments 5-6, 1-3, and 13-14 of Fig. 2. These discontinuities are non-evolutional[14] and therefore cannot exist. Such flows, however, can be realized in the case of gas flowing in a nozzle. Analogously, segments 13-12, 8-7 and 15-5, 16-10 can be realized in a nozzle without going through the velocity of sound.

5. ATTACHED SHOCK WAVE

Let us proceed to an analysis of a shock wave attached to the vertex of a wedge in the stream.

We consider two-dimensional perturbations δv_x, δv_y, δp, and δs of the velocity, pressure, and entropy. The perturbations of the entropy and of the velocity curl propagate along the characteristic

$$x / t = v_x, \qquad y / t = v_y, \tag{8}$$

while the perturbations of the pressure and the velocity potential propagate along the characteristic cone

$$(x / t - v_x)^2 + (y / t - v_y)^2 = c^2. \tag{9}$$

We consider first the evolutionality conditions of an infinitesimal shock-wave segment located away from the wedge. In the region 1 in front of the shock wave the flow is supersonic; therefore perturbations arising on the shock wave cannot go into this region. These perturbations can be outgoing only in the region 2 behind the shock wave. On the surface of the shock wave, three independent boundary conditions are satisfied, and therefore the number of outgoing shock waves should also be three.

The entropy wave and the wave of the velocity curl are always outgoing. Were the characteristic cone (9) with vertex located on the discontinuity surface, to be located entirely in the region 2 when $t > 0$, then two additional perturbations would propagate along it. Thus, the number of outgoing waves would be four, and the shock wave would be non-evolutional. Were this cone to be wholly situated in region 1, then the number of outgoing waves would be two, i.e., the shock wave would also be non-evolutional. Therefore the characteristic cone

*In view of the confusion in the terminology, we must stop to discuss the term "weak detonation." Courant and Friedrichs[15] call a weak detonation one in which $M_1 > 1$ and $M_2 > 1$, and prove that it cannot exist (an analogous proof was first published by Zel'dovich[16]). Other authors[17-20] speak of "weak detonation," now taken to mean a condensation discontinuity in a supersonic stream. Finally, Ubbelohde[21] and Lewis[22] also speak of the possibility of a weak detonation, but define it as an overcompressed detonation ($M_1 > 1$, $M_2 < 1$) in which the release of energy is small.

(9) should be situated partially in region 1 and partially in region 2. This means that the normal component of the velocity of the medium behind the shock wave should be subsonic

$$v_{2n} < c_2. \quad (10)$$

Let us consider now the evolutionality conditions of an infinitesimal shock-wave segment adjacent to the vertex of the wedge. Since the angle of inclination of the shock wave is determined uniquely by the size of the wedge and by the velocity of the medium in region 1, the intensity of the shock wave is not an independent perturbation. Therefore four boundary conditions are satisfied on the discontinuity surface, and the number of outgoing waves should be four.

Let us locate the x axis on the wall in region 2. Then the points located on the wall will satisfy in the three-dimensional space (x, y, t) the relations

$$y = 0, \qquad x > 0, \qquad t > 0 \quad (11)$$

(here $v_{2x} = v_2$ and $v_{2y} = 0$).

The intersection of the characteristic cone (9) with the surface of the wall (11) defines two characteristics $x/t = v_2 + c_2$ and $x/t = v_2 - c_2$ $(y = 0)$, which, by virtue of the foregoing, should be outgoing, i.e., should be located in the region $t > 0$, $x > 0$. From this it follows, in turn, that an attached shock wave is evolutional only when the total velocity of the medium behind the shock wave is greater than the velocity of sound:

$$v_2 > c_2. \quad (12)$$

Thus, flow about a wedge gives rise to weak shock waves, as is indeed observed in practice.

As shown by Thomas,[23] when $v_2 < c_2$ an infinitesimal perturbation of an oblique shock wave causes a sharp change in the entire flow pattern. As is well known, the angle of inclination of the flow in an oblique shock wave cannot exceed a certain maximum value χ_{max}. It follows from the foregoing that detachment of an oblique shock wave occurs at a smaller angle of inclination χ_s. This angle is determined from the condition that the velocity of the gas v_2 behind the shock wave must equal the velocity of sound c_2.

6. INTERSECTION OF DISCONTINUITIES

Let us consider the correct reflection of a shock wave from a solid wall (see Fig. 3; OA — incident wave, OB — reflected wave, α_1 — angle of incidence, α_2 — angle of reflection). The angle of incidence α_1 is specified, and the frame of reference is chosen such as to make the point

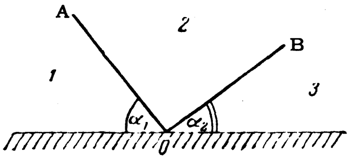


FIG. 3

O stand still. In the regions 1, 2, and 3, the perturbations δp, δs, δv_x, and δv_y are possible; in addition, the velocity of the point O and the angle of reflection α_2 can experience perturbations δU and $\delta\alpha_2$. These perturbations are interrelated by ten boundary conditions: four boundary conditions on the incident wave OA, four boundary conditions on the reflected wave OB, and the two conditions making the velocity of the medium in the regions 1 and 3 parallel to the wall ($\delta v_{1y} = 0$, $\delta v_{3y} = 0$; the x axis is directed along the wall).

By eliminating from these ten boundary conditions the six quantities δp_2, δs_2, δv_{2x}, δv_{2y}, δU, and $\delta\alpha_2$, we obtain four independent boundary conditions, which relate the perturbations in the regions 1 and 3. The evolutionality conditions consist in the fact that the number of characteristics outgoing from the point O should also be four. Repeating the arguments of Sec. 5, we find that the flow in region 3 should be supersonic, $v_3 > c_3$. This means that the shock wave always belongs to the weak family.

Application of the evolutionality conditions to the Mach reflection of shock waves is expected to eliminate the ambiguity of the solution.

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