CONTRIBUTION TO THE THEORY OF HIGHLY COMPRESSED MATTER. II

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We investigate the interaction between electrons and ions in highly compressed matter. The possibility of superconductivity is discussed. We find the spectrum and the damping of the electron excitations of highly compressed hydrogen.

WE have studied earlier [1] a number of properties of highly compressed matter. In particular, we showed that the nuclei form a crystalline lattice at high density; we found the long-wavelength lattice vibrations spectrum. The present paper is mainly devoted to a study of the spectrum of electron excitations in highly compressed matter.

1. ELECTRON GREEN'S FUNCTION IN A CRYSTAL. ROLE OF THE STATIC LATTICE FIELD

It is well known (see [2]) that it is necessary to find the poles of the Fourier transform of the appropriate Green's function

$$G_{\alpha\beta}(\mathbf{x}-\mathbf{x}';\ t-t') = -i\langle T(\tilde{\psi}_\alpha(\mathbf{x},\ t)\,\tilde{\psi}^+_\beta(\mathbf{x}',\ t'))\rangle, \quad (1.1)$$

to obtain the excitation spectrum in an isotropic system; in (1.1) $\tilde{\psi}_\alpha$ is the Heisenberg operator and $\langle \ldots \rangle$ indicates averaging over the ground state. In the case under consideration the system is not isotropic, but possesses translational symmetry. Because of this we derive anew Lehmann's formula [3] applicable to such a system. We assume that the number of electrons is given, but at the same time we shall use a Hamiltonian $\hat{H}-\mu\hat{N}$ where μ is the value of the energy on the Fermi surface. The electron energy is then calculated from the level μ.

Introducing as usual [3,2] summation over intermediate states we find for $t > t'$*

$$G(\mathbf{x},\mathbf{x}';\ t-t') = -i\sum_m \langle\Phi_0^*\tilde{\psi}(\mathbf{x},t)\,\Phi_m\rangle\langle\Phi_m^*\tilde{\psi}^+(\mathbf{x}',t')\,\Phi_0\rangle$$

$$= -i\sum_m \langle\Phi_0^*\psi(\mathbf{x})\,\Phi_m\rangle\langle\Phi_m^*\psi^+(\mathbf{x})\,\Phi_0\rangle$$

$$\times \exp\{-i(E_m-E_0)(t-t')\} \quad (1.2a)$$

and for $t < t'$

$$G(\mathbf{x},\mathbf{x}';\ t-t') = i\sum_m \langle\Phi_0^*\psi^+(\mathbf{x}')\,\Phi_m\rangle\langle\Phi_m^*\psi(\mathbf{x})\,\Phi_0\rangle$$

$$\times \exp\{i(E_m-E_0)(t-t')\}, \quad (1.2b)$$

where the summation is over all possible states of the system, $E_m - E_0$ are the corresponding excitation energies (for a Hamiltonian $\hat{H}-\mu\hat{N}$), and $\psi(\mathbf{x})$ are the Schrödinger operators.

Because of the translational symmetry, the matrix elements of the kind $\langle\Phi_0^*\psi(\mathbf{x})\,\Phi_m\rangle$ must possess the properties of the Bloch wave functions of an electron in a periodic field, i.e., it must be possible to write them in the form

$$\langle\Phi_0^*\psi(\mathbf{x})\,\Phi_m\rangle = V^{-1/2}e^{i\mathbf{k}\mathbf{x}}u_{n\mathbf{k}}(\mathbf{x}),$$

$$\langle\Phi_0^*\psi^+(x)\,\Phi_m\rangle = V^{-1/2}e^{i\mathbf{k}\mathbf{x}}v_{n\mathbf{k}}(\mathbf{x}), \quad (1.3)$$

where $\mathbf{k}$ is the quasi-momentum and $u_{n\mathbf{k}}$ and $v_{n\mathbf{k}}$ are periodic functions of the coordinates. The index m, enumerating the excited states, corresponds to the collection of the numbers n and $\mathbf{k}$.

From (1.2) and (1.3) we get for the Fourier component of G with respect to $t-t'$

$$G(\varepsilon;\ \mathbf{x},\mathbf{x}') = \frac{1}{V}\sum_{\mathbf{k},n} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\left\{\frac{u_{n\mathbf{k}}(\mathbf{x})\,u^*_{n\mathbf{k}}(\mathbf{x}')}{\varepsilon-\xi_n(\mathbf{k})+i\delta} + \frac{v_{n\mathbf{k}}(\mathbf{x})\,v^*_{n\mathbf{k}}(\mathbf{x}')}{\varepsilon+\xi_n(\mathbf{k})-i\delta}\right\}, \quad (1.4)$$

where $\xi_n(\mathbf{k}) = E_{n\mathbf{k}} - E_0$. The values of $\mathbf{k}$ in this sum are restricted to the basis cell in the reciprocal lattice. If we recollect the well-known commutation properties of the Schrödinger operators, we see easily that the functions $u_{n\mathbf{k}}$ and $v_{n\mathbf{k}}$ satisfy the following conditions

$$\frac{1}{V}\sum_{n,\mathbf{k}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\{u_{n\mathbf{k}}(\mathbf{x})\,u^*_{n\mathbf{k}}(\mathbf{x}') + v_{n\mathbf{k}}(\mathbf{x})\,v^*_{n\mathbf{k}}(\mathbf{x}')\} = \delta(\mathbf{x}-\mathbf{x}'),$$

$$\frac{1}{V}\sum_{n,\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}+\mathbf{x}')}\{v_{n\mathbf{k}}(\mathbf{x}')u^*_{n\mathbf{k}}(\mathbf{x}) + v_{n\mathbf{k}}(\mathbf{x})\,u^*_{n\mathbf{k}}(\mathbf{x}')\} = 0. \quad (1.5)$$

*We use units in which $\hbar = 1$. We have omitted for the sake of simplicity the spin indices.

We can expand the functions $u_{nk}(\mathbf{x})$ and $v_{nk}(\mathbf{x})$, which are periodic functions of the coordinates, in Fourier series

$$u_{nk}(\mathbf{x}) = \sum_{\mathbf{K}} u_{nk}(\mathbf{K})\, e^{i\mathbf{K}\mathbf{x}}, \quad v_{nk}(\mathbf{x}) = \sum_{\mathbf{K}} v_{nk}(\mathbf{K})\, e^{i\mathbf{K}\mathbf{x}}, \quad (1.6)$$

where the summation over $\mathbf{K}$ is over all periods of the reciprocal lattice. Substituting (1.6) into (1.4) and performing the Fourier transformation with respect to $\mathbf{x}$ and $\mathbf{x}'$ we get

$$G(\varepsilon;\, \mathbf{k}+\mathbf{K},\, \mathbf{k}'+\mathbf{K}') = G(\varepsilon;\, \mathbf{K},\, \mathbf{K}';\, \mathbf{k})\,(2\pi)^3\,\delta(\mathbf{k}-\mathbf{k}'),$$
$$G(\varepsilon;\, \mathbf{K},\, \mathbf{K}';\, \mathbf{k}) = \sum_n \left\{ \frac{u_{nk}(\mathbf{K})\, u^*_{nk}(\mathbf{K}')}{\varepsilon - \xi_n(\mathbf{k}) + i\delta} + \frac{v_{nk}(\mathbf{K})\, v^*_{nk}(\mathbf{K}')}{\varepsilon + \xi_n(\mathbf{k}) - i\delta} \right\}. \quad (1.7)$$

The spectrum of the electron excitation is according to (1.7), as also in the case of an isotropic Fermi system, determined by the poles of the Green's function, and these poles are independent of $\mathbf{K}$ and $\mathbf{K}'$. It will therefore be convenient for us to consider henceforth the diagonal element, i.e., $G(\epsilon;\, \mathbf{K}, \mathbf{K};\, \mathbf{k})$ and introduce instead of the quasi-momentum $\mathbf{k}$ the momentum $\mathbf{p}: \mathbf{p} = \mathbf{k} + \mathbf{K}$. Such a function corresponds completely to the usual Green's function in an isotropic medium $G(\epsilon, \mathbf{p})$.

The function $G(\epsilon, \mathbf{p})$ has in the first approximation the form [2]

$$G^{(0)}(\varepsilon, \mathbf{p}) = \frac{1}{\varepsilon - \xi(\mathbf{p}) + i\delta\,\mathrm{sign}\,\xi(\mathbf{p})}, \quad (1.8)$$

where $\xi(\mathbf{p}) = \mathbf{p}^2/2m - p_0^2/2m$, $\delta \to +0$ (for p near to p_0 the function $\xi \approx u(p - p_0)$ where $u = p_0/m$). In the following we consider the change in this function under the influence of the interaction of the electrons with one another and with the ions in the lattice. We shall as usual write G in the form $(\epsilon - \xi + \Delta\mu - \Sigma)^{-1}$ and study the irreducible diagrams which give a contribution to the "self-energy part" Σ.

Before doing this, we consider the interaction of the electrons with the static lattice field. The diagrams for the G-function depicted in Fig. 1 are responsible for this interaction. We denote by a cross the vertex

$$Q(\varepsilon, \mathbf{p};\, \varepsilon, \mathbf{p}+\mathbf{K}) = -4\pi Z e^2\,(N/V)\,K^{-2}, \quad (1.9)$$

where $\mathbf{K}$ is a reciprocal lattice vector. We first get rid of the vertices with $\mathbf{K} = 0$. We replace the Coulomb interaction law by $e^{-\alpha r}/r$. The contribution to $G(\epsilon, \mathbf{p})$ which is introduced by the lattice vertices with $\mathbf{K} = 0$ can be expressed by means of the self-energy part $\Sigma_{(1)} = -4\pi Z e^2 N/\alpha^2 V$. One sees easily that this part is exactly compensated by the diagram of Fig. 2, which arises from

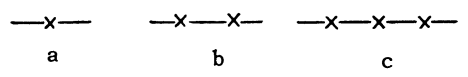


FIG. 1.

FIG. 2

the electron-electron interaction (the wavy line corresponds here to a Coulomb vertex $4\pi e^2/k^2$, where $\mathbf{k}$ is the momentum transfer). Indeed, the latter gives the self-energy part

$$\Sigma_{(2)} = -\frac{4\pi e^2}{\alpha^2}\, i \int G(\varepsilon, \mathbf{p})\, e^{i\varepsilon\tau}\, \frac{d^3\mathbf{p}\,d\varepsilon}{(2\pi)^4} = \frac{4\pi e^2}{\alpha^2}\cdot\frac{N_e}{V} = \frac{4\pi e^2}{\alpha^2}\, Z\, \frac{N}{V}.$$

Both these and other diagrams can thus be dropped.

At first sight it seems that as soon as we get rid of the vertices with $\mathbf{K} = 0$, the corrections to $G(\mathbf{p}, \epsilon)$ from the interaction with the static lattice field will be at least of second order. However, in actual fact this is not always correct. We consider the simplest diagram of Fig. 1b assuming that the changes in momentum at the vertices compensate one another. We then get the following additional term in Σ

$$\Sigma = \left(4\pi Z e^2\, \frac{N}{V}\right)^2 \sum_{0<|\mathbf{K}|} \frac{1}{K^4}\, \frac{1}{\varepsilon - \xi(\mathbf{p}-\mathbf{K}) + i\delta\,\mathrm{sign}\,\xi(\mathbf{p}-\mathbf{K})}. \quad (1.10)$$

We are interested in the vicinity of the pole of G, in other words, in the point $\epsilon = \xi(\mathbf{p})$, and the region near the Fermi surface, i.e., $|\xi| \ll p_0^2/2m$, where p_0 is the limiting Fermi momentum, will be the most important one. One sees easily that for several values of the momentum $\mathbf{p}$ one (or several) of the differences $\xi(\mathbf{p}) - \xi(\mathbf{p}-\mathbf{K})$ becomes very small and the corresponding term in Σ very large.*

The equations $\xi(\mathbf{p}) = \xi(\mathbf{p}-\mathbf{K})$ determine surfaces in momentum space (the boundaries of the Brillouin zones). It is clear that near such boundaries Eq. (1.10) is no longer suitable. It is well known that intersections of the Fermi surface and the Brillouin zone boundaries make this surface more complicated, and in particular lead to the formation of open surfaces. If we assume that highly compressed matter has a body centered lattice (see [1]), the smallest distance to the boundary is equal to $\sqrt{2}\pi/a$, where a is the cube edge, or $\sqrt{2}\pi\,(2V/N)^{-1/3} = 3.52\,(N/V)^{1/3}$. If we

*This fact is well known from the theory of an electron in a weak periodic field (see [4]).

compare this with the Fermi momentum $p_0 = (3\pi^2 ZN/V)^{1/3} = 3.09\,(ZN/V)^{1/3}$, it is clear that for hydrogen the whole of the Fermi surface can be contained within the basis cell of the reciprocal lattice. The position is, however, already different for helium ($p_0 = 3.89\,(N/V)^{1/3}$). This conclusion remains valid also in the case where highly compressed matter has a face-centered cubic lattice.

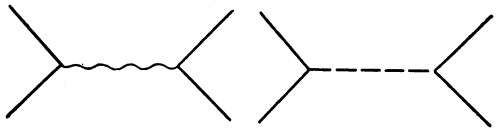
One can show that in the case where the intersection takes place, in the regions near the intersections [at distances on the order of p_0e^2/u (where $u = p_0/m$)] the radius vector of the Fermi surface changes by an amount of the same order, and the velocity on the Fermi surface changes even by an amount of the order of p_0/m.

In the following we shall restrict ourselves for the sake of simplicity to a study of the electron spectrum of compressed hydrogen. Since there are no dangerous intersections the correction to Σ from the static lattice field will be a quantity of second order. We shall neglect such quantities in the following.

2. INTERACTION BETWEEN THE ELECTRONS

For what follows it is necessary to study the interaction between the electrons. The main characteristic of this interaction is the so-called vertex part Γ, in which all Feynman diagrams with four electron ends occur. Apart from the free Green's functions (1.8), the elements of such diagrams are the elementary vertices, due both to direct interaction of the electrons with one another and to their interaction with phonons.

The vertex corresponding to the electron Coulomb interaction is equal to $\Gamma_{01} = 4\pi e^2/\mathbf{k}^2$ and will be depicted by a wavy line in the diagrams (Fig. 3a).



a b

FIG. 3

The electron-phonon vertex depends on the choice of the phonon field operators. If we take as the phonon operators $(NM/V)^{1/2}\mathbf{u}(\mathbf{x}, t)$, where $\mathbf{u}$ is the ion displacement and M the ion mass, we can easily obtain an expression for the elementary electron-phonon vertex by expanding the electron-ion interaction operator; it turns out to be equal to

$$\gamma_\alpha(\varepsilon, \mathbf{p}; \varepsilon+\omega, \mathbf{p}+\mathbf{k}+\mathbf{K}; \omega, \mathbf{k}) = 4\pi e^2 i\sqrt{N/VM}\,(\mathbf{k}+\mathbf{K})_\alpha/(\mathbf{k}+\mathbf{K})^2, \quad (2.1)$$

where $\mathbf{k}$ lies within the confines of the basis cell of the reciprocal lattice.* In this expression the fact is manifest that the electrons possess not momentum, but quasi-momentum, which is conserved only accurate to an arbitrary reciprocal lattice period $\mathbf{K}$.

The simplest diagram for Γ due to the electron-phonon interaction is illustrated in Fig. 3b and is equal to

$$\Gamma_{02}(\varepsilon_p, \mathbf{p}; \varepsilon_q+\omega, \mathbf{q}+\mathbf{k}+\mathbf{K}; \varepsilon_p+\omega, \mathbf{p}+\mathbf{k}+\mathbf{K}'; \varepsilon_q, \mathbf{q}) = 4\pi e^2\omega_0^2 D_{\alpha\beta}(\omega, \mathbf{k})(\mathbf{k}+\mathbf{K})_\alpha(\mathbf{k}+\mathbf{K}')_\beta/(\mathbf{k}+\mathbf{K})^2(\mathbf{k}+\mathbf{K}')^2, \quad (2.2)$$

where $\omega_0 = \sqrt{4\pi e^2 N/MV}$ and $D_{\alpha\beta}(\omega, \mathbf{k})$ is the Fourier component of the phonon Green's function (it corresponds to the dotted line) given by the equation

$$D_{\alpha\beta}(\mathbf{R}_i - \mathbf{R}_k, t-t') = -i\,(NM/V)\langle T\,(u_\alpha(\mathbf{R}_i, t)\,u_\beta(\mathbf{R}_k, t'))\rangle = \frac{1}{V}\sum_{\mathbf{k}}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}D_{\alpha\beta}(\mathbf{k}, \omega)\exp\{i\,[\mathbf{k}(\mathbf{R}_i - \mathbf{R}_k) - \omega(t-t')]\}. \quad (2.3)$$

Here $\langle \ldots \rangle$ indicates an average over the ground state; $\mathbf{R_i}$ are the ion coordinates; the summation over $\mathbf{k}$ in the last formula is confined to the basis cell of the reciprocal lattice.

From the definition of the D-function we can easily obtain the relation

$$D_{\alpha\beta}(\mathbf{k}, \omega) = \sum_s \frac{v_\alpha(\mathbf{k}, s)\,v_\beta^*(\mathbf{k}, s)}{\omega^2 - \omega^2(\mathbf{k}, s) + i\delta}, \quad (2.4)$$

where $\omega(\mathbf{k}, s)$ and $\mathbf{v}(\mathbf{k}, s)$ are the natural frequency and polarization vector of the s-th branch of the phonon spectrum, while

$$\sum_\alpha v_\alpha(\mathbf{k}, s)\,v_\alpha^*(\mathbf{k}, s) = 1.$$

In the case when the momentum transferred is not very small, we can restrict ourselves in first approximation to the simplest diagrams of Fig. 3a and b:

$$\Gamma_0(\varepsilon_p, \mathbf{p}; \varepsilon_q+\omega, \mathbf{q}+\mathbf{k}+\mathbf{K}; \varepsilon_p+\omega, \mathbf{p}+\mathbf{k}+\mathbf{K}'; \varepsilon_q, \mathbf{q}) = (4\pi e^2/(\mathbf{k}+\mathbf{K})^2)\,[\delta_{\mathbf{KK}'} + \omega_0^2 D_{\alpha\beta}(\mathbf{k}) \times (\mathbf{K}+\mathbf{k})_\alpha(\mathbf{K}'+\mathbf{k})_\beta/(\mathbf{k}+\mathbf{K}')^2]. \quad (2.5)$$

However, since the Coulomb forces have a long range, such a vertex part has a singularity for small momentum transfers. This singularity occurs according to Eq. (2.5) only in those Γ for which at least one of the $\mathbf{K}$ vanishes. The most

*We recall that the whole of this consideration is only applicable to hydrogen, so that we assume everywhere that $Z = 1$.

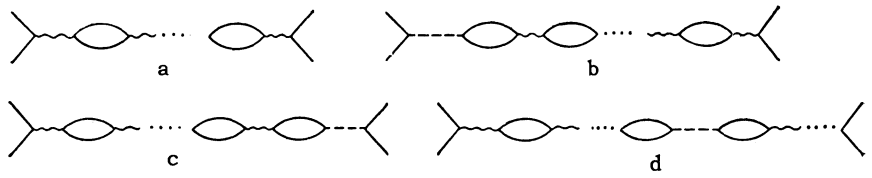


FIG. 4

important case is the one when $\mathbf{K} = \mathbf{K}' = 0$ for when $\mathbf{K} \neq 0$ and $\mathbf{K}' = 0$ the vertex is smaller by at least a factor $k/K_{min} \sim k/p_0$. We consider therefore only the case $\mathbf{K} = \mathbf{K}' = 0$, $k \ll p_0$. To describe this vertex correctly we must take terms of higher order into account.

One sees easily that the main role will be played by the corrections to Γ_0 corresponding to the diagrams of Fig. 4. In each such diagram the increase in the power of e^2 is compensated by a corresponding power of the large quantity $1/\mathbf{k}^2$. The basic element of such diagrams is the loop formed by electron lines. Such a loop corresponds to the expression*

$$\Pi(\omega, k) = 2i\int \frac{d\varepsilon}{2\pi}\frac{d^3\mathbf{p}}{(2\pi)^3} G_0(\mathbf{p}, \varepsilon)\, G_0(\mathbf{p}+\mathbf{k}, \varepsilon+\omega)$$
$$= \frac{p_0 m}{\pi^2}\left[1 - \frac{\omega}{2uk}\ln\left(\frac{\omega + uk + i\delta\,\mathrm{sign}\,\omega}{\omega - uk + i\delta\,\mathrm{sign}\,\omega}\right)\right], \qquad (2.6)$$

where $\delta \to +0$.

Summing all diagrams which do not contain phonon lines (Fig. 4a) we get

$$4\pi e^2/(k^2 + 4\pi e^2\Pi). \qquad (2.7)$$

In the case $\omega \ll uk$ this formula gives $4\pi e^2/(k^2+\kappa^2)$, where κ is the reciprocal of the Debye radius and is equal to

$$\kappa = \sqrt{4p_0 m e^2/\pi}. \qquad (2.8)$$

The summation of the loops is simply equivalent to taking the Debye screening into account.

We now turn to the diagrams containing a phonon line. It was shown in [1] that the lattice vibration spectrum in the region of the small momenta consists of three acoustical branches, one of which corresponds to longitudinal vibrations while the other two correspond to the transverse vibrations (apart from small terms of the order of e^2/u). Since the electrons interact only with the longitudinal phonons [one sees this easily from Eqs. (2.2) and (2.4); see also the last footnote] the expression $k_\alpha k_\beta k^{-2} D_{\alpha\beta}(\omega, \mathbf{k})$ will in the case $\omega \ll uk$ simply correspond to

$$1/(\omega^2 - \omega_l^2(k) + i\delta), \qquad (2.9)$$

where ω_l is the frequency of the longitudinal phonons, which was found in [1]

$$\omega_l = \omega_0\,[k^2/(k^2+\varkappa^2)]^{1/2}. \qquad (2.10)$$

We obtained Eq. (2.10) with account of the Debye screening. All electron loops strung along the D-line are thus already taken into account in Eqs. (2.9) and (2.10) when $\omega \ll uk$. If $\omega \gtrsim uk$, Eq. (2.9) is no longer valid. However, as in (2.7), the whole of the difference consists in that one must substitute the more general expression $4\pi e^2\Pi$ for κ^2. If we perform this substitution formally in Eq. (2.10) for ω_l and substitute this into (2.9) we obtain the complete D-function also for the case $\omega \gtrsim uk$.

There remains now for us to sum all diagrams of Figs. 4b, c, d. One sees easily that these diagrams differ from the diagram of Fig. 3b by the replacement of both electron-phonon vertices by more general expressions which take screening into account. The summation of the necessary diagrams causes each electron-phonon vertex to be simply multiplied by $k^2/(k^2 + 4\pi e^2\Pi)$. The total expression for Γ has thus for small transfers $\mathbf{k}$ and ω the form

$$\Gamma(\mathbf{k}, \omega) = \frac{4\pi e^2}{k^2 + 4\pi e^2\Pi}\left[1 + \frac{\omega_l^2}{\omega^2 - \omega_l^2 + i\delta}\right].$$

Substituting Eq. (2.10) for ω_l and taking the substitution $\kappa^2 \to 4\pi e^2\Pi$ into account we find finally

$$\Gamma(\mathbf{k}, \omega) = \frac{4\pi e^2\omega^2}{(\omega^2 - \omega_0^2)k^2 + 4\pi e^2\Pi\omega^2 + i\delta}. \qquad (2.11)$$

Let us consider the limiting cases. When $\omega \ll uk$ this expression becomes

$$\Gamma(\omega \ll uk) = \frac{4\pi e^2\omega^2}{\omega^2(k^2+\varkappa^2) - \omega_0^2 k^2 + i\delta}. \qquad (2.12)$$

When $k \to 0$ (in the following $c_l = \omega_0/\kappa$)

$$\Gamma \to 4\pi e^2/\varkappa^2 \quad \text{when } uk \gg \omega \gg c_l k,$$
$$\Gamma \to 4\pi e^2\omega^2/\omega_0^2 k^2 \quad \text{when } \omega \ll c_l k. \qquad (2.13)$$

The last formula is not completely exact. If we take into account corrections from "transverse" terms in the D-function it is clear that it is valid only if $\omega \gg c_t k$ (c_t is the velocity of the trans-

*If the loop π arises after a phonon line, it may depend on $k + K$. However, in that case, according to Eq. (2.1), there occurs in the corresponding electron-phonon vertex a factor K_α/K^2 instead of k_α/k^2, and this leads to a decrease of the diagram by the factor $k/K \sim k/p_0$.

verse phonons which is of the order of $\sqrt{e^2/u}\,c_l$). When $\omega \ll c_t k$ we have $\Gamma \sim 4\pi e^2 \kappa^{-2} (c_t/c_l)^2$. Equation (2.12) at $\omega = \omega_l(k)$ has a pole corresponding to longitudinal phonons (in actual fact there are also poles from the transverse phonons, but they occur with small coefficients of the order of e^2/u).

When $\omega \gg uk$ we have $\Pi \to -Nk^2/Vm\omega^2$ and neglecting the term $\omega_0^2 k^2$ as compared to $4\pi e^2 \Pi \omega^2$ ($\sim \omega_0^2 k^2 M/m$) in the denominator of (2.11) we get

$$\Gamma_\omega \equiv \Gamma(\omega \gg uk) = \frac{4\pi e^2 \omega^2}{k^2(\omega^2 - 4\pi e^2 N/mV + i\delta)}. \qquad (2.14)$$

The pole in Γ_ω at $\omega_p = \sqrt{4\pi e^2 N/mV}$ corresponds to plasma oscillations. The dispersion of these oscillations arises from the next term in the expansion in Eq. (2.6)

$$\Pi \approx -Nk^2/Vm\omega^2 - p_0^5 k^4/5\pi^2 m^3 \omega^4.$$

Taking this last term into account we get for the pole

$$\omega_p^2 = \omega_p^2(0) + \tfrac{3}{5} u^2 k^2. \qquad (2.15)$$

From (2.14) we find the following limiting formula

$$\Gamma \to -mV\omega^2/Nk^2, \qquad \omega_p \gg \omega \gg uk. \qquad (2.16)$$

3. SUPERCONDUCTIVITY

We consider now whether superconductivity is possible in highly compressed matter. To solve this problem we apply the simple and clear method of Cooper,[5] by means of which the possibility of the formation of bound electron pairs was first demonstrated. According to Cooper the equation for the wave function of a bound electron pair can be written in the momentum representation in the form

$$(2\xi(\mathbf{p}) - E) a_{\mathbf{p}} + \frac{1}{V} \sum_{|\mathbf{p}'| > p_0} U_{\mathbf{pp}'} a_{\mathbf{p}'} = 0. \qquad (3.1)$$

We substitute for the effective interaction $U_{\mathbf{pp}'}$ the electron-electron vertex part $\Gamma(p_1 p_2; p_3 p_4)$ in which $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}$, $\mathbf{p}_3 = -\mathbf{p}_4 = \mathbf{p}'$, $\epsilon_1 = \epsilon_2 = \xi(\mathbf{p})$, $\epsilon_3 = \epsilon_4 = \xi(\mathbf{p}')$, i.e., $\mathbf{k} = \mathbf{p} - \mathbf{p}'$, $\omega = \xi(\mathbf{p}) - \xi(\mathbf{p}')$.

In the integral term in Eq. (3.1) the domain of integration over $d^3\mathbf{p}$ is divided into two. In the first region $|\mathbf{p} - \mathbf{p}'| \ll p_0$ and in the second region $|\mathbf{p} - \mathbf{p}'| \sim p_0$. Since the region $|\mathbf{p} - \mathbf{p}'| \lesssim \kappa$ makes a relatively small contribution, we can assume in the first region that $|\mathbf{p} - \mathbf{p}'| \gg \kappa$ and use Eq. (2.5) with $\mathbf{K} = \mathbf{K}' = 0$, which in the present case gives

$$U_{\mathbf{pp}'} = \frac{4\pi e^2}{(\mathbf{p} - \mathbf{p}')^2} \frac{[\xi(\mathbf{p}) - \xi(\mathbf{p}')]^2}{[\xi(\mathbf{p}) - \xi(\mathbf{p}')]^2 - \omega_0^2}, \qquad \mathbf{p} - \mathbf{p}' | \ll p_0. \qquad (3.2)$$

This potential can approximately be written in the form ($\kappa \ll |\mathbf{p} - \mathbf{p}'| \ll p_0$)

$$U_{\mathbf{pp}'} = \begin{cases} 4\pi e^2/(\mathbf{p} - \mathbf{p}')^2 & \text{for } |\xi(\mathbf{p}) - \xi(\mathbf{p}')| > \omega_0 \\ 0 & \text{for } |\xi(\mathbf{p}) - \xi(\mathbf{p}')| < \omega_0 \end{cases}. \qquad (3.3)$$

When we substitute this potential into the second term of Eq. (3.1) we must bear in mind that the coefficients $a_{\mathbf{p}'}$ need not depend on the direction of $\mathbf{p}'$. We can thus integrate over the angle between $\mathbf{p}$ and $\mathbf{p}'$. Taking it into account that we shall in the following be interested in the values $|\mathbf{p}| \approx |\mathbf{p}'| \approx p_0$ we obtain in that case

$$\frac{4\pi e^2}{p_0^2}\left(\ln\frac{p_0}{\kappa} + c_1\right)\frac{1}{V}\sum_{p'} a_{p'},$$
$$|\xi(\mathbf{p}) - \xi(\mathbf{p}')| > \omega_0, \quad \xi(\mathbf{p}) > 0, \qquad (3.4)$$

where c_1 is a constant of the order of unity.

We now consider the integral over the second region $|\mathbf{p} - \mathbf{p}'| \sim p_0$.

It is now necessary to take into account the contribution of all phonon branches to the D-function (and also the vertices with $\mathbf{K}, \mathbf{K}' \neq 0$). According to Eqs. (2.5), the phonon term in Γ has then in the region $|\xi(\mathbf{p}) - \xi(\mathbf{p}')| \lesssim \omega_0$ the same order of magnitude as the electron term, while in the region $|\xi(\mathbf{p}) - \xi(\mathbf{p}')| \gg \omega_0$ it is appreciably less than the electron term (see [1]). Bearing in mind that the interaction potential depends in the region $|\mathbf{p} - \mathbf{p}'| \sim p_0$ weakly on the angle between $\mathbf{p}$ and $\mathbf{p}'$ we can approximately write it in the form

$$U_{\mathbf{pp}'} = \begin{cases} 4\pi e^2 c_2/p_0^2, & |\xi(\mathbf{p}) - \xi(\mathbf{p}')| \lesssim \omega_0 \\ 4\pi e^2 c_3/p_0^2, & |\xi(\mathbf{p}) - \xi(\mathbf{p}')| \gg \omega_0 \end{cases}. \qquad (3.5)$$

Here, c_2 and c_3 are constants of the order of unity, and $c_3 > 0$. As to the constant c_2, its sign depends on the relation between the two terms in (2.5) in the region $k \sim p_0$, which can be found only by evaluating the phonon spectrum in the short-wavelength region.

When we substitute the potential (3.5) into the integral term of Eq. (3.1) we get two terms. The term which contains a summation over the region $|\xi(\mathbf{p}) - \xi(\mathbf{p}')| > \omega_0$ is similar to expression (3.4) and contributes to the constant c_1 which combines with $\ln(p_0/\kappa)$.

After all transformations Eq. (3.1) becomes of the form

$$(2\xi(\mathbf{p}) - E) a_{\mathbf{p}} + A\frac{1}{V}\sum_{\substack{|\xi(\mathbf{p}) - \xi(\mathbf{p}')| < \omega_0 \\ \xi(\mathbf{p}') > 0}}$$
$$\times a_{\mathbf{p}'} + B\frac{1}{V}\sum_{\substack{|\xi(\mathbf{p}) - \xi(\mathbf{p}')| > \omega_0 \\ \xi(\mathbf{p}') > 0}} a_{\mathbf{p}'} = 0, \qquad (3.6)$$

where

$$A = 4\pi e^2 c_2/p_0^2, \qquad B = (4\pi e^2/p_0^2)(\ln(p_0/\kappa) + c_1).$$

In the various regions of $\xi(\mathbf{p})$ we have

$$(2\xi(\mathbf{p}) - E)\, a_p = -A\alpha - B\beta, \quad \xi(\mathbf{p}) < \omega_0,$$
$$(2\xi(\mathbf{p}) - E)\, a_p = -B(\alpha + \beta), \quad \xi(\mathbf{p}) > \omega_0, \qquad (3.7)$$

where

$$\alpha = \frac{1}{V} \sum_{0<\xi(\mathbf{p}')<\omega_0} a_{p'}, \qquad \beta = \frac{1}{V} \sum_{\xi(\mathbf{p}')>\omega_0} a_{p'}.$$

We find from these equations a_p and after that α and β. As a result we get the following equations for α and β

$$\alpha\left(1 + \frac{1}{2}\frac{p_0^2}{\pi^2 u} A \ln\frac{\omega_0}{\Delta}\right) + \beta \cdot \frac{1}{2}\frac{p_0^2}{\pi^2 u} B \ln\frac{\omega_0}{\Delta} = 0,$$
$$\alpha \cdot \frac{1}{2}\frac{p_0^2}{\pi^2 u} B \ln\frac{p_0 u}{\omega_0} + \beta\left[1 + \frac{1}{2}\frac{p_0^2}{\pi^2 u} B \ln\frac{p_0 u}{\omega_0}\right] = 0,$$

where $2\Delta = -E$ is the pair binding energy.

Assuming that $(p_0^2/\pi^2 u)\, B \ln(p_0 u/\omega_0) \ll 1$ we find the following equation to determine Δ

$$1 + \frac{1}{2}\frac{p_0^2}{\pi^2 u} A \ln\frac{\omega_0}{\Delta} = 0. \qquad (3.8)$$

As we have already noted earlier the sign of the constant A can be either positive or negative. The question, whether or not this equation has a solution, i.e., whether or not there are bound pairs, thus remains an open one. We are thus led to the conclusion that the possibility of the appearance of superconductivity is determined by the properties of the short-wavelength phonons, and the problem posed here can therefore not be solved without completely determining the phonon spectrum. We can only state on the basis of Eq. (3.8) that if superconductivity occurs the order of magnitude of the quantity Δ is given by the relation

$$\Delta \sim \omega_0 \exp(-\pi u / e^2).$$

This means that superconductivity is an exponentially small effect and that the magnitude of the gap decreases under compression ($u = p_0/m = m^{-1}(3\pi^2 N/V)^{1/3}$).

4. ELECTRON SPECTRUM

We now consider how the interaction of the electrons with one another and with the phonons influences the electron excitation spectrum. We shall evaluate all quantities in the first non-vanishing order in e^2. We shall then neglect superconducting effects, i.e., we shall assume the distance from the Fermi boundary to be large compared to Δ. We have illustrated in Fig. 5 the first-order diagrams for the self-energy part, where the electron line corresponds to the complete G-function (it was shown in Sec. 1 that one need not take into account the diagram of Fig. 2).

FIG. 5

The expression for Σ can thus be written in the form*

$$\Sigma(\mathbf{p}, \varepsilon) = i\int \frac{d^3\mathbf{p}_1}{(2\pi)^3}\frac{d\varepsilon_1}{2\pi} \times \Gamma(\mathbf{p}_1 - \mathbf{p}, \varepsilon_1 - \varepsilon)\frac{1}{\varepsilon_1 - \xi(\mathbf{p}_1) + \Delta\mu - \Sigma(\mathbf{p}_1, \varepsilon_1)}, \qquad (4.1)$$

where Γ corresponds to expression (2.5). When small values of the momentum $\mathbf{p}_1 - \mathbf{p}$ are important in the integral one must take into account diagrams of the next order. One sees easily that it is sufficient in that case to take into account the diagrams of Fig. 6, i.e., take for Γ in Eq. (4.1) expression (1.9). Since we are interested in the excitation spectrum we shall in the following only be interested in the vicinity of the pole of G, i.e., we shall put $\epsilon \approx \xi(\mathbf{p})$.

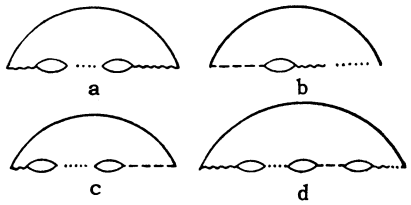


FIG. 6

We first split off from Γ the simple Coulomb term $\Gamma_1 = 4\pi e^2/(\mathbf{p}_1 - \mathbf{p})^2$. We denote the corresponding part of Σ by Σ_1. One sees easily that the values $\epsilon_1 = \xi(\mathbf{p}_1) \sim p_0 u$ are important in the integral in (4.1). We can thus neglect the term $\Sigma(\mathbf{p}_1, \epsilon_1)$ in the denominator of the integrand. Integrating,† we get

$$\Sigma_1(\mathbf{p}, \varepsilon) = -\int_{|\mathbf{p}_1|<p_0} \frac{d^3\mathbf{p}_1}{(2\pi)^3}\frac{4\pi e^2}{(\mathbf{p} - \mathbf{p}_1)^2}$$
$$= \frac{e^2}{\pi}\left(-p_0 + \frac{p^2 - p_0^2}{2p}\ln\frac{p + p_0}{|p - p_0|}\right). \qquad (4.2)$$

The first term within the bracket corresponds to the value Σ_1 at $p = p_0$ and must be included in μ. The second term in (4.2) has for $|p - p_0| \ll p_0$ the form

*Non-diagonal components $G(\varepsilon; k; K, K')$ may occur under the integral sign in (4.1) together with the corresponding term in Γ, but we can neglect them since these components themselves are at least of first order in e^2/u.

†The first-order diagram given in Fig. 5a with $G = G^0$ is important here. The integral (4.1) contains for such a diagram a factor $e^{i\epsilon_1\tau}$ where $\tau \to +0$. Thanks to this, the contour of the integral over ε_1 is closed in the upper half-plane.

$$\Sigma_1' = (e^2\xi/\pi u)\ \ln(2p_0 u/\xi). \tag{4.3}$$

We shall see below that $\Sigma(\mathbf{p}, \epsilon) - \Delta\mu \ll \xi$. We can therefore in the integrand in (4.1) replace the Green's function by the free one. We get thus

$$\Sigma(\mathbf{p}, \varepsilon) - \Sigma_1(\mathbf{p}, \varepsilon) = i\int \frac{d^3\mathbf{p}_1}{(2\pi)^3}\frac{d\varepsilon_1}{2\pi}[\Gamma(\mathbf{p}_1 - \mathbf{p}, \varepsilon_1 - \varepsilon) - \Gamma_1(\mathbf{p}_1 - \mathbf{p}, \varepsilon_1 - \varepsilon)]\ \frac{1}{\varepsilon_1 - \xi(\mathbf{p}_1) + i\delta\ \mathrm{sign}\ \xi(\mathbf{p}_1)}. \tag{4.4}$$

We perform several transformations in this integral (see the paper by Migdal [6]). Instead of ϵ_1 and $\cos(\widehat{\mathbf{pp}_1})$ we introduce new integration variables $\epsilon_1 - \epsilon = \omega$ and $|\mathbf{p}_1 - \mathbf{p}| = k$. The integral is then transformed to

$$\Sigma - \Sigma_1 = \frac{i}{(2\pi)^3 p}\int_0^\infty kdk \int_{|p-k|}^{p+k} p_1 dp_1 \times \int_{-\infty}^{\infty} d\omega\,[\Gamma(k, \omega) - \Gamma_1(k)]\ \frac{1}{\varepsilon + \omega - \xi(p_1) + i\delta\ \mathrm{sign}\ \xi(p_1)}. \tag{4.5}$$

We now write $\Gamma - \Gamma_1$ as the sum of three terms

$$\Gamma - \Gamma_1 = \Gamma_2 + \Gamma_3 + \Gamma_4, \tag{4.6}$$

where

$$\Gamma_2 = -\frac{4\pi e^2}{k^2}\frac{4\pi e^2\,\Pi}{k^2 + 4\pi e^2\,\Pi},$$

$$\Gamma_3 = \frac{4\pi e^2\omega_0^2 k^2}{[(\omega^2 - \omega_0^2)k^2 + 4\pi e^2\,\Pi\omega^2 + i\delta][k^2 + 4\pi e^2\,\Pi]},$$

$$\Gamma_4 = \Gamma - \frac{4\pi e^2\omega^2}{(\omega^2 - \omega_0^2)k^2 + 4\pi e^2\,\Pi\omega^2 + i\delta}. \tag{4.7}$$

Correspondingly, we also break up $\Sigma - \Sigma_1$ into three parts: $\Sigma - \Sigma_1 = \Sigma_2 + \Sigma_3 + \Sigma_4$.

We shall see that values of $k \ll p_0$ are important in the integrals for Σ_2 and Σ_3. We can thus simplify the integral in (4.5). We shall assume that $|p - p_0| \ll p_0$ and introduce a new variable $\xi = u(p_1 - p_0)$. The integrals for Σ_2 and Σ_3 then become of the form

$$\Sigma_n = \frac{i}{(2\pi)^3 u}\int_0^\infty kdk \int_{\xi - uk}^{\xi + uk} d\xi_1 \int_{-\infty}^{\infty} d\omega\,\Gamma_n(\omega, k)\ \frac{1}{\varepsilon + \omega - \xi_1 + i\delta\ \mathrm{sign}\ \xi_1}.$$

It is important here that Γ_2 and Γ_3 decrease at large ω as $1/\omega^2$. We can thus integrate over ξ_1 before integrating over ω. Integrating we get

$$\Sigma_n = \frac{i}{(2\pi)^3 u}\int_0^\infty kdk \int_{-\infty}^{\infty} d\omega\,\Gamma_n(\omega, k)\ln\left|\frac{\xi - \varepsilon - \omega - uk}{\xi - \varepsilon - \omega + uk}\right| + \frac{\pi}{(2\pi)^3 u}\int_0^\infty kdk \int_{\xi - \varepsilon - uk}^{\xi - \varepsilon + uk} d\omega\,\Gamma_n(\omega, k)\ \mathrm{sign}(\varepsilon + \omega).$$

We are interested in the pole of the G-function, i.e., the case $\epsilon \approx \xi$. The first term of the foregoing equation then tends to zero. This follows from the fact that Γ_2 and Γ_3 are even functions ω [see (4.7) and Eq. (2.6) for Π], while $\ln|(\omega + uk)/(\omega - uk)|$ is an odd function. There remains thus from the integration over ξ_1 only the residue around the pole. Taking into account the fact that Γ_2 and Γ_3 are even, we find

$$\Sigma_n = \frac{\mathrm{sign}\,\varepsilon}{(2\pi)^3 u}\left[\int_0^{|\varepsilon|/u} kdk \int_0^{uk} d\omega + \int_{|\varepsilon|/u}^{\infty} kdk \int_0^{|\varepsilon|} d\omega\right]\Gamma_n = \frac{\mathrm{sign}\,\varepsilon}{(2\pi)^2 u}\int_0^{|\varepsilon|} d\omega \int_{\omega/u}^{\infty} kdk\,\Gamma_n(\omega, k). \tag{4.8}$$

We consider Σ_2 first. Since the general case is difficult we study two limiting cases: $|\epsilon| \ll u\kappa$ and $|\epsilon| \gg u\kappa$, where κ is the reciprocal of the Debye radius. In the case $\epsilon \ll u\kappa$ we can in Eq. (4.8) for Σ_2 substitute Π for $\omega \ll k$. This follows from the fact that when $k \gtrsim \kappa$ the frequency $\omega \ll uk$, while for $k \ll \kappa$ the quantity Γ_2 does no longer depend on Π ($4\pi e^2\Pi \sim \kappa^2$). When $\omega \ll uk$ we have, up to terms of the order ω/uk, $4\pi e^2\Pi \approx \kappa^2$. If we substitute this value into the integral we find

$$\Sigma_2 \approx \mathrm{Re}\,\Sigma_2 = -\frac{\mathrm{sign}\,\varepsilon}{(2\pi)^2 u}\int_0^{|\varepsilon|} d\omega \int_{\omega/u}^{\infty} \frac{kdk\,4\pi e^2\kappa^2}{k^2(k^2 + \kappa^2)} = -\frac{e^2}{\pi u}\varepsilon\left(\ln\frac{\kappa u}{|\varepsilon|} + 1\right). \tag{4.9}$$

To find the imaginary part of Σ_2 we must take into account the next term of the expansion of Π in terms of ω/uk. According to (2.6) $4\pi e^2\Pi \approx \kappa^2(1 + i\pi|\omega|/2uk)$. Substituting this into the equation for Γ_2 we obtain

$$\mathrm{Im}\,\Sigma_2 = -\frac{\mathrm{sign}\,\varepsilon}{(2\pi)^2 u}\int_0^{|\varepsilon|} d\omega \int_{\omega/u}^{\infty} dk\,\frac{4\pi e^2\kappa^2\pi\omega}{2u(k^2+\kappa^2)} = -\frac{\pi e^2\,|\varepsilon|}{16u^2\kappa}. \tag{4.10}$$

We see easily that in the opposite limiting case $|\epsilon| \gg \kappa u$ the region $k \sim \kappa$, $\omega \sim u\kappa$ is important in the integral (4.8) for Σ_2. Because of this we must consider the upper limit of the integral over ω to be infinite. After that we change the order of integration over ω and k. If we introduce new variables $z = \omega/uk$ and $y = k/\kappa$ one can easily integrate over y and we get

$$\Sigma_2 = -\frac{1}{2}e^2\kappa(\beta_1 + i\beta_2)\ \mathrm{sign}\,\varepsilon, \tag{4.11}$$

where β_1 and β_2 are dimensionless constants which are respectively equal to the real and the imaginary part of the integral

$$\beta_1 + i\beta_2 = \int_0^1 dz\left(1 - z\,\mathrm{th}\,z + \frac{i\pi}{2}z\right)^{-1/2} \tag{4.12}*$$

*th = tanh.

(we take here the value of the radical with the positive imaginary part).

We now turn to Σ_3. We can also here distinguish two limiting cases: $|\epsilon| \ll \omega_0$ and $|\epsilon| \gg \omega_0$. According to (4.8) we can in the first case substitute

$$\Gamma_3 \approx \frac{4\pi e^2 \omega_0^2 k^2}{[-\omega_0^2 k^2 + 4\pi e^2 \Pi\omega^2 + i\delta][k^2 + 4\pi e^2 \Pi]}.$$

The real part of Σ_3 is obtained if we take the principal value of the integral over k, which corresponds to the region $p_0 \gtrsim k \gtrsim \kappa$. Bearing in mind that $\omega < |\epsilon| \ll \omega_0 \ll u\kappa$ we can replace $4\pi e^2\Pi$ by κ^2. Apart from logarithmic terms, we then find*

$$\mathrm{Re}\ \Sigma_3 = -(e^2/\pi u)\ \varepsilon \ln (p_0/\varkappa). \tag{4.13}$$

The imaginary part of Σ_3 occurs because of the residue of the pole arising from the first bracket in the denominator of Γ_3 and also because of the imaginary correction to Π in the second bracket in the denominator of Γ_3. The pole corresponds to the point $k^2 = \kappa^2\omega^2/\omega_0^2$ (we note that in that point $k \sim \omega/c_1 \gg \omega/u$). The residue from this pole gives

$$(\mathrm{Im}\ \Sigma_3)_1 = -e^2\varepsilon^3/6u\omega_0^2. \tag{4.14}$$

One sees easily that the second term of $\mathrm{Im}\ \Sigma_3$ exactly compensates the contribution from $\mathrm{Im}\ \Sigma_2$ in this region:

$$(\mathrm{Im}\ \Sigma_3)_2 = \pi e^2 \varepsilon\ |\varepsilon| / 16u^2\varkappa. \tag{4.15}$$

In the second limiting case $|\epsilon| \gg \omega_0$ we must use the complete expression (4.7) for Γ_3. Since the integration over ω and k is mainly over the region $\omega \sim \omega_0$, $k \sim \kappa$ we may perform the substitution $4\pi e^2\Pi \approx \kappa^2$ when evaluating the real part of Σ_3. Moreover, we can in first approximation put the limit of the integration over ω equal to infinity. If after this we interchange the order of integration over ω and k, we get

$$\int_0^\infty kdk \int_0^{uk} d\omega \approx \int_0^\infty kdk \int_0^\infty d\omega,$$

since $k \sim \kappa$. One sees easily that the principal value of $\int_0^\infty d\omega$ vanishes. This leads to the following interesting result. If we combine the real parts of the different terms of Σ of the form (4.4), (4.9), and (4.13) with $\epsilon \approx \xi$, it turns out that in the case $\epsilon \ll \omega_0$ all logarithmic terms cancel one another.† On the other hand, when $\epsilon \gg \omega_0$ we have $\mathrm{Re}\ \Sigma_3 \approx 0$ and there thus remains in $\mathrm{Re}\ \Sigma$ the term $(e^2/\pi u)\,\epsilon \ln (p_0/\kappa)$. One can observe the appearance of a logarithmic term if one calculates $\mathrm{Re}\ \Sigma_3$, including logarithmic terms, for the case $|\epsilon| \sim \omega_0$. We get then

$$\mathrm{Re}\ \Sigma_3 = \frac{\mathrm{sign}\ \varepsilon}{(2\pi)^2 u} \int_0^{|\varepsilon|} d\omega \int_{\omega/u}^\infty kdk\, \frac{4\pi e^2\omega_0^2 k^2}{[(\omega^2 - \omega_0^2)k^2 + \varkappa^2\omega^2][k^2 + \varkappa^2]}$$

$$\approx \frac{\mathrm{sign}\ \varepsilon e^2}{\pi u} \int_0^{|\varepsilon|} d\omega\, \frac{\omega_0^2}{\omega^2 - \omega_0^2} \ln \frac{p_0}{\varkappa}$$

$$= -\frac{e^2}{2u\pi}\,\omega_0 \ln \left|\frac{\varepsilon + \omega_0}{\varepsilon - \omega_0}\right| \ln \frac{p_0}{\varkappa}.$$

Combining all logarithmic terms in Σ we find

$$\frac{e^2}{\pi u}\left(\varepsilon - \frac{\omega_0}{2} \ln \left|\frac{\varepsilon + \omega_0}{\varepsilon - \omega_0}\right|\right) \ln \frac{p_0}{\varkappa}. \tag{4.16}$$

We now turn to the imaginary part of Σ_3 for $|\epsilon| \gg \omega_0$. It arises from the residue in Γ_3 which only occurs when the condition $\omega < \omega_0$ is satisfied. We find thus

$$\mathrm{Im}\ \Sigma_3 = -\frac{2\pi^2 e^2}{(2\pi)^2 u}\ \mathrm{sign}\ \varepsilon \int_0^{\omega_0} \frac{\omega^2}{\omega_0^2 - \omega^2} d\omega.$$

This integral does in actual fact not diverge, since the pole in k can not lie above p_0, i.e., $\kappa^2\omega^2/(\omega_0^2 - \omega^2) \ll p_0^2$. Restricting ourselves to the order of the logarithmic terms we get

$$\mathrm{Im}\ \Sigma_3 = -(e^2\omega_0/2u)\ \mathrm{sign}\ \varepsilon \ln (p_0/\varkappa). \tag{4.17}$$

As far as additional terms in $\mathrm{Im}\ \Sigma_3$ due to the imaginary part of Π are concerned, one sees easily that they are of relative order $\omega_0/u\kappa \ll 1$ in comparison with (4.17).

There now only remains the last term, Σ_4. We have chosen the function Σ_4 especially in such a way that it vanishes when $k \ll p_0$. Only values $k \sim p_0$ will therefore be important in the integral. We can write in this region Γ_4 in the form [see (2.5)]

$$\Gamma_4(\mathbf{k}, \omega) = \frac{4\pi e^2}{k^2}\,\omega_0^2 \left(\frac{1}{k^2} \sum_s \frac{|\mathbf{v}(s, \mathbf{k})\,\mathbf{k}|^2}{\omega^2 - \omega^2(s, \mathbf{k}) + i\delta} - \frac{1}{\omega^2 - \omega_0^2 + i\delta}\right). \tag{4.18}$$

The momentum $\mathbf{k}$ can then take on any value, and we must substitute in $\mathbf{v}(\mathbf{k})$ and $\omega(\mathbf{k})$ the value of this vector, which is reduced to the basis cell of the reciprocal lattice by the subtraction of the appropriate vector $\mathbf{K}$. The integral for Σ_4 can be written in a form similar to (4.6), but we must take into account that expression (4.18) is anisotropic and that we must therefore still integrate over $d\varphi$ in (4.5). Since for an exact calculation

*Knowledge of the short-wavelength part of the phonon spectrum is required to attain high accuracy. This applies to Eqs. (4.16) and (4.17).

†We shall show later that Σ_4 does not contain such terms.

we need to know $\omega(s, \mathbf{k})$ and $v(s, \mathbf{k})$ in the short-wavelength region $k \sim p_0$, we can only estimate its order of magnitude.

In the region $k \sim p_0$ the integral over p_1 in (4.5) is taken between the limits 0 and $\sim 2p_0$. The situation here corresponds exactly to the case considered by Migdal.[6] One sees easily that the principal value of the integral over ξ_1 gives a correction to μ ($\sim \omega_0 e^2/u$), and the part obtained from the residue of the pole can be written in the form

$$\frac{\pi}{(2\pi)^4 u}\int d\varphi \int_0^{\sim 2p_0} kdk \int_{-\varepsilon}^{\varepsilon} d\omega \Gamma_4(\omega, \mathbf{k}).$$

In the case $|\epsilon| \ll \omega_0$ the imaginary part of that integral vanishes and the real part is of the order $e^2\epsilon/u$. When $\epsilon \gg \omega_0$ there are both a real and an imaginary part. Both are of the order $e^2\omega_0/u$.

We have thus determined all terms which make up the self-energy part Σ. The pole of the G-function is obtained from the solution of the equation $\epsilon - \xi + \Delta\mu - \Sigma = 0$, i.e., $\epsilon = \xi + \Sigma(\epsilon = \xi) - \Delta\mu$. It is well known[2] that the real and the imaginary parts of the pole of the G-function determine the energy of the excitations and their damping:

$$\varepsilon(p) = \xi(p) + \mathrm{Re}\,\Sigma - \Delta\mu, \qquad \gamma = -\mathrm{Im}\,\Sigma.$$

Combining all results obtained in the foregoing we get

$$\varepsilon(p) = \begin{cases} \xi(p)(1 + e^2\alpha_1/\pi u), & \xi \ll \omega_0 \\ \xi(p)[1 + (e^2/\pi u)(\ln(2p_0/\varkappa) - 1)], & \omega_0 \ll \xi \ll \varkappa u \\ \xi(p)[1 + (e^2/\pi u)\ln(2p_0 u/\xi(p))], & \varkappa u \ll \xi \ll u p_0 \\ \xi(p)[1 + (e^2 m/\pi p)\ln((p + p_0)/|p - p_0|)], & \xi \sim u p_0 \end{cases} \tag{4.19}$$

$$\gamma = \frac{e^2}{u} \cdot \begin{cases} \xi^3/6\omega_0^2, & \xi \ll \omega_0 \\ 1/16\,\pi\xi|\xi|/u\varkappa + \omega_0 \operatorname{sign}\xi\,[1/2\ln(p_0/\varkappa) + \alpha_2], & \omega_0 \ll \xi \ll u\varkappa. \\ 1/2\,\beta_2 u\varkappa \operatorname{sign}\xi, & \xi \gg \varkappa u \end{cases} \tag{4.20}$$

The constants α_1 and α_2 depend here on the parameters of the short-wavelength part of the phonon spectrum, while the constant β is expressed in terms of the integral (4.12). The change in the "velocity on the Fermi surface" which is given by Eq. (4.19) corresponds in the region $\omega_0 \ll \xi \ll u\kappa$ to the equation of Gell-Mann[7] for the electronic specific heat.

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102