PION-NUCLEON AMPLITUDE WITH ACCOUNT OF ππ INTERACTION

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Dispersion relations for the invariant πN amplitudes similar to those used by Bowcock et al[7] are considered. The relations contain additional subtractions in the energy s (one subtraction) and momentum transfer t (two subtractions). They are regarded as integral equations with kernels dependent on the S, P, and D wave ππ scattering amplitudes. All quadratures obtained in the solution of the integral equations are evaluated for a simple phase shift model which is a generalization of the effective interaction range approximation. Simple analytic expressions for the contribution of ππ interaction to the πN amplitude are obtained. It is shown that in previous calculations[3-9] the convergence of the integrals along the cut $t \geq 4\mu^2$, where μ is the pion mass, is not sufficiently rapid; conclusions based on these calculations are therefore not reliable.

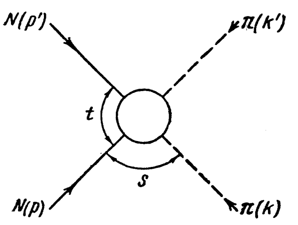
1. INTRODUCTION

SEVERAL attempts have recently been made to calculate the πN amplitudes for small values of the invariants s and t on the basis of the Mandelstam representation.[1,2] The solution of this problem would relate the ππ-scattering, πN-scattering, NN-scattering, and nucleon electromagnetic form factors. The simplest way of solving this problem is to use dispersion relations to extrapolate the πN amplitudes from the $\pi + N \rightarrow \pi + N$ scattering channel where the imaginary parts of the amplitudes are known from experiment.[3-8] Then the use of unitarity in the two-meson approximation for the $N + \bar{N} \rightarrow \pi + \pi$ annihilation channel turns these relations into simple linear integral equations; the solution of these equations is in the form of quadratures dependent on the ππ scattering amplitudes.

The main defect of the previous calculations is the insufficiently rapid decrease of the integrands in the integrations along the annihilation cut $t \geq 4\mu^2$, where t is the square of the total energy in the annihilation channel and μ is the pion mass. As a result, the calculations are inconsistent since, on the one hand, the use of unitarity in the two-meson approximation assumes that we are concerned with only small values of t, while, on the other hand, the integrals obtained in the solution of the equations depend strongly on the behavior of the amplitudes for large t. Moreover, in several papers[3-6] numerical integration was used to carry out the calculations; this makes the consideration of the various problems requiring a knowledge of the πN amplitudes very complicated. We note also that Efremov, Meshcheryakov, and Shirkov[6] made use of a patently incorrect simplification in their equations (see below); as a result, their results must be revised.

Bowcock, Cottingham, and Lurie[7] (see also [8] and [9]) obtained simple analytic expressions for the πN amplitudes in a model with a sharp ππ resonance; however, this was achieved at the price of not using the correct solution of the integral equations. For this reason it is not clear whether the contributions they obtain from the cut $t \geq 4\mu^2$ are consistent with the two-meson approximation or not, i.e., whether they are a consequence of ππ interaction or of heavier intermediate states in the unitarity condition.

In this work we attempt to improve the method used in [7] by introducing additional subtractions and correctly solving the integral equations thus obtained. In a simple ππ phase model which is a generalization of the effective interaction range approximation, all the integrals are evaluated and simple analytic expressions are obtained for the contributions from the cut $t \geq 4\mu^2$. In this model the ππ interaction is specified by a number of parameters which are unknown at present. The scattering length[3,4,6] and Breit-Wigner resonance[7-9] approximations used previously are special cases of our model. The πN amplitudes we obtain can easily be used in various problems involving nucleons and pions; this will be done in separate papers.



2. EQUATIONS FOR THE INVARIANT πN AMPLITUDES

We use the system of units in which $\hbar = c = \mu = 1$ and the following notations for the invariants (see Fig. 1):

$$t = (p + p')^2,\ \nu = (p - p')(k - k')/4m,$$
$$s = (p + k)^2 = m^2 + 1 - t/2 + 2m\nu,$$
$$u = (p + k')^2 = m^2 + 1 - t/2 - 2m\nu,$$
$$p^2 = p'^2 = m^2 = 1/\varepsilon^2, \qquad \varepsilon = 0.15, \qquad k^2 = k'^2 = 1. \tag{1}$$

In the $\pi + N \to \pi + N$ channel the invariant s is related to the total energy ω of the incident pion in the laboratory system of coordinates by

$$\omega = (s - m^2 - 1)/2m.$$

In considering the πN amplitudes in the annihilation channel, it is convenient to use the variables x and $\bar\theta$ (the square of the pion 3-momentum and the angle between the vectors **p** and -k, respectively, in the center-of-mass system in this channel), which are related to t and ν by

$$t = 4(1 + x), \qquad \nu = \sqrt{t/4m^2 - 1}\sqrt{t/4 - 1}\cos\bar\theta. \tag{1'}$$

The pion-nucleon amplitude is specified by the invariant functions $A^{(\pm)}$ and $B^{(\pm)}$ (in the usual notation[10,11]), which satisfy dispersion relations in ν (or, equally, in s) for fixed momentum transfer t. We use these relations as a starting point. Instead of $A^{(\pm)}$ we consider the functions with definite helicity[12,13]

$$F^{(\pm)} = (1 - t/4m^2)A^{(\pm)} + \nu B^{(\pm)}, \tag{2}$$

which, as will be apparent in the following, have some advantages over the $A^{(\pm)}$.

The dispersion relation for $F^{(+)}$ is

$$F^{(+)}(\nu, t) = F_p^{(+)}(\nu, t) + F^{(+)}(0, t) + \frac{1}{\pi}\int_{(m+1)^2}^{\infty} \operatorname{Im} F^{(+)}(s', t)\left[\frac{1}{s' - s(\nu, t)} + \frac{1}{s' - u(\nu, t)} - \frac{2}{s' - s(0, t)}\right]ds', \tag{3}$$

where the notation for the pole terms is

$$F_p^{(\pm)} = \nu B_p^{(\pm)}; \qquad B_p^{(\pm)} = g_r^2\left(\frac{1}{m^2 - s} \mp \frac{1}{m^2 - u}\right). \tag{4}$$

with $g_r^2 = 4\pi g^2$, $g^2 = 14.5$.

We consider the analytic properties in t of $\operatorname{Im} F^{(+)}(s', t)$ and of the subtraction term $F^{(+)}(0, t)$. The subtraction term has the cuts $t \geq 4$ and $t \leq -4m$, and $\operatorname{Im} F^{(+)}(s', t)$ has cuts for $t \geq t(c_{13}) \geq 4$ and $t \leq t(c_{12}) < -4m$, where c_{12} and c_{13} are the boundary curves in Mandelstam's[1] notation. The contributions from the left-hand cuts can be expanded in powers of t; this gives a series in powers of $|t/t_{\text{eff}}| < |t|/4m = \epsilon|1 + x|$. Keeping just two terms of this expansion and combining them with similar terms from the right-hand cuts (which is equivalent to using a dispersion relation in t with two subtractions), we obtain

$$F^{(+)}(\nu, t) = F_p^{(+)}(\nu, t) + \mathcal{F}^{(+)}(0, t) + \frac{1}{\pi}\int_{(m+1)^2}^{\infty} \operatorname{Im}\mathcal{F}^{(+)}(s', t)\left[\frac{1}{s' - s(\nu, t)} + \frac{1}{s' - u(\nu, t)} - \frac{2}{s' - s(0, t)}\right]ds' + \frac{t^2}{\pi}\int_4^{\infty} F_{ann}^{(+)}(\nu, t')\frac{dt'}{(t' - t)t'^2}, \tag{5}$$

$$\mathcal{F}^{(+)}(0, t) = F^{(+)}(0,0) + t\left[\frac{\partial}{\partial t}F^{(+)}(0, t)\right]_{t=0},$$
$$\operatorname{Im}\mathcal{F}^{(+)}(s', t) = \operatorname{Im}F^{(+)}(s', 0) + t\left[\frac{\partial}{\partial t}\operatorname{Im}F^{(+)}(s', t)\right]_{t=0} \tag{6}$$

which is accurate to order $\epsilon^2(1 + x)^2$.

The last term in Eq. (5) gives the exact contribution (with two subtractions) of the cut $t \geq 4$; $F_{\text{ann}}^{(+)}$ signifies the annihilation part of the absorptive part of the amplitude $F^{(+)}$. For $t \leq 16$ only two-meson states contribute to $F_{\text{ann}}^{(+)}$; their contribution can be obtained by extrapolating $\operatorname{Im} F^{(+)}$ from the physical region of the $N + \bar N \to \pi + \pi$ channel.[1,14] The extrapolation can be carried out by means of an expansion in Legendre polynomials $P_l(\cos\bar\theta)$.[13] This expansion in annihilation harmonics is equivalent to a series in powers of $[\nu/\nu(c_{13})]^2$ and converges in the region between the boundary curves* c_{13}:

$$-\nu(c_{13}) \leqslant \nu \leqslant \nu(c_{13}), \qquad \nu(c_{13}) \geqslant 2.75.$$

For accuracy of order $\leq (\nu/2.75)^2$ we can keep just the lowest harmonic; by using unitarity† this

*The minimum value (equal to 2.75) of ν on the curve c_{13} occurs for $t = 4.5m$.

†We note that the unitarity condition for the functions (2) has the same form as in the scalar theory and leads to the simple expressions (7) and (10).

can be expressed in terms of the S wave phase shift $\delta_0(x)$ for $\pi\pi$ scattering:

$$F_{ann}^{(+)}(\nu, x) = e^{-i\delta_0(x)} \sin \delta_0(x)\ F_0^{(+)}(x). \qquad (7)$$

In Eq. (7) and in the following we use the notation

$$(F_l^{(\pm)}(x);\ B_l^{(\pm)}(x)) = \frac{1}{2}\int_{-1}^{1} P_l(\cos\overline{\theta})\,(F^{(\pm)}(\cos\overline{\theta}, x);\ B^{(\pm)}(\cos\overline{\theta}, x))\, d\cos\overline{\theta} \qquad (8)$$

for the annihilation harmonics.

For the other invariant amplitudes we obtain

$$B^{(+)}(\nu, t) = B_p^{(+)}(\nu, t) + \frac{1}{\pi}\int_{(m+1)^2}^{\infty} \operatorname{Im}\mathscr{B}^{(+)}(s', t)\left[\frac{1}{s'-s(\nu, t)} - \frac{1}{s'-u(\nu, t)}\right] ds' + \frac{t^2}{\pi}\int_4^{\infty} B_{ann}^{(+)}(\nu, t')\frac{dt'}{(t'-t)\,t'^2},$$

$$F^{(-)}(\nu, t) = F_p^{(-)}(\nu, t) + \nu[\mathscr{F}^{(-)}(\nu_0, t) - \mathscr{F}_p^{(-)}(\nu_0, t)]/\nu_0|_{\nu_0=0} + \frac{1}{\pi}\int_{(m+1)^2}^{\infty} \operatorname{Im}\mathscr{F}^{(-)}(s', t)\left[\frac{1}{s'-s(\nu,t)} - \frac{1}{s'-u(\nu,t)} - \frac{4\nu}{(s'-s(0,t))^2}\right] ds' + \frac{t^2}{\pi}\int_4^{\infty} F_{ann}^{(-)}(\nu, t')\frac{dt'}{(t'-t)\,t'^2},$$

$$B^{(-)}(\nu, t) = B_p^{(-)}(\nu, t) + \mathscr{B}^{(-)}(0, t) - \mathscr{B}_p^{(-)}(0, t) + \frac{1}{\pi}\int_{(m+1)^2}^{\infty} \operatorname{Im}\mathscr{B}^{(-)}(s', t)\left[\frac{1}{s'-s(\nu,t)} + \frac{1}{s'-u(\nu,t)} - \frac{2}{s'-s(0,t)}\right] ds' + \frac{t^2}{\pi}\int_4^{\infty} B_{ann}^{(-)}(\nu, t')\frac{dt'}{(t'-t)\,t'^2}, \qquad (9)$$

where $\mathscr{F}^{(-)}$ and $\mathscr{B}^{(\pm)}$, as in Eq. (6), denote two terms in the expansions of $F^{(-)}$ and $B^{(\pm)}$ in powers of t. We note that for the amplitudes $F^{(-)}$ and $B^{(-)}$ it would be possible to make no subtractions in the energy variable, but in this case the original dispersion relations would contain arbitrary terms $\nu a(t)$ and $b(t)$ of the same form as the subtraction terms.[11] Thus, the introduction of subtractions to improve the convergence of the integrals over the energy does not increase the number of unknown parameters.

The lowest harmonics of the functions $F_{ann}^{(-)}$ and $B_{ann}^{(\pm)}$ are expressed in terms of the P and D wave $\pi\pi$ scattering phase shifts $\delta_1(x)$ and $\delta_2(x)$ by

$$B_{ann}^{(+)}(\nu, x) = 3\nu e^{-i\delta_2(x)} \sin\delta_2(x)\,[B_1^{(+)}(x) - B_3^{(+)}(x)]\,/\,W\sqrt{-x},$$

$$F_{ann}^{(-)}(\nu, x) = 3\nu e^{-i\delta_1(x)} \sin\delta_1(x)\ F_1^{(-)}(x)/W\sqrt{-x},$$

$$B_{ann}^{(-)}(\nu, x) = e^{-i\delta_1(x)} \sin\delta_1(x)\ [B_0^{(-)}(x) - B_2^{(-)}(x)],$$

$$W = \sqrt{1-\varepsilon^2(1+x)}. \qquad (10)$$

The extrapolation of the functions (10) from the physical region of the $N + \overline{N} \to \pi + \pi$ reaction to the region $x \sim 1$ is made on the upper side of the cut $x \geq 0$ $(t \geq 4)$, where we take $W(-x)^{1/2} = -i\,|W(-x)^{1/2}|$.

Equations (5) and (9) are similar to those used in [7]; they differ only in the presence of additional subtractions. The subtraction terms in (5) and (9) can be expressed in terms of the S, P, and D pion-nucleon scattering lengths (see the Appendix); in $F^{(\pm)}$ they can be expressed in terms of only the S and P scattering lengths. The s' integration is over the physical region for πN scattering along the line $t = 0$, $s' \geq (m+1)^2$; the main contribution comes from energies of several hundred Mev, where the πN amplitudes are rather accurately known.

For numerical calculations it is essential to express the first term in the expansion of $\operatorname{Im} F^{(\pm)}(s', t)$ in powers of t in terms of the total cross sections $\sigma_t(\pi^{(\pm)}p)$ for $\pi^{(\pm)}p$ interactions:

$$\operatorname{Im} F^{(\pm)}(s, 0) = \operatorname{Im}\mathscr{F}^{(\pm)}(s, 0) = E_{c.m.}p_{c.m.}\,[\sigma_t(\pi^-p) \pm \sigma_t(\pi^+p)]/2m. \qquad (11)$$

Here $E_{c.m.}$ and $p_{c.m.}$ are the total energy and momentum in the center-of-mass system. This procedure gives significantly higher accuracy than calculating with the πN phase shifts.

The relations (5) and (9) are thus equations of the form

$$(F^{(\pm)}(\nu, t);\ B^{(\pm)}(\nu, t)) = (\widetilde{F}^{(\pm)}(\nu, t);\ \widetilde{B}^{(\pm)}(\nu, t)) + \frac{t^2}{\pi}\int_4^{\infty}\frac{(F_{ann}^{(\pm)}(\nu, t');\ B_{ann}^{(\pm)}(\nu, t'))\,dt'}{(t'-t)\,t'^2} \qquad (12)$$

where $\widetilde{F}^{(\pm)}$ and $\widetilde{B}^{(\pm)}$ are known functions without singularities on the line $t = 4$ and $F_{ann}^{(\pm)}$ and $B_{ann}^{(\pm)}$ are certain integrals of $F^{(\pm)}$ and $B^{(\pm)}$. By solving these equations we obtain the amplitudes for small values of the invariants with an accuracy of order

$$|t|^2/16m^2, \qquad |\nu|^2/8, \qquad (13)$$

due to the expansions used above.*

Similar equations were obtained by Ishida et al.[4] They differ from (5) and (9), however, in that they contain integrals of $\operatorname{Im} A^{(\pm)}$ and $\operatorname{Im} B^{(\pm)}$ in an unphysical region along the line $u = \text{const}$, $s \geq (m+1)^2$; it is therefore difficult to estimate with sufficient reliability the contribution from

*We emphasize that the parameters in (13) define the accuracy of the solution of the problem within the two-meson approximation; the study of the conditions under which the latter is applicable is the self-consistency question. From general considerations one can expect that higher-mass intermediate states in the unitarity condition will give additional contributions of order $(t/16)^2$ and $(t/4m)^2$.

that part of the region of integration where they used an incorrect expansion in Legendre polynomials.

3. SOLUTION OF THE EQUATIONS

When Eq. (12) is written in terms of the annihilation harmonics (8) with the substitutions (7) and (10), it has the form of a well known integral equation[15,16] whose solution can be written in terms of the auxiliary function (meson form factor)

$$\varphi_l(x) = \exp\left[\frac{x-\xi}{\pi}\int_0^\infty \frac{\delta_l(x')\,dx'}{(x'-x)(x'-\xi)}\right]. \tag{14}$$

A subtraction at some point ξ is used to improve the convergence for $x' \to \infty$. In order to calculate (14) we approximate the $\pi\pi$ phase shifts by

$$x^l\sqrt{x}\,\mathrm{ctg}\,\delta_l(x) = X(x), \tag{15*}$$

where $X(x)$ is a polynomial of arbitrary degree. The phase shift (15) corresponds to the $\pi\pi$ amplitude

$$\lambda_l(x) = ix^l\sqrt{-x}/(X(x) + x^l\sqrt{-x}),$$

$$\lambda_l(x + i0) = e^{i\delta_l(x)}\sin\delta_l(x), \tag{16}$$

defined on the whole plane with a cut for $x \geq 0$ and poles at the points $x = x_k$ ($k = 1, 2, \ldots n$) which are the roots of the equations $X(x) + x^l(-x)^{1/2} = 0$, $\mathrm{Re}(-x)^{1/2} \geq 0$. Generally speaking, these poles have no physical significance, but just represent the effects of the left-hand cut $x \leq -1$, which the "true" $\pi\pi$ amplitude with the correct analytic properties must have.

Substituting (15) into (14) and transforming the integral into a contour integral around the cut $x \geq 0$, we obtain

$$\varphi_l(x) = \frac{\Pi_{k=1}^n(x-x_k)}{X(x) + x^l\sqrt{-x}}. \tag{17}$$

We consider now the solution for $F^{(+)}$, which can be written in the form†

$$F_0^{(+)}(x) = \widetilde{F}_0^{(+)}(x) + \frac{(1+x)^2}{2\pi i}\varphi_S(x)$$
$$\times \int_0^\infty \frac{\widetilde{F}_0^{(+)}(x')\,[\varphi_S^{-1}(x'-i0) - \varphi_S^{-1}(x'+i0)]\,dx'}{(x'-x)(1+x')^2}, \tag{18}$$

*ctg = cot.

†The solution (18) is unique if we require that $F_0^{(+)}(x)/x^2\varphi_S(x)$ go to zero as $|x|$ becomes infinite; this is fulfilled if we require that the function $F_0^{(+)}(x)/\varphi_S(x)$, which has no cut for $x \geq 0$, satisfy a dispersion relation with two subtractions (cf.[13]).

$$\widetilde{F}_0^{(+)}(x) = \mathscr{F}^{(+)}(0, x)$$
$$+ \varepsilon g_r^2\left[-1 + \frac{\varepsilon(1+2x)}{4W\sqrt{-x}}\ln\frac{\varepsilon(1+2x) + 2W\sqrt{-x}}{\varepsilon(1+2x) - 2W\sqrt{-x}}\right]$$
$$+ \frac{2}{\pi}\int_1^\infty \frac{\mathrm{Im}\,\mathscr{F}^{(+)}(\omega, x)}{\omega + \varepsilon(1+x)}\left[\frac{\omega + \varepsilon(1+x)}{2W\sqrt{-x}}\right.$$
$$\left.\times \ln\frac{\omega + \varepsilon(1+x) + W\sqrt{-x}}{\omega + \varepsilon(1+x) - W\sqrt{-x}} - 1\right]d\omega. \tag{19}$$

For the higher harmonics $F_l^{(+)}(x) = \widetilde{F}_l^{(+)}(x)$. The subtraction term in (19) can be expressed in terms of the experimental πN amplitude with an accuracy of $\epsilon^2(1 + x)$ (see the Appendix); it is

$$\mathscr{F}^{(+)}(0, x) = \varepsilon g_r^2\widetilde{\alpha}(x),\quad \widetilde{\alpha}(x) = 0.95 + 0.2(1 + x). \tag{20}$$

The last term in (19) can be neglected, since it is a correction of order $\leq \epsilon^2 x$ to the subtraction term, as can be seen by expanding the integrand in powers of

$$\left|\frac{W\sqrt{-x}}{\omega + \varepsilon(1+x)}\right|^2 \sim \left|\frac{W\sqrt{-x}}{\omega_{\text{eff}} + \varepsilon(1+x)}\right|^2 \sim \varepsilon x \tag{21}$$

and calculating the lowest order term (the value $\omega_{\text{eff}} \approx 2.4$ corresponds to the 33 resonance).

The integration over x which remains in (18) can be put into a form analogous to the integration in (14) by noting that

$$\frac{-2x^l\sqrt{-x}}{\Pi_{k=1}^n(x-x_k)} \tag{22}$$

is the analytic continuation of the function $\varphi_l^{-1}(x - i0) - \varphi_l^{-1}(x + i0)$ to the whole plane with the cut $x \geq 0$ and coincides with it on the upper side of the cut. Summing all the harmonics in reverse, we obtain finally

$$F^{(+)}(\nu, x) = \widetilde{F}^{(+)}(\nu, x)$$
$$+ i\varepsilon g_r^2\lambda_S(x)\left\{\widetilde{\alpha}(x) - 1 + \frac{\varepsilon(1+2x)}{2\sqrt{-x}}\right.$$
$$\left.\times \ln\left(1 + 2x + \frac{2\sqrt{-x}}{\varepsilon}\right) + L_{n+1}^{(S)}(x)/\sqrt{-x}\right\}, \tag{23}$$

where the polynomial $L_{n+1}^{(S)}(x)$ of degree $n + 1$ is found from the conditions that the expression in the curly brackets vanish at the points $x = -1$ and $x = x_k$ ($k = 1, 2, \ldots, n$) along with its first derivative at the point $x = -1$.* The conditions at $x = -1$ are equivalent to the presence of two subtractions in (18). In the integration of the logarithmic term in (19), W has been replaced by unity

*For actually calculating the polynomial $L_{n+1}^{(S)}$, it is convenient to write it in the form $L_1(x) + (1 + x)^2 L_{n-1}(x)$, where the first degree polynomial $L_1(x)$ is first determined from the conditions at $x = -1$, i.e., it is independent of the x_k.

and the contribution from the cut $x \lesssim -m^2$ ($t \lesssim -4m^2$) which remains after the integral in (18) is transformed into a contour integral has been dropped; this gives an accuracy of $\epsilon^2(1+x)$.

We call attention to the compensation of the subtraction and pole (of order ϵg_r^2) terms in (19) and (23); these appear in the combination* $\tilde{\alpha}(x) - 1 \approx \epsilon(1+x)$. A similar compensation also occurs in the subsequent terms of the expansion (7) which depend on D-wave and higher $\pi\pi$ phase shifts. This makes the contribution of $\pi\pi$ interaction to $F^{(+)}$ smaller by an additional factor ϵ and increases the relative error due to the corrections that have been neglected in calculating this contribution. This compensation shows that in no case may one neglect the contribution of $B^{(+)}$ to the absorptive part in the equation for $A^{(+)}$, as was done by Efremov, Meshcheryakov, and Shirkov[6] [see Eq. (5.3) in their first paper]. Notwithstanding this fundamental distortion of the equation, they obtained the same S-wave $\pi\pi$ scattering length as was obtained in papers by Sato et al[3] and Ishida et al,[4] where this contribution was included. This is apparently a consequence of the fact that in both cases the main contribution comes from the region of impermissibly large values $x \gtrsim m$, where this compensation does not occur.

The solutions for the other invariant amplitudes can be obtained in a similar way. Thus, for $B^{(+)}$ we expand the additional absorptive part in powers of $\epsilon(1+2x)/2x^{1/2}$ and keep just the lowest order term to obtain

$$B^{(+)}(\nu, x) = \tilde{B}^{(+)}(\nu, x) + 5i\nu\varepsilon g_r^2\lambda_D(x)\{x^{-1} + L_{n+1}^D(x)/x^2\sqrt{-x}\}, \tag{24}$$

where the polynomial $L_{n+1}^D(x)$ is determined in a manner analogous to that in which $L_{n+1}^{(S)}(x)$ is determined. For simplicity we denote the number of poles in the D-wave $\pi\pi$ scattering amplitude by the same letter n that we use for the S wave amplitude. Since the absorptive part vanishes like $x^{5/2}$ near $x = 0$, the expansion in powers of $\epsilon(1+2x)/2x^{1/2}$ does not, in practice, introduce errors near $x = 0$; it provides an accuracy of $\epsilon x^{1/2}$ for the integration region $x \gtrsim 1$. Corrections of order ϵ to the contribution of the D-wave scattering to $B^{(+)}$ need not be taken into account, since they are of the same order of magnitude as the D-wave contributions to $F^{(+)}$, which we neglected.

The solutions for $F^{(-)}$ and $B^{(-)}$, which depend on the P-wave $\pi\pi$ scattering, are expressed in terms of the annihilation harmonics

$$\frac{1}{W\sqrt{-x}}\tilde{F}_1^{(-)'}(x) = \frac{1}{3\nu_0}[\mathcal{F}^{(-)}(\nu_0, x) - \mathcal{F}_P^{(-)}(\nu_0, x)]\Big|_{\nu_0=0} + \frac{\varepsilon^2 g_r^2(1+2x)}{2W^2x}\left[1 - \frac{\varepsilon(1+2x)}{4W\sqrt{-x}}\ln\frac{\varepsilon(1+2x)+2W\sqrt{-x}}{\varepsilon(1+2x)-2W\sqrt{-x}}\right], \tag{25}$$

$$\tilde{B}_0^{(-)'}(x) - \tilde{B}_2^{(-)'}(x) = \mathcal{B}^{(-)}(0, x) - \mathcal{B}_P^{(-)}(0, x) + \frac{3\varepsilon g_r^2}{4W^2}\left\{\frac{1+\varepsilon^2/4x}{W\sqrt{-x}}\ln\frac{\varepsilon(1+2x)+2W\sqrt{-x}}{\varepsilon(1+2x)-2W\sqrt{-x}} - \frac{\varepsilon(1+2x)}{x}\right\}, \tag{26}$$

where we have dropped the terms similar to the last term in (19), since they are corrections of order $\epsilon^2 x$ to (25) and (26). We obtain finally

$$F^{(-)}(\nu, x) = \tilde{F}^{(-)}(\nu, x) + 3i\nu\lambda_P(x)\{\tilde{f}(x) + L_{n+1}^{(P)}(x)/x\sqrt{-x}\}, \tag{27}$$

$$B^{(-)}(\nu, x) = \tilde{B}^{(-)}(\nu, x) + i\lambda_P(x)\{\tilde{b}(x) + M_{n+1}^{(P)}(x)/x\sqrt{-x}\}, \tag{28}$$

where $\tilde{f}(x)$ and $\tilde{b}(x)$ denote the functions (25) and (26) with the additional replacements

$$W \to 1,$$

$$\ln\frac{\varepsilon(1+2x)+2W\sqrt{-x}}{\varepsilon(1+2x)-2W\sqrt{-x}} \to 2\ln\left(1+2x+\frac{2\sqrt{-x}}{\varepsilon}\right) \tag{29}$$

The polynomials in (27) and (28) are determined analogously to those in (23) and (24).

We call attention to the fact that (25), like (19), is decreased by a factor of order ϵ. However, this is due to the smallness of both the subtraction and pole terms and not to their compensation; therefore this factor should apparently also occur in the corrections we have neglected in obtaining the original equations [see the derivation of Eqs. (5) and (9)]. In this case, the main error in (25) is due to the error in the subtraction term which amounts to a correction of order $\epsilon(1+x)/3$ to the pole term and must be set equal to zero for the accuracy stated above (see the Appendix).

Comparison of the amplitudes in Eqs. (23), (24), (27), and (28) with Eq. (12) shows that the solution of the integral equations actually amounts to a calculation of the contribution to the πN amplitudes from the annihilation cut $t \geq 4$ due to $\pi\pi$ interaction.

We now discuss the convergence of the integrals over the annihilation cut which we obtained in the solution; this convergence, along with the errors in the integrands, determines the accuracy of the calculations of the $\pi\pi$-interaction terms. The solution of Eq. (12) in the two-meson approxima-

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*This result was obtained previously for the amplitudes at the point $\nu = 0$, $t = 4\mu^2$ where they determine the peripheral interaction.[17,18]

tion, i.e., with the absorptive parts (7) and (10), is logically justifiable only if the result is independent of the behavior of the amplitudes in the region where they are not known, which is the region $x \gtrsim m$ ($t \gtrsim 4m$). Clearly this requires sufficiently rapid convergence of integrals of the type (18) in the region $x' < m$, and thus a corresponding behavior of the harmonics like (19) and the auxiliary functions like (22). In this latter region we have effectively

$$\widetilde{F}_0^{(+)} \sim x, \qquad [\widetilde{B}_1^{(+)} - \widetilde{B}_3^{(+)}]/W\sqrt{-x} \sim 1/x,$$
$$\widetilde{F}_1^{(-)}/W\sqrt{-x} \sim \text{const}, \qquad \widetilde{B}_0^{(-)} - \widetilde{B}_2^{(-)} \sim 1/\sqrt{x}, \tag{30}$$

and therefore convergence of the type $dx'/x'^{5/2}$, for example, which would give a contribution of order $\epsilon^{3/2}$ from the region $x' \gtrsim m$, requires the presence of at least one pole in the $\pi\pi$ amplitude (16). Only poles which lie sufficiently near improve the function (22) in the region $x' < m$. This shows the importance for the "true" $\pi\pi$ amplitude of the left-hand cut $x \leq -1$, which is represented in the model (15), (16) by unphysical poles.

It is also not hard to see that the previously-considered simpler scattering length[3,4,6] and sharp $\pi\pi$ resonance[7 9] models lead, in a scheme with one subtraction in t, to integrals of the form $dx'/x'^{1/2}$ for $x' < m$; a large contribution must therefore come from the impermissible region of integration $m \lesssim x' \lesssim m^2$, where the quantities in (30) are replaced by factors that converge somewhat more rapidly. For these models, the introduction of a second subtraction, like that used in the present work, leads to convergence of the form $dx'/x'^{3/2}$ in the region $x' < m$ and makes possible the calculation of the $\pi\pi$ terms with an accuracy of $\epsilon^{1/2}$. Then corrections of order $\epsilon(1 + 2x)/2x^{1/2}$ must be neglected in (25) along with corrections of order $\epsilon^2(1 + 2x)^2/4x$ in (26); this is equivalent to substituting in Eqs. (27) and (28)

$$\widetilde{f}(x) = \varepsilon^2 g_r^2 (1 + 2x)/2x,$$
$$\widetilde{b}(x) = \mathcal{B}^{(-)}(0, 0) - \mathcal{B}_p^{(-)}(0, 0) + \frac{3\varepsilon g_r^2}{4}\left\{\frac{2}{\sqrt{-x}}\ln\left(1 + 2x + \frac{2\sqrt{-x}}{\varepsilon}\right) - \frac{\varepsilon(1 + 2x)}{x}\right\}. \tag{31}$$

4. CONCLUSIONS

1. The method used in the present work allows us to obtain the invariant πN amplitudes in the two-meson approximation with an accuracy of about $(t/4m)^2$; consequently, in the calculation of the contributions from the $\pi\pi$ interaction we have neglected corrections of order $\epsilon^2(1 + x)^2$ and ϵ^2. However, the amplitude $F^{(+)}$ [Eq. (23)] is decreased by an extra factor ϵ due to compensation of the $\pi\pi$ terms, and its relative accuracy is therefore determined by the parameters ϵ and ϵx. The $\pi\pi$ term in the amplitude $B^{(+)}$ [Eq. (24)] is small because of the small D-wave $\pi\pi$-scattering amplitude and therefore corrections of order ϵ and ϵx have been dropped in calculating it. The relative accuracy of the $\pi\pi$ terms in (27) and (28) is determined by the parameters ϵ^2 and $\epsilon^2 x^2$.

2. The accuracy described in item 1 is attained only for "good" meson form factors which provide sufficiently rapid convergence of the integrals along the annihilation cut $t \geq 4$ and thus lead to relatively insignificant contributions ($\lesssim \epsilon$ for $F^{(+)}$ and $B^{(+)}$ and $\lesssim \epsilon^2$ for $F^{(-)}$ and $B^{(-)}$) from the region of integration $t \gtrsim 4m$. The behavior of the form factors is improved by singularities of the $\pi\pi$ amplitude in the unphysical region within a radius $|x| < m$. In the worst case the $\pi\pi$ terms have an approximate accuracy of $\epsilon^{1/2}$ or $(\epsilon x)^{1/2}$.

3. Peripheral πN interaction is due to the contribution from the S-wave $\pi\pi$-scattering amplitude. The reduction of this contribution to $F^{(+)}$ because of the compensation described above must make it more difficult to fulfill the conditions under which the asymptotic formulas[17] can be applied for the partial amplitudes with large angular momentum l. The disagreement[17] of the theoretical and experimental $l = 2$ phases is apparently a consequence of the fact that the role of terms neglected in the calculation, namely $B^{(-)}$, which is due to P-wave $\pi\pi$ scattering, and further peripheral terms such as four-meson terms and pole-terms, is sharply increased by the compensation.

4. In previous calculations[3-9] the integrals do not converge rapidly enough, and the $\pi\pi$ terms must depend in an essential way on the amplitudes in the region $|t| \gtrsim 4m$ in which the behavior of the amplitudes is not known at present. Therefore, conclusions concerning the $\pi\pi$ interaction drawn on the basis of these calculations must be considered unreliable.*

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*Ball and Wong[19] have also shown that the results of Frazer and Fulco[5] are unreliable and that subtractions must be introduced.

APPENDIX

CALCULATION OF SUBTRACTION TERMS

The subtractions at $\nu = 0$ which enter the original dispersion relations can be expressed in terms of subtractions at $s = s_0 = (m+1)^2$ by

$$F^{(+)}(0, t) - F_p^{(+)}(0, t) = F^{(+)}(s_0, t) - F_p^{(+)}(s_0, t) - \frac{2}{\pi}\left(1+\frac{t}{4m}\right)^2 \int_1^\infty \frac{\operatorname{Im} F^{(+)}(\omega, t)\, d\omega}{(\omega-1)(\omega+t/4m)(\omega+1+t/2m)}, \tag{A.1}$$

$$\frac{1}{\nu_0}[F^{(-)}(\nu_0, t) - F_p^{(-)}(\nu_0, t)]|_{\nu_0=0} = \frac{F^{(-)}(s_0, t) - F_p^{(-)}(s_0, t)}{1+t/4m} - \frac{2}{\pi}\left(1+\frac{t}{4m}\right)^2 \int_1^\infty \frac{\operatorname{Im} F^{(-)}(\omega, t)\, d\omega}{(\omega-1)(\omega+t/4m)^2(\omega+1+t/2m)}. \tag{A.2}$$

The analogous relations for $B^{(-)}$ are obtained by replacing $F^{(+)}$ in (A.1) by $B^{(-)}$. By keeping just two terms in the expansion of (A.1) and (A.2) in powers of t, we obtain the relations we need for the functions $\mathcal{F}^{(\pm)}$ and $\mathcal{B}^{(-)}$. Thus, for example, for $\mathcal{F}^{(+)}$ we have

$$\mathcal{F}^{(+)}(0, t) = \mathcal{F}^{(+)}(s_0, t) - \mathcal{F}_p^{(+)}(s_0, t) - \frac{2}{\pi}\int_1^\infty \frac{\operatorname{Im}\mathcal{F}^{(+)}(\omega, t)\, d\omega}{\omega(\omega^2-1)} - \frac{t}{2\pi m}\int_1^\infty \frac{\operatorname{Im}\mathcal{F}^{(+)}(\omega, 0)(2\omega+1)\, d\omega}{\omega^2(\omega+1)^2}. \tag{A.3}$$

The main contribution of the integral terms in (A.3) can be expressed in terms of the πN cross section[20] by using (11); this gives a value $-0.10 - 0.02(1+x)$. In calculating the remaining part of the integral terms we consider just the resonant 33 phase shift[21]; this gives $-0.08(1+x)$. The first term in (A.3) is expressed in terms of the S and P scattering lengths by

$$\mathcal{F}^{(+)}(s_0, t)/4\pi = \tfrac{1}{3}(1+\varepsilon)(2a_3 + a_1) + \tfrac{4}{3}(1+x)(1+\varepsilon)\left[2a_{33} + a_{31} + \tfrac{1}{2}(2a_{13} + a_{11})\right]. \tag{A.4}$$

The lengths a_3, a_1, a_{33}, and a_{31} are quite accurately known,[21 22] but reliable values of a_{13} and a_{11} are not available and we neglect them. This gives an error which we estimated by using the 200- to 300-Mev data[22] and assuming the P phases to vary with energy as $p^3_{\text{c.m.}}$; this gave a contribution to (A.4) of order $\epsilon^2(1+x)$. Substituting also the pole term, which gives the main contribution to (A.3), we obtain finally

$$\mathcal{F}^{(+)}(0, x) = \varepsilon g_r^2 \tilde{\alpha}(x), \quad \alpha(x) = 0.95 + 0.2(1+x) \tag{A.5}$$

with a relative error of order $\epsilon^2(1+x)$.

In a completely analogous way we obtain

$$\frac{1}{\nu_0}[\mathcal{F}^{(-)}(\nu_0, x) - \mathcal{F}_p^{(-)}(\nu_0, x)]|_{\nu_0=0} = \varepsilon g_r^2[a + b(1+x)], \quad |a|, |b| \sim \varepsilon^2. \tag{A.6}$$

In the subtraction term $\mathcal{B}^{(-)}(0, x) - \mathcal{B}_p^{(-)}(0, x)$ the main contribution comes from the term

$$\mathcal{B}^{(-)}(s_0, x)/4\pi = \tfrac{1}{6}\varepsilon(a_1 - a_3) + \tfrac{2}{3}m(a_{33} - a_{31} + a_{11} - a_{13}) - (1+x)[\varepsilon(a_{33} - a_{13}) - 4m(d_{35} - d_{33} + d_{13} - d_{15})]. \tag{A.7}$$

We neglect the dependence of (A.7) on the D lengths d. To estimate the error thus introduced we use the analysis of the data at energies above 300 Mev[23] and assume that the D phases vary with energy as $p^5_{\text{c.m.}}$. This gives a D-length contribution of order $\epsilon^2(1+x)$ to (A.7). Therefore the main error comes from the contribution of $a_{11} - a_{13}$ which we do not take into account and which apparently amounts to not more than 20%. We obtain finally

$$[\mathcal{B}^{(-)}(0, x) - \mathcal{B}_p^{(-)}(0, x)]/4\pi = 0.9 + 0.3(1+x) \tag{A.8}$$

with an error of $A + b(1+x)$, $|A| \lesssim 0.2$, $|b| \sim \epsilon^2$.

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