FERMI SYSTEMS WITH ATTRACTIVE AND REPULSIVE INTERACTIONS

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We have used quantum field theoretical methods to consider a uniform, infinite system of fermions with pair interactions. The interaction consists of two parts: an attraction U of range b and a strong repulsion V of range a, with $b \gg a$. The ground-state energy expression is expanded in terms of two parameters— $a\rho^{1/3}$ and $1/b\rho^{1/3}$ —for the intermediate range of densities ρ ($b^3\rho \gg 1$ and $a^3\rho \ll 1$). All diagrams, the contribution of which to the energy is not less than the contribution which in the gas approximation is cubic in a, are taken into account. We show under what conditions there are states of the system for which $E_{av} < 0$, $\partial E_{av}/\partial\rho = 0$, and $\partial^2 E_{av}/\partial\rho^2 > 0$ (E_{av} is the energy per particle). We obtain in the two-parameter approximation a set of equations that enable us to find the ground-state energy. We compare this set of equations, obtained in the present paper, with Brueckner's equation which corresponds to the gas approximation.

1. INTRODUCTION

WE obtained in a previous paper,[1] in the approximation of weak correlations, a set of equations for the ground state of nuclei (or of nuclear matter). We then essentially left out of the discussion the nature of the forces that must act between isolated particles in order that the weak correlation approximation be valid. A similar situation occurs also in other papers[2] on nuclear matter or on atomic nuclei as many-body systems.

It is well known that at distances of 1.5 to 2.5 fermi units the nucleon-nucleon interaction potential is negative. A study of experimental data on nucleon-nucleon scattering and on high-energy pick-up processes and similar ones enables one to assume that there are strong repulsive forces acting at distances of 0.4 to 0.5 fermi units.[3,4] We base our assumptions upon these data and assume that the nucleon-nucleon interaction potential consists of two parts: an attraction U and a repulsion V with ranges b and a, respectively, where $U \ll V$, $a < b$.

A consistent consideration of nuclear matter as a many-body system is made difficult by the fact that there is no small parameter which could be used reliably to estimate the perturbation-theory diagrams and to sum them. Indeed, the quantity that characterizes the contribution from different diagrams of the same order is $p_0 b$, where p_0 is the Fermi-momentum. If $p_0 b \ll 1$, the so-called gas approximation is valid,[5] and if $p_0 b \gg 1$ and U is not large, the high-density approximation is valid.[6] For the inner regions of heavy nuclei (and also for nuclear matter) p_0 is such that $p_0 b \approx 3$ and the large value of V makes it impossible to use the high-density approximation.

However, the fact that the assumed interaction consists of two parts enables us to introduce not one but two parameters, $p_0 a < 1$ and $p_0 b > 1$, and to expand the ground-state energy, the energy of single-particle excitations, and so on in powers of $p_0 a$ and $1/p_0 b$. The approximate equations obtained by this method for the single-particle and pair wave functions enable us to verify the validity of the weak correlations approximation.

Since the system considered here, with two kinds of interaction, is of interest not only as a model for nuclear matter, we shall consider the problem from a more general point of view, restricting ourselves only to the following assumption about the forces: $|V| \gg |U|$, $a \ll b$ and $ap_0 \ll 1$, $bp_0 \gg 1$. For the sake of simplicity we restrict ourselves to homogeneous systems of infinite extent in space, we determine their binding energy, and we show under what condition a state can be realized for which $\partial E_{av}/\partial\rho = 0$.

2. A SYSTEM WITH TWO KINDS OF INTERACTIONS. ESTIMATES OF DIAGRAMS

We shall use the apparatus of quantum field theory as a general method of considering systems with a large number of particles. We write the S-

matrix of the system under consideration in the form

$$S = T\exp\left\{-\frac{i}{2}\int [V(x_1 - x_2) + U(x_1 - x_2)]\,\psi^+(\mathbf{r}_1, t_1)\,\psi^+(\mathbf{r}_2, t_2) \times \psi(\mathbf{r}_2, t_2)\,\psi(\mathbf{r}_1, t_1)\,d\mathbf{r}_1\,d\mathbf{r}_2\,dt_1 dt_2\right\},$$

$$V(x_1 - x_2) + U(x_1 - x_2) = [V(\mathbf{r}_1 - \mathbf{r}_2) + U(\mathbf{r}_1 - \mathbf{r}_2)]\,\delta(t_1 - t_2),$$

$$\hbar = m = 1. \qquad (1)$$

Regarding the ground state of a system of non-interacting particles as the vacuum state and changing from the T- to the N-product, we can use a diagram technique.* Then:

1. In each vertex there enter three lines, one of which corresponds to a factor $-iU(\mathbf{q})$ or $-iV(\mathbf{q})$ while the other two are either internal or external lines that correspond to the motion of virtual and real particles and holes.

2. An internal line corresponds to a propagator which in the momentum representation is equal to

$$G^{-1}(\mathbf{p}) = p_e - p^2/2 + i\delta\theta(\mathbf{p}),$$
$$\theta(\mathbf{p}) = 1 - 2n_{\mathbf{p}} = \begin{cases} 1, & |\mathbf{p}| > p_0 \\ -1, & |\mathbf{p}| < p_0 \end{cases}, \qquad (2)$$

where p_e is the energy and n_p the occupation number of non-interacting particles in the ground state.

3. The law of conservation of four-momentum is valid at each vertex.

4. The diagram of n-th order in U and V is multiplied by

$$i^n \frac{k\ldots(k-m+1)}{2^n m!\,n!}(-1)^P,$$

where P is the number of closed particle-hole loops, and k and m the number of U- or V-lines ($k > m$) which are arranged over the diagram in such a way that interchange of U and V does not change the matrix element. We depict in each diagram V by a dotted line and U by a wavy line.

We consider first diagrams containing V only.

Since $ap_0 \ll 1$, we can restrict ourselves to the ladder (gas) approximation, which corresponds to multiple scattering of one particle of an isolated pair by the other particle of the pair (Fig. 1). Since $Va^2 \gg 1$ one must everywhere perform the summation over V instead of the separate diagrams of Fig. 1. We denote the sum depicted in Fig. 2 by V_{eff}.

Of all the diagrams containing only U, the main contribution is made by the diagrams of Fig. 3, where in each order of the perturbation theory the maximum number of particles (which is equal to the order) is excited. The ratio of any diagram of Fig. 1 to another of the same order is less than $(ap_0)^{-1}$, and the same ratio for the diagrams of Fig. 3 is of higher order in bp_0.* We denote the sum of diagrams in Fig. 3 by U_{eff} and depict it by a double wavy line.

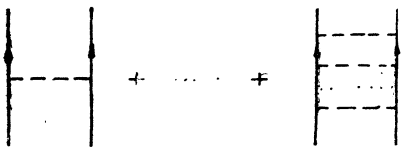

FIG. 1

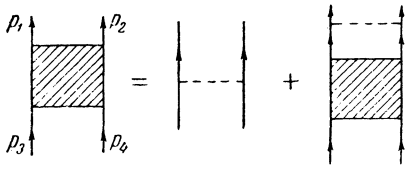

FIG. 2

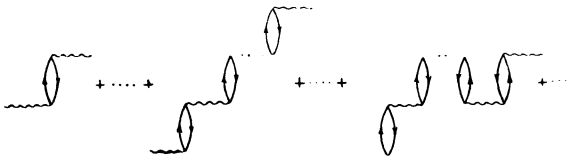

FIG. 3

From the calculation given in Galitskii's paper[5] it follows that we can assume—at any rate to get estimates—that the effective short-range interaction V_{eff} is the same as the scattering amplitude of hard spheres of radius a.† One can assume that the scattering amplitude is constant for not too large values of the momentum. Moreover, if $V_{eff} \approx 4\pi a$, then $p_0 V_{eff} \ll 1$. The scattering amplitude is independent of the magnitude of V for sufficiently large V. It will become clear in the following that all expressions for the total energy, the single-particle Green's function, the effective interaction, and so on, contain in the integrand steeply decreasing functions of the type $(x-a)^{-1}$ and the integration over the intermediate momenta is thus limited to momenta not larger than a few times p_0, and this means that for estimates one must everywhere substitute a instead of V_{eff}.

We shall take into account all diagrams whose contribution to the ground-state energy E_0 is larger than that from the term proportional to a^3 in the gas approximation. We must thus drop all

*For details see references [5], [7], and [8].

*We shall give a more detailed and accurate comparison of different diagrams in another paper, where we shall use as an example the value of p_0 corresponding to nuclear matter, and where V, U, a, and b are taken from nucleon-nucleon interaction data.

†In the system of units in which $m = \hbar = 1$, V_{eff} is nearly equal to f, i.e., to the scattering amplitude.

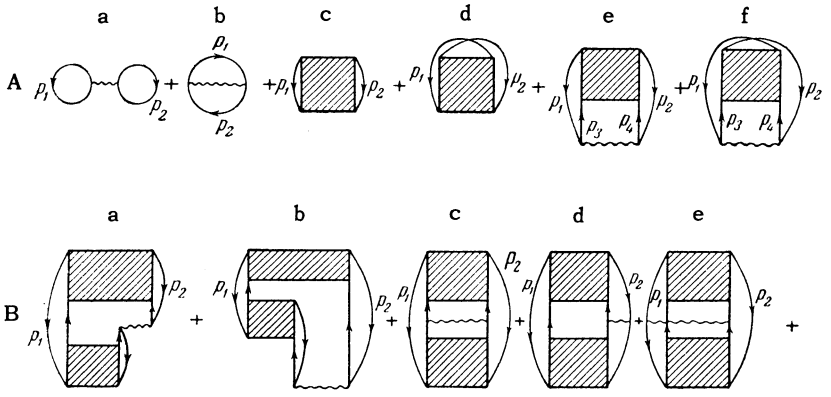

FIG. 4

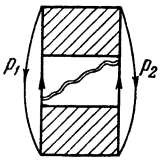

FIG. 5

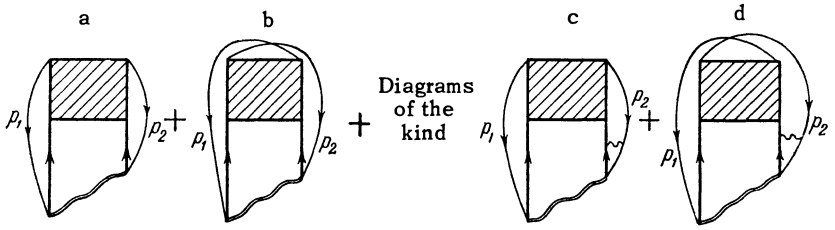

FIG. 6

diagrams which contain powers of V_{eff} larger than the second. All irreducible diagrams* which determine E_0 and which do not contain higher powers of U than the first are depicted in Fig. 4.

We now consider irreducible diagrams containing higher powers of U. The main contribution is in each order in U given by the diagrams containing the smallest number of integrations of U(q) over the momentum transfer q, since U(q) is different from zero only for small values of q. Thus, of all the possible complications when we go from the n-th to the (n+1)st order in U, we must take into account the diagrams depicted in Fig. 3. We must then take into account, apart from the diagrams of Fig. 4B, the five diagrams that are obtained from Fig. 4B by replacing U by U_{eff} (for instance, the diagram of Fig. 5).

In first order in V_{eff} one must take into account the diagrams of Fig. 6, as becomes clear from the estimates given in the following. In zeroth order in V_{eff} one must take into account, apart from the correlation energy diagrams (Fig. 7a), also the exchange diagrams. One must, of course, take into account irreducible diagrams in which instead of the Green's function (2) one must substitute the Green's function that takes the self energy into account†

*Diagrams are called irreducible if they do not contain parts of the self-energy type.

†A consideration of $\Sigma(p)$ will be the subject of a subsequent paper.

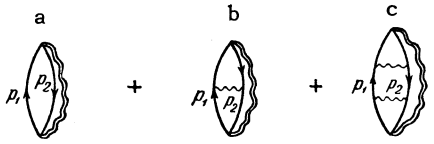

FIG. 7

$$G^{-1}(p) = p_e - p^2/2 - \Sigma(\mathbf{p}) + i\delta\theta(\mathbf{p}). \qquad (3)$$

Before we go over to estimate the reduced diagrams, we must note that should it ever become necessary to take into account terms proportional to V_{eff}^3, the main difficulty would lie in the appreciable increase in the number of mixed diagrams.

To elucidate the conditions for the existence of a state of the system for which $\partial E_{av}/\partial\rho = 0$, it is necessary to know how the matrix elements depend on p_0. Since the dependence on p_0 is different for different potentials, we perform the estimates for three potentials: $U = (U_{01}/r\nu_1)e^{-\nu_1 r}$, $U = U_{02}e^{-\nu_2 r}$, and a potential which is a square well of depth U_0 and width b. For the diagrams of Fig. 4A we have*

$$a:\ U_{01}p_0^6/\nu_1^3,\quad U_{02}p_0^6/\nu_2^3,\quad U_0 b^3 p_0^6;\qquad b:\ U_{01}p_0^3,\quad U_{02}p_0^3,\ U_0 p_0^3;$$

$$c, d:\ ap_0^6;\qquad e, f:\ U_{01}ap_0^5/\nu_1,\quad U_{02}ap_0^3\nu_2\ln(p_0/\nu_2),\quad U_0 ap_0^4,$$

and for the diagrams of Fig. 4B

$$a, b, c, d, e:\ \sim a^2p_0^6U_{01}/\nu_1,\quad a^2p_0^4U_{02}\nu_2,\quad a^2p_0^5U_0.$$

*We shall give elsewhere more detailed estimates of the diagrams for nuclear matter.

It is somewhat more difficult to estimate the terms containing U_{eff}.

For the correction to the correlation energy, which is described by the diagram of Fig. 7a, we find by summing the diagrams of Fig. 3 in the approximation $p_0 b \gg 1$ (see [6] and [9])

$$E_{corr} = \frac{3}{8\pi p_0^2} \int_0^{p_0} q^3 dq \int_{-\infty}^{+\infty} ds \, [\ln (1 - B) + B], \quad (4)$$

$$B = \frac{p_0 |U(q)|}{2\pi^2} \left\{ \left(1 - s \operatorname{arctg} \frac{1}{s}\right) + \frac{q}{4p_0} \left[(1 + s^2) \ln \left(1 + \frac{1}{s^2}\right) - \frac{2 + s^2}{1 + s^2} \right] \right\}. \quad (5)^*$$

If for our estimate we replace V_{eff} by $4\pi a$, we get for the diagram of Fig. 6a

$$E^a_{corr} = \frac{3a}{2p_0^2} \int_0^{p_0} q^3 \frac{dq}{|U(q)|} \int_{-\infty}^{+\infty} ds \left[\ln (1 - B) + B + \frac{B^2}{2} \right]. \quad (6)$$

It is expedient to distinguish two cases in (4) and (6), namely $B \gg 1$ and $B \ll 1$. If $B \ll 1$ we find from (4) after some straightforward but tedious calculations

$$E_{corr} = -\frac{1}{16\pi^4} \int_0^{p_0} q^3 dq \, U^2(q) \left[(1 - \ln 2) + \frac{3q}{16 p_0} (4 \ln 2 - 1) \right]. \quad (7)$$

For all long-range potentials with finite range, $U(q)$ tends with increasing q to zero at least as fast as $1/q^2$. The main contribution is thus given by that part of B which does not contain a factor q, and this makes it possible to drop in E^a_{corr} the term proportional to q. We get then

$$E^a_{corr} = -\frac{0.23}{16\pi^6} a p_0 \int_0^{p_0} q^3 dq U^2(q). \quad (8)$$

The exchange diagram of Fig. 6 contributes the same. An estimate yields

$$E_{corr} \approx (U_{01}^2/\nu_1^2) \ln (p_0/\nu_1), \quad U_{02}^2/\nu_2^2, \quad U_0^2 b^2 \ln p_0 b;$$
$$E^a_{corr} \approx a p_0 (U_{01}^2/\nu_1^2) \ln (p_0/\nu_1), \quad a p_0 U_{02}^2/\nu_2^2, \quad a p_0 U_0^2 b^2 \ln (p_0 b). \quad (9)$$

We consider now the case when $B > 1$. One can easily verify that the main contribution to E_{corr} and E^a_{corr} will be made by the integration over q from a q_m such that $B(q_m) \approx 1$ to p_0 (i.e., $q_m \sim \nu_1, \nu_2, 1/b$). To obtain estimates when $B > 1$ we must thus replace the quantities $\ln (p_0 b)$, $\ln (p_0/\nu_1)$ in (9) by $\ln (p_0/q_m)$.

When choosing the diagrams occurring in the correlation energy (Fig. 3 and the diagram of Fig. 7a) we have noted that in each order the ratio of the retained diagram to any dropped one is larger than $p_0 b$. This means that the contribution from the diagram of Fig. 7b and from the diagrams of Fig. 6c and d is smaller by approximately $1/p_0 b$ than the contribution from the diagrams of Figs. 7a and 6a, while that from the diagram of Fig. 7c is less than that from the diagram of Fig. 7a by a factor $1/p_0^2 b^2$.

The estimates performed here enable us to reach some qualitative conclusions.

1. If $a \gg U_0 b^3$ the attraction U is taken into account only in first order and the energy is evaluated from the hard-sphere formula:[10]

$$E_{av} = \frac{3}{10} p_0^2 \left[1 + \frac{20}{27\pi} U_0 b^3 p_0 + \frac{10}{9\pi} p_0 a + \frac{4}{21\pi^2} (11 - 2 \ln 2) p_0^2 a^2 \right] - \frac{1}{2} U_0. \quad (10)$$

2. If $a \approx U_0 b^3$ the expression for the energy is given by Eq. (10). Most interest is the case where $a \ll U_0 b^3$ since it corresponds to the possibility of forming a state with negative total energy.

3. GROUND STATE ENERGY AND EQUATION OF STATE OF A SYSTEM WITH TWO KINDS OF INTERACTION

We evaluate the energy pertaining to one nucleon in the ground state when $U_0 b^3 \gg a$.* For the diagrams of Fig. 4A a and b we find

$$E^a = \rho \int U(r) \, dr,$$
$$E^b = -\frac{1}{2\rho (2\pi)^6} \int n_{p_1} n_{p_2} U(|p_1 - p_2|) \, dp_1 \, dp_2 \approx -\frac{U_0}{2}, \quad (11)$$

where $\rho = p_0^3/6\pi^2$ is the density of the particles with spins in the same direction.

We now find the contributions from the diagrams of Fig. 4A c and d. The difficulty in solving this problems lies in the difference between (3) and the free-particle Green's function (2). In the present paper we shall not attempt to solve this problem exactly, and assume only that $\Sigma(p)$ can be approximated in the momentum range $0 < p < 1/a$ by a quadratic parabola; this corresponds to replacing the kinetic energy $p^2/2$ in the Green's function by $p^2/2\mu_1$.† The effective interaction V_{eff} is expressed in terms of the amplitude for hard sphere-hard sphere scattering:[5]

* arctg = $\tan^{-1}$.

*We assume that the interaction potentials are spin-independent. $G_0(p)$ contains $\delta_{ss'}$, where s' and s are the spin components at the ends of the line G in the diagrams. In all results given here we have summed over s.

†One can hope that the results given here will not change appreciably when the dispersion law differs from a quadratic one.

$$V_{\text{eff}} \sim [f + O(f^2)].$$

If one replaces $p^2/2$ in the single-particle Green's functions by $p^2/2\mu_1$, i.e., if one takes into account that the scattering takes place in a dispersive medium, then $V_{\text{eff}} \sim \mu_1^{-1}[f + O(f^2)]$, as one can verify using the derivation of V_{eff} given by Galitskii.[5]

Since the function f is equal to a for the momenta $pa \ll 1$ which are of interest to us, and is independent of μ_1, the effective interaction V_{eff} is proportional to a/μ_1 and this expresses the change in the effective repulsive potential by a factor μ_1 if the presence in the system of a long-range attraction is taken into account. Such an approximation is unsatisfactory for $p \approx p_0$. However, if integration over a wide range is important in the matrix element (short-range potential) the contribution from the $p \approx p_0$ region is negligibly small. We can thus evaluate it assuming that the above-mentioned method of taking the change in the single-particle Green's function into account is sufficiently accurate. The contribution from the diagrams of Fig. 4A c and d is then equal to

$$E^{cd} = p_0^2(ap_0/3\pi\mu_1) + p_0^2(2a^2p_0^2/35\pi^2\mu_1)(11 - 2\ln 2). \quad (12)$$

If, however, the $p \approx p_0$ region is important, we must use a different approximation:

$$\Sigma(p) = \Sigma(p_0) + q\Sigma'(p_0)$$

or

$$\tfrac{1}{2}p^2 \approx \tfrac{1}{2}p_0^2 + qp_0 \to \tfrac{1}{2}p_0^2 + qp_0/\mu_2 \qquad (q = p - p_0). \quad (13)$$

One can verify that μ_2 differs less from unity than μ_1.

We now consider the diagrams of Fig. 4A e and f. If we restrict ourselves to the term proportional to a in V_{eff}, we find after integrating over the fourth component of the momentum

$$E^{ef} = -\frac{6ap_0}{(2\pi)^6\mu_1}\int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 n_{\mathbf{p}_1} n_{\mathbf{p}_2}(1 - n_{\mathbf{p}_3})$$
$$\times (1 - n_{\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3})\, U'(p_0 |\mathbf{p}_1 - \mathbf{p}_3|)$$
$$\times [\tfrac{1}{2}p_3^2 + \tfrac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3)^2$$
$$- \tfrac{1}{2}p_1^2 - \tfrac{1}{2}p_2^2 + \Sigma(p_3) + \Sigma(|\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3|)$$
$$- \Sigma(p_1) - \Sigma(p_2) + 2i\delta(n_{\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3} + n_{\mathbf{p}_3} - n_{\mathbf{p}_1} - n_{\mathbf{p}_2})]^{-1}. \quad (14)$$

Terms of the type

$$\int \frac{n_{\mathbf{p}_1} n_{\mathbf{p}_2} n_{\mathbf{p}_3} n_{\mathbf{p}_4} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)}{p_1^2 + p_2^2 - p_3^2 - p_4^2}\, d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{p}_3\, d\mathbf{p}_4$$

which we obtain on integrating, vanish because the domain of integration is the same for all variables while the denominator is antisymmetric. To display explicitly the way the energy depends on the density, the momenta are expressed in units p_0. Writing $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_3$ and using the fact that U(q) decreases steeply with increasing q we expand the integrand [apart from U(q)] in a power series in q, retaining the first two terms. After straightforward but very tedious calculations we get

$$E^{ef} = -\frac{ap_0\mu_2}{2\mu_1\pi^3}\int_0^1 dq\, U'(qp_0)\, q^3$$
$$\times\left[(1 - \ln 2) + \frac{3}{16}q(4\ln 2 - 1)\right],$$
$$U'(p_0q) \equiv p_0^3 U(p_0q). \quad (15)$$

The contribution from the diagrams of Fig. 7a and those of Fig. 6 a, b is obtained by analogy with (4) and (6):

$$E_{\text{corr}} = \frac{3p_0^2}{8\pi^2\mu_2}\int_0^1 q^3 dq \int_{-\infty}^{+\infty} ds\, [\ln(1 - \mu_2 B) + \mu_2 B], \quad (16)$$

$$E^a_{\text{corr}} = \frac{3ap_0^2}{2\mu_1\mu_2}\int_0^1 \frac{q^3 dq}{|U(qp_0)|}\int_{-\infty}^{+\infty} ds\left[\ln(1 - \mu_2 B) + \mu_2 B + \frac{\mu_2^2 B^2}{2}\right]; \quad (17)$$

the quantity B is defined in (5).

If $ap_0^2 b > 1$ and E^a_{corr} is of the order of the part E^{cd} which is proportional to a^2, then (11), (12), and (15) to (17) determine the ground state energy of the system $E_0 = E_{av}N$, where N is the number of particles in the system up to terms of the order a^2.

As an example we give the expression for E_0 for an exponential attractive potential $U_0 e^{-\nu r}$ for the case where $B < 1$:

$$E_0/N = \frac{3}{10}p_0^2 + \frac{4}{3\pi}U_0\frac{p_0^3}{\nu^3} - \frac{1}{2}U_0$$
$$+ \frac{3}{2}U_0\frac{\nu}{p_0}\left\{\frac{1}{\pi}\ln\frac{2p_0}{\nu} + \frac{1}{\pi} + \frac{\nu}{p_0}\right\}$$
$$+ \frac{1}{3\pi\mu_1}ap_0^3 + \frac{2}{35\pi^2\mu_1}(11 - 2\ln 2)p_0^4a^2$$
$$- U_0 a\nu\frac{\mu_2}{\mu_1\pi^2}\left\{4(1 - \ln 2)\left(2\ln\frac{p_0}{\nu} - 1\right)\right.$$
$$\left. + \frac{3}{2}\left(1 - \frac{3}{4}\pi\frac{\nu}{p_0}\right)(4\ln 2 - 1)\right\} - \frac{U_0^2\mu_2}{\pi^2\nu^2}\left\{\frac{1}{3}(1 - \ln 2)\right.$$
$$\left. + \frac{3}{64}(4\ln 2 - 1)\left(\frac{\pi}{2}\frac{\nu}{p_0} - 1\right) + 0.08\frac{ap_0}{\mu_1\pi^2}\right\}. \quad (18)$$

The equation of state of the gas considered is obtained from (18) by differentiating with respect to ρ:

$$P = \rho^2\partial E_{av}/\partial\rho. \quad (19)$$

For a density for which $(U_0/\nu^2)(p_0/\nu) > 1$ the

pressure is basically determined by the attractive potential. Therefore $P < 0$, corresponding to a tendency of the system to compress without limit. However, the presence of a repulsion, the energy of which increases very steeply with increasing density (together with the term $\sim p_0^2 a^2$ in (18), an important contribution to E_0 is starting to be made by the terms proportional to $p_0^3 a^3$ and by all higher powers of $p_0 a$) and this leads to the fact that at high densities the rate of growth ($\partial E/\partial\rho$) of the energy of the attraction and of the repulsion compensate one another and the pressure is determined by the kinetic energy.

We shall, however, be interested in the case where $P = 0$, and we shall now consider that case.

4. SPONTANEOUSLY CONDENSED SYSTEMS

If the system is not closed, the condition $P = 0$ may not be satisfied for its internal energy, and there is no connection between ρ, a, b, V, and U. Closed systems will spontaneously condense to such a density that $P = 0$, and there is thus a well-defined connection between ρ and the parameters of the potentials. In the present section we shall study systems for which

$$\partial E_{av}/\partial\rho = 0, \qquad \partial^2 E_{av}/\partial\rho^2 > 0. \tag{20}$$

These are precisely of greatest interest to us, since they apply to practically all atomic nuclei.[1] Condition (20) was proposed in [1] on the basis of a set of equations obtained to describe the ground state of atomic nuclei. In the present paper (20) appears as an additional condition.

Let us consider what requirements are imposed on the potentials U and V by the existence of a minimum of the ground state energy with respect to ρ when $ap_0 \ll 1$ and $bp_0 \gg 1$. It is of certain interest to show that a many-particle system with two-body interactions taken from experimental data* can be in the state (20). It is well known that (20) is not satisfied when there are only attractive forces (Wigner's theorem). Exchange forces by themselves lead to the possibility of satisfying (20).

If the interaction potential is negative and contains together with an exchange part also a non-exchange part, (20) is not satisfied. We consider under what conditions strong repulsions at small distances can lead to (20) being satisfied. It is clear that one can always find a ρ for which (20) is valid if V is sufficiently large. However, we shall in the present paper be interested in the possibility of satisfying (20) just for an intermediate density when E_{av} is determined with sufficient accuracy by the sum of the contributions from the diagrams of Figs. 4 to 7 or simply by (18). One sees easily that in order that condition (20) be satisfied it is necessary that the expression for the energy E_{av} contain terms that increase more rapidly than ρ with increasing density.

*For example, for nuclear matter from nucleon-nucleon scattering experiments.

Estimates of the diagrams which were given in the foregoing, and also Eq. (18), show that the only term which increases more rapidly than ρ is the term proportional to $\rho^{4/3}$:

$$E_s = \tfrac{12}{35}\mu_1^{-1}(6\pi^2)^{1/3}a^2(11 - 2\ln 2)\rho^{4/3} \approx 13a^2\rho^{4/3}/\mu_1. \tag{21}$$

The remaining terms, either those corresponding to the presence in the system of only a long-range attraction or the mixed ones, cannot contain ρ in powers higher than the first, as can be seen from the estimates.

We shall see under what conditions condition (20) can be satisfied. We find from (18) and (20)

$$\begin{aligned}
&\frac{3}{5} + \frac{4}{\pi}\xi\eta - \frac{3}{2}\xi\frac{1}{\eta^3}\left\{\frac{1}{\pi}\ln 2\eta + \frac{2}{\eta}\right\} \\
&\quad + \frac{\zeta}{\pi\mu_1} + \frac{8}{35\pi^2\mu_1}(11 - 2\ln 2)\zeta^2 \\
&\quad - \frac{\mu_2}{\mu_1\pi^2}\xi\zeta\frac{1}{\eta^3}\left\{8(1 - \ln 2) + \frac{9\pi}{8\eta}(4\ln 2 - 1)\right\} \\
&\quad - \frac{\mu_2}{\pi^2}\frac{\xi^2}{\eta^2}\left\{0.08\frac{\zeta}{\mu_1\pi^2} - \frac{3}{64}(4\ln 2 - 1)\frac{\pi}{2}\frac{1}{\eta}\right\} = 0,
\end{aligned} \tag{22}$$

$$\begin{aligned}
&-\frac{3}{5} + \frac{6}{\pi}\xi\frac{1}{\eta^3}\ln 2\eta - \frac{3}{2\pi}\xi\frac{1}{\eta^3} + 15\xi\frac{1}{\eta^4} \\
&\quad + \frac{8}{35\pi^2\mu_1}(11 - 2\ln 2)\zeta^2 + \frac{0.16\mu_2}{\mu_1\pi^2}\zeta\xi^2\frac{1}{\eta^2} \\
&\quad + \frac{\mu_2}{\mu_1\pi^2}\xi\zeta\frac{1}{\eta^3}\left\{24(1 - \ln 2) + \frac{9\pi}{2\eta}(4\ln 2 - 1)\right\} \\
&\quad - \frac{3}{32}\frac{\mu_2}{\pi}\xi^2\frac{1}{\eta^3}(4\ln 2 - 1) > 0,
\end{aligned} \tag{23}$$

where

$$\xi = U_0/v^2, \qquad \eta = p_0/v, \qquad \zeta = p_0 a.$$

Let U be the Wigner potential. Then $\xi < 0$ and one can show that μ_1 and $\mu_2 > 1$. The calculation given here had a meaning only because $\zeta \ll 1$ and $\eta \gg 1$. But under those conditions (23) is not satisfied since all terms in (23), apart from those proportional to ζ^2 and $\zeta\xi^2/\eta^2$, are negative. It is perfectly clear that the average energy E_{av} of the ground state of the system considered must be negative. We find from (18) and (22)

$$\begin{aligned}
E_{av} \approx &\frac{1}{5}p_0^2 - \frac{U_0}{2} + \frac{2}{\pi}\frac{U_0}{\eta}\ln\eta + \frac{3}{2\pi}U_0\frac{1}{\eta} \\
&- \frac{8\mu_2}{\mu_1}(1 - \ln 2)U_0\frac{\zeta}{\eta}\ln\eta,
\end{aligned} \tag{24}$$

from which we see that $E_{av} > 0$.

We consider a system such as a neutron fluid, since we neglect the Coulomb interaction and do not take the isotopic spin of the particles into account. The result obtained here means thus that it is impossible for a neutron fluid of intermediate density to exist if its interaction potential is the Wigner potential. This conclusion does, of course, not exclude the possibility that there exists a bound state at densities so high that $\zeta \approx 1$ and $\eta \gg 1$.

We now see what changes occur if we take the Majorana potential into account along with the Wigner potential. We shall assume for the sake of simplicity that their radial dependences are the same and we denote the respective parts in the mixture by the letters W and M. (18) is then changed and using (20) we find instead of (24)

$$E_{av} \approx \frac{1}{5} p_0^2 - \left(\frac{W}{2} - M\right) U_0 + \frac{2}{\pi}\left(\frac{W}{2} - M\right) U_0 \frac{\ln \eta}{\eta} + \frac{3}{2\pi}\left(W - \frac{M}{2}\right) U_0 \frac{1}{\eta} - \frac{2}{105\mu_1}\zeta^2 p_0^2 - \frac{8\mu_2}{\mu_1}(1 - \ln 2)\left(W - \frac{M}{2}\right) U_0 \frac{\zeta}{\eta} \ln \eta < 0 \qquad (25)$$

or

$$M > W/2, \qquad 5 \mid W/2 - M \mid \mid \xi \mid > \eta^2 (1 - 2\zeta^2/21\,\mu_1). \qquad (26)$$

If $M \geq W/2$ we have $\mu_1 \leq 1$. For sufficiently large M ($M \gtrsim W$) the quantity μ_1 is appreciably less than unity.* If, moreover, $\eta\zeta \gg 1$, we have from (23)

$$\zeta^2 > 2.7\,\mu_1, \qquad (27)$$

and we get from (22) an equation for η, which in first approximation can be written as

$$0.6 + 1.27\,(W - M/2)\,\xi\eta + 0.32\,\zeta/\mu_1 + 0.22\,\zeta^2/\mu_1 = 0. \qquad (28)$$

The appearance of the factor μ_1 leads to terms $\sim a^2$ being able to contribute to the energy to approximately the same extent as the kinetic energy even if ζ is appreciably less than unity. (One must therefore take into account terms proportional to a^3, for instance when studying nuclear matter.) It is clear that the introduction of an exchange interaction does not only enable us to achieve $E_{av} < 0$, but also increases the repulsive energy so that (27) can be satisfied. One can show that E_{av} and the density corresponding to (20) depend strongly on the radius of the core (of the short-range potential) a, while the above-mentioned strong dependence arises from the requirement that (20) be satisfied.

The presence of two kinds of particles in the system increases the repulsive energy. Indeed, if $\zeta \ll 1$, the repulsion is important only in states for which the spatial part of the pair wave function is symmetric. The statistical weight of such a state is equal to unity for identical particles, while it is equal to four for different particles. The repulsive energy per particle is thus increased threefold. For smaller values of η when $\eta\zeta \approx 1$ we get from (23) instead of (17)

$$-1.9\,(M - W/2)\,\xi\zeta^3 \ln 2\zeta + 0.22\zeta^2/\mu_1 > 0.6 \qquad (29)$$

and from (22) we get an estimate for ξ

$$\xi \approx -0.25/\mu_1 (W - M/2). \qquad (30)$$

Equations (30) and (25) go together. We note that (27) is satisfied if ζ is not too small. One must then take into account the contribution from the diagrams of Fig. 4B, which leads to a decrease of the kinetic energy, as is clear from estimates.

Up to now we have considered the possibility that there exists a bound state when $\zeta \ll 1$ and $\eta \gg 1$. However, one can verify by a direct calculation that the contribution of the correlation energy is altogether very small even if $\eta \gtrsim 1$. Equation (18) is in that case incorrect since it does not take into account the contribution of the diagrams of Figs. 6c, d and 7 b, c. However, for estimates one can neglect them in $\partial^2 E_{av}/\partial\rho^2$ compared to the exchange terms in (23), i.e.,

$$(W/2 - M)\,\xi\,(\ln 2\eta - 0.25 + 0.8/\eta) > 0.31\eta^3.$$

There is thus a possibility for the existence of a bound state when $\zeta \ll 1$ and $\eta \gg 1$, if exchange forces are taken into account.

Although the density-dependence of the matrix elements containing U is different for different potentials, one can conclude on the basis of estimates given earlier that the results of the present section are qualitatively independent of the form of U. If we take into account that the nucleon-nucleon interaction consists of two parts, it is possible to show that (20) is satisfied for nuclear matter at intermediate densities only if there is a strong repulsion at small distances (a core) and also an appreciable admixture of exchange forces in U. The equilibrium density ρ will depend strongly on the core radius.

5. A COMPARISON WITH BRUECKNER'S METHOD

Recently Brueckner's method,[2] which is based upon replacing the true interaction by a t-matrix which is obtained from an integral equation, has been used very widely for studying systems with interactions of the kind considered, in

*μ_1 depends, generally speaking, on the density in such a way that $\partial^2(\rho^{4/3}/\mu_1)/\partial\rho^2 < \mu_1 \partial^2\rho^{4/3}/\partial\rho^2$. In that case, however, $\partial^2(\rho^{4/3}/\mu_1)/\partial\rho^2 > \partial^2\rho^{4/3}/\partial\rho^2$.

particular for nuclear matter. The ground state energy, the single-particle energy, and the basis functions,[2,1] and also the t-matrix are determined from the following relations

$$E_0 = -\sum_{i=1}^{N}\left(i\left|\frac{\Delta}{2}\right|i\right)+\frac{1}{2}\sum_{i,k=1}^{N}(ik|t|Aik),$$

$$(ik|t|lq) = (ik|V+U|lq) + \sum_{s,p>F}\frac{(ik|V+U|sp)(sp|t|lq)}{E_l+E_q-E_s-E_p},$$

$$E_l = -\left(l\left|\frac{\Delta}{2}\right|l\right)+\sum_{q\leqslant F}(lq|t|Alq), \qquad (31)$$

where the notation is the same as in [1].

The equation for the t-matrix is obtained in the ladder approximation[11] and is valid only at low density. Since, however, p_0b is about 3 to 3.5 for nuclear forces, such an equation has no exact theoretical foundation. Doubt was therefore often expressed as to the validity of the method, and astonishment in connection with the good agreement with experiment both for the average energy per particle in nuclear matter and for the density for which the energy is a minimum. Apart from the agreement with experiment, arguments in favor of this method are based either on qualitative considerations (see a discussion in [12] on the influence of the Pauli principle and the "healing length") or on calculations of some of the dropped perturbation-theory diagrams.

Hugenholtz[13] has, however, already shown that in the high density approximation (corresponding to $p_0b \approx 3$ to 3.5) the main contribution is not made by the ladder diagrams (compare estimates of the diagrams of Figs. 4a, b, c and 7a). Moreover, in Brueckner's method some formal difficulties are raised by the possibility that the t-matrix may have poles below the Fermi surface, which would make it wrong to drop terms containing higher powers in t than in (31). Many attempts have been made to get rid of these poles but they have not been successful. The appearance of poles need not cause surprise since in the low-density approximation the presence of an attraction (however small) leads to a logarithmic divergence in the scattering amplitude near the Fermi surface. It is, however, not clear whether this singularity is real in the case of nuclear matter, for instance, or whether it appears because of an application of the Brueckner method.*

In the present paper a different method is proposed, which as far as we can see is free from the above-mentioned deficiencies. For comparison we obtain the set of equations of the type (31) in the two-parameter approximation. The contribution from the diagrams of Fig. 4A c, d can in the notation of (31) be written in the form

$$E^{cd} = \frac{1}{2}\sum_{i,k=1}^{N}(ik|Q|Aik),$$

where Q is determined from the integral equation

$$(ik|Q|lq) = (ik|V|lq) + \sum_{s,p>F}\frac{(ik|V|sp)(sp|Q|lq)}{E_l+E_q-E_s-E_p}. \qquad (32)$$

Summing the contribution from the diagrams of Fig. 4A a to f, we find

$$E_0 = -\sum_{i=1}^{N}\left(i\left|\frac{\Delta}{2}\right|i\right)+\frac{1}{2}\sum_{i,k=1}^{N}(ik|\tau|Aik),$$

$$(ik|\tau|lq) = (ik|V+U|lq) + \sum_{s,p>F}\frac{(ik|V+U|sp)(sp|Q|lq)}{E_l+E_q-E_s-E_p},$$

$$E_l = -\left(l\left|\frac{\Delta}{2}\right|l\right)+\sum_{q\leqslant F}(lq|\tau|Alq). \qquad (33)$$

Equations (33) can be generalized in such a way that one can take into account the diagrams of Figs. 7a and 6a, b. We leave for the moment the problem of the application of these equations to a nucleus or to nuclear matter; this requires considerably more accurate estimates of diagrams than the ones given here. At the present more accurate estimates are being made.

In conclusion I consider it a pleasant duty to express my deep gratitude to Professor L. A. Sliv and also to G. M. Shklyarevskii for many discussions and valuable hints and to D. A. Kirzhnits for interesting discussions about the problems of this paper.

*The problem under what conditions poles appear in the pair scattering amplitude will be considered elsewhere.

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