DETERMINATION OF THE $K^+ \rightarrow 2\pi + \gamma$ *DECAY AMPLITUDE FROM DISPERSION RELATIONS AND UNITARITY*

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The effect of $\pi\pi$ interaction on the amplitude for the radiative θ^+ decay is taken into account. An expression for the amplitude is obtained in terms of P-wave $\pi\pi$-scattering phase shifts. The Omnes-Muskhelishvili method for solving the integral equation is generalized to the case when the inhomogeneous part of the equation is a singular function.

THE radiative θ^+ decay $K^+ \rightarrow \pi^+ + \pi^0 + \gamma$ has been discussed by Good,[1] without taking into consideration the final state interaction between the pions. In the present paper the interaction between the π mesons is taken into account and the πK interaction is neglected.

1. GENERAL FORM OF THE AMPLITUDE

The amplitude for the indicated process consists of two parts, corresponding to electric and magnetic radiations:

$$M = (k_\mu w_\nu - k_\nu w_\mu)\, p_\mu q_\nu \varphi + i\varepsilon_{\mu\nu\alpha\beta}\,(k_\alpha w_\beta - k_\beta w_\alpha)\, p_\mu q_\nu \chi. \quad (1)$$

Here k and w are respectively the 4-momentum and polarization of the photon; p and q are the 4-momenta of the π^+ and π^0 mesons; and φ and χ are functions of the invariants in the problem.

Internal bremsstrahlung of the K^+ and π^+ mesons contributes only to the "electric" part of the amplitude, i.e., to the function φ. The internal bremsstrahlung diagram is shown in Fig. 1. The corresponding amplitude is given by

$$M_T = \left(\frac{(Qw)}{(Qk)} - \frac{(pw)}{(pk)}\right) emG, \quad (2)$$

where G is the $K^+_{2\pi}$ decay coupling constant, m is the mass and Q is the 4-momentum of the K^+ meson. M_T may be expressed in the form

$$M_T = q_\mu p_\nu (k_\mu w_\nu - w_\mu k_\nu)\, \varphi_0; \quad (3)$$

$$\varphi_0 = emG/(pk)(Qk) = 4meG/(s - m^2)(t - \mu^2), \quad (4)$$

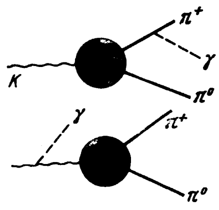

FIG. 1

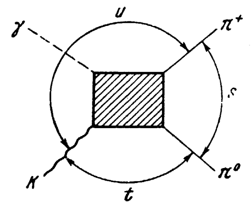

FIG. 2

where μ is the mass of the π^+ meson, and s and t are the invariants corresponding to the $\pi^+\pi^0$ and π^0K^+ channels (see Fig. 2):

$$s = (p+q)^2, \qquad t = (k+p)^2. \quad (5)$$

In addition to internal bremsstrahlung one has radiation from the block of heavy particles ("vertex" radiation). This radiation has an electric and a magnetic part. The part of the amplitude corresponding to this radiation depends only weakly on the invariants and may therefore be written in the following form

$$M_B = Ap_\mu q_\nu (k_\mu w_\nu - k_\nu w_\mu) + Dp_\mu q_\nu i\varepsilon_{\mu\nu\alpha\beta}(k_\alpha w_\beta - k_\beta w_\alpha). \quad (6)$$

Here A and D are constants connected with the electric and magnetic parts.

The electric part is of greatest interest because of the importance of the interference between the "vertex" radiation and the pole term. On the other hand the interference between the magnetic and electric radiations vanishes after summation over polarizations.[1] Thus, what is taken into account below are chains of the kind illustrated in Fig. 3, as well as chains involving internal bremsstrahlung of the kind illustrated in Fig. 4.* We leave out contributions arising from radiation from the

*Chains with an odd number of pions give contributions that are even less than those that are referred to below as small, because virtual decays of K^+ mesons into an even number of pions are forbidden by isotopic spin selection rules.

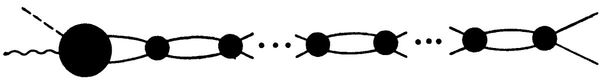

FIG. 3

$\pi\pi$-interaction vertex, i.e., radiation from the entire block at the strong interaction vertex (see Fig. 5). The contribution from such diagrams is however less than the contribution from the diagrams shown in Fig. 3, because in the diagrams of Fig. 5 the $K^+_{2\pi}$ decay has to take place first and it is forbidden by the $\Delta T = 1/2$ rule. As noted by Good,[1] this forbiddenness is lifted for that part of the amplitude which is connected with radiation from the heavy particles block in the "decay" vertex. As a consequence the ratio of the contributions from diagrams of Fig. 5 and Fig. 3 should be proportional to the ratio of the $K^0_{2\pi}$ and $K^+_{2\pi}$ decay constants, i.e., $\sim 1/20$. We ignore any effects due to the $K\pi$ interaction; some estimates are made in this connection at the end of the paper.

The calculation of the magnetic part of the amplitude is made difficult by the existence of diagrams shown in Fig. 6. Here the $K_{3\pi}$ decay takes place at the weak vertex, which is isotopically allowed; these diagrams are difficult to take into account.

2. SPECTRAL DECOMPOSITION AND UNITARITY CONDITION FOR THE ELECTRIC PART OF THE AMPLITUDE

The spectral decomposition for the electric part of the amplitude may be written in the form

$$\varphi(s, t) = \varphi_0(s, t) + A + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \rho_1(s')\left(\frac{1}{s'-s-i\varepsilon} - \frac{1}{s'}\right)ds' + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \rho_2(t')\left(\frac{1}{t'-t-i\varepsilon} - \frac{1}{t'}\right)dt' + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \rho_3(u')\left(\frac{1}{u'-u-i\varepsilon} - \frac{1}{u'}\right)du'. \tag{7}$$

These dispersion relations have been written with a subtraction. The subtraction constant A is connected with radiation from the block of heavy particles.

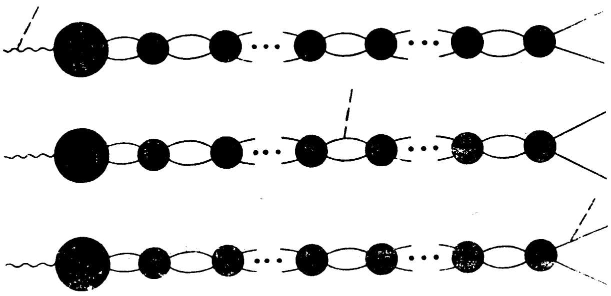

FIG. 4

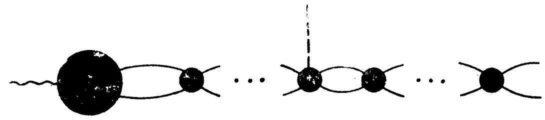

FIG. 5

The branch points in the invariants t and u lie significantly further than $4\mu^2$. This makes it possible to ignore the last two integrals in Eq. (7) (μ/m approximation). An estimate of these terms will be given below. (We may note here that the magnetic part of the amplitude cannot be calculated precisely because the branch point in the t-channel lies at also $4\mu^2$.) Thus we obtain, in first approximation, the following expression for the electric part of the amplitude

$$\varphi(s, t) = \varphi_0(s, t) + A + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \rho_1(s')\left[\frac{1}{s'-s-i\varepsilon} - \frac{1}{s'}\right]ds'. \tag{8}$$

In writing Eq. (7) we have assumed that the channel with four pions, which gives rise to a branch point at $s = 16\mu^2$, can be neglected. If this channel cannot be neglected than instead of Eq. (8) we get the more general expression

$$\varphi(s, t) = \varphi_0(s, t) + A + \frac{1}{\pi}\iint \frac{\rho(s, t)\, st\, dt'\, ds'}{s't'(s'-s)(t'-t)}. \tag{9}$$

By averaging Eqs. (4) and (7) over t for a fixed s we get

$$\varphi_0(s) = \frac{-4mGe\sqrt{s}}{(s-m^2)^2\sqrt{s-4\mu^2}} \ln\left(\frac{\sqrt{s}+\sqrt{s-4\mu^2}}{2\mu}\right)^2;$$

$$\varphi_1(s) = A + \varphi_0(s) + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \rho_1(s')\left[\frac{1}{s'-s-i\varepsilon} - \frac{1}{s'}\right]ds'. \tag{10}$$

The unitarity conditions connect A and φ_0 with each other, but do not couple the electric and magnetic parts of the amplitude. The unitarity conditions give

$$\rho_1(s)\, p_\mu q_\nu (k_\mu w_\nu - k_\nu w_\mu) = (k_\mu w_\nu - k_\nu w_\mu)\sqrt{1-4\mu^2/s} \times \frac{1}{4\pi}\int \varphi(s)\, p'_\mu q'_\nu \lambda_{\pi\pi}\, d\Omega = (k_\mu w_\nu - k_\nu w_\mu)\, p_\mu q_\nu e^{-i\delta_1}\sin\delta_1 \varphi(s), \tag{11}$$

where δ_1 is the P-wave $\pi\pi$ scattering phase shift. Here we took into account the $\pi\pi$ scattering channel (see Fig. 3). If also the second and third channels ($K\pi$ scattering) are taken into account then the unitarity relations for φ become

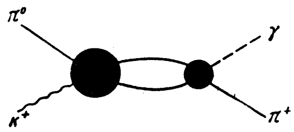

FIG. 6

$$\rho_1 = e^{-i\delta_1}\sin\delta_1\varphi(s); \tag{12}$$

$$\rho_2 = e^{-i\delta_1'}\sin\delta_1'\varphi(t), \qquad \rho_3 = e^{-i\delta_1'}\sin\delta_1'\varphi(u). \tag{13}$$

Here δ_1' is the P-wave $K\pi$ scattering phase shift.

In first approximation we obtain according to Eq. (10) the following integral equation for the function $\varphi_1(s)$

$$\varphi_1(s) = \varphi_0(s) + A + \frac{1}{\pi}\int_{4\mu^2}^{\infty}\left[\frac{1}{s'-s-i\varepsilon} - \frac{1}{s'}\right]e^{-i\delta_1}\sin\delta_1\varphi_1 ds'. \tag{14}$$

3. SOLUTION OF INTEGRAL EQUATION

In order to solve Eq. (14) it is sufficient to solve an equation of the following type

$$\varphi(s) = \frac{q(s)}{[s-\nu-i\varepsilon]^n} + \frac{1}{\pi}\int_a^\infty \frac{h^*(s')\varphi(s')}{s'-s-i\varepsilon}ds', \tag{15}$$

$$h^* = e^{-i\delta_1}\sin\delta_1.$$

The function $q(s)$ appearing in Eq. (15) is regular in the region $a \leq s < \infty$. For $n = 0$ this type of equation has been solved by Omnes[2] using Muskhelishvili's method. In generalizing the Omnes-Muskhelishvili method to the solution of Eq. (15) with $n \neq 0$ we found it necessary to consider integrals of the type

$$\int_a^\infty F(s')[(s'-\nu-i\varepsilon)^n(s'-s-i\varepsilon)]^{-1}ds', \tag{16}$$

where $F(s')$ is a regular function.

Let us evaluate first the integral

$$J = \int_L F(s')(s'-\nu)^{-n}ds' \tag{17}$$

where the contour L is shown in Fig. 7 and the function $F(s)$ is assumed to have the property that $\operatorname{Im} F(s) = 0$ if $\operatorname{Im} s = 0$. The imaginary part of this integral is given by

$$\operatorname{Im} J = \pi \operatorname{res} J = \pi F^{(n-1)}(\nu)[(n-1)!]^{-1}. \tag{18}$$

If $n = 1$ then

$$\operatorname{Re} J = \mathrm{P}\int_a^\infty F(s')(s'-\nu)^{-1}ds'.$$

However already for $n = 2$ the principal value integral

$$\mathrm{P}\int_a^\infty F(s')(s'-\nu)^{-n}ds'$$

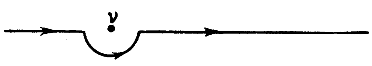

FIG. 7

diverges (equals $+\infty$). The real part of the integral J may be calculated in this case directly by integration along the contour; on the other hand it is equal to the generalized principal value integral, which we denote by PP, defined by

$$\mathrm{PP}\int_a^\infty F(s')(s'-\nu)^{-n}ds' = \lim_{\varepsilon\to 0}\mathrm{P}\int_a^\infty F(s')\left[\prod_{k=1}^{n}(s'-\nu-k\varepsilon)\right]^{-1}ds' = \operatorname{Re} J. \tag{19}$$

Carrying out the integration around the pole we find

$$\mathrm{PP}\int_a^{2\nu-a}\frac{ds'}{(s'-\nu)^n} = \frac{(-1)^{n-1}-1}{(n-1)(\nu-a)^{n-1}}; \qquad n>1, \quad a<\nu. \tag{20}$$

Expanding the function $F(s)$ in a power series and making use of Eq. (20) we get

$$\begin{aligned} J &= \int_L \frac{F(s')ds'}{(s'-\nu)^n} = \int_a^\infty \frac{F(s')ds'}{(s'-\nu-i\varepsilon)^n} \\ &= i\pi\frac{F^{(n-1)}(\nu)}{(n-1)!} + \sum_{k=0}^{n-2}F^{(k)}(\nu)\frac{(-1)^{n-k-1}-1}{(n-k-1)\,k!(\nu-a)^{n-k-1}} \\ &+ \mathrm{P}\int_a^{2\nu-a}\left[F(s') - \sum_{k=0}^{n-2}\frac{(s-\nu)^k}{k!}F^{(k)}(\nu)\right]\frac{ds'}{(s'-\nu)^n} \\ &+ \int_{2\nu-a}^\infty F(s')\frac{ds'}{(s'-\nu)^n}. \end{aligned} \tag{21}$$

For integrals of the type

$$J_{1,2} = \int_{L_{1,2}}\frac{F(s')ds'}{(s'-\nu)^n(s'-s)} = \int_a^\infty \frac{F(s')}{(s'-\nu-i\varepsilon)^n(s'-s\mp i\varepsilon)}ds' \tag{22}$$

along contours shown in Fig. 8, we find after some elementary operations the following expressions

$$\begin{aligned} J_2 &= \int_a^\infty \frac{F(s')}{(s'-\nu-i\varepsilon)^n(s'-s-i\varepsilon)}ds' = \mathrm{PP}\int_a^\infty \frac{F(s')}{(s'-\nu)^n(s'-s)}ds' \\ &- i\pi\left[\frac{F(s)}{(s-\nu)^n} + \sum_{l=0}^{n-1}F^{(n-l-1)}(\nu)[(s-\nu)^{l+1}(n-l-1)!]^{-1}\right], \end{aligned} \tag{23}$$

$$\begin{aligned} J_1 &= \int_a^\infty \frac{F(s')}{(s'-\nu-i\varepsilon)^n(s'-s-i\varepsilon)}ds' = \mathrm{PP}\int_a^\infty \frac{F(s')ds'}{(s'-\nu)^n(s'-s)} \\ &+ i\pi\left[\frac{F(s)}{(s-\nu)^n} - \sum_{l=0}^{n-1}F^{(n-l-1)}(\nu)[(s-\nu)^{l+1}(n-l-1)!]^{-1}\right]. \end{aligned} \tag{24}$$

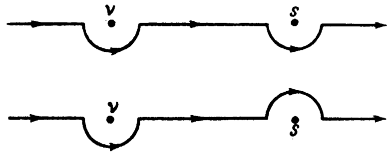

FIG. 8

Here the generalized principal-value integral, defined by the limiting procedure of Eq. (19), may also be expressed in terms of principal value integrals

$$\mathrm{PP}\int_a^\infty \frac{F(s')\,ds'}{(s'-\nu)^n(s'-s)}$$
$$=\sum_{k=0}^{n-2}\frac{F^{(k)}(\nu)}{k!}\Big\{\sum_{l=0}^{n-k-2}\frac{1-(-1)^{n-k-l-1}}{(s-\nu)^{l+1}(\nu-a)^{n-k-l-1}}$$
$$+\sum_{l=n-k}^{n-1}\frac{(\nu-a)^{k+l-n+1}[(-1)^{k+l-n+1}-1]}{(k+l-n+1)}$$
$$+\frac{1}{(s-\nu)^{n-k}}\ln\frac{\nu-a-(s-\nu)}{\nu-a+(s-\nu)}$$
$$+\sum_{l=0}^{k-1}(s-\nu)^{k-1-l-n}(\nu-a)^{l+1}[1-(-1)^{l+1}](l+1)^{-1}\Big\}$$
$$+\mathrm{P}\int_a^{2\nu-a}\Big[F(s')-\sum_{k=0}^{n-2}\frac{F^{(k)}(\nu)(s-\nu)^k}{k!}\Big]\frac{ds'}{(s'-\nu)^n(s'-s)}$$
$$+\mathrm{P}\int_{2\nu-a}^\infty\frac{F(s')\,ds'}{(s'-\nu)^n(s'-s)}. \tag{25}$$

In order to solve Eq. (15) we introduce, following Omnes,[2] the new function

$$F(z)=\frac{1}{2\pi i}\int_L\frac{h^*(s')\,ds'}{s'-z} \tag{26}$$

and denote correspondingly

$$F(s_+)=\frac{1}{2\pi i}\int_L\frac{h^*(s')\,\varphi(s')}{s'-s-i\varepsilon}\,ds',$$
$$F(s_-)=\frac{1}{2\pi i}\int_L\frac{h^*(s')\,\varphi(s')\,ds'}{s'-s+i\varepsilon}. \tag{27}$$

Then the desired function φ can be expressed in terms of $F(z)$ as follows

$$\varphi(s)=[F(s_+)-F(s_-)]/h^*. \tag{28}$$

Equation (15) becomes:

$$e^{-2i\delta}F(s_+)-F(s_-)=h^*(s)f_0(s),\quad f_0(s)=q(s)(s-\nu-i\varepsilon)^{-n}. \tag{29}$$

The solution of Eq. (29) obtained by Omnes[2] is valid also in the case when the function $f_0(s)$ is singular:

$$F(z)=\Phi(z)\,\Omega(z),\qquad \Omega(z)=e^{u(z)},$$
$$u(z)=\frac{1}{\pi}\int_L\delta_1(z')(z'-z)^{-1}dz', \tag{30}$$

and if the last integral diverges then we take

$$u(z)=\frac{1}{\pi}\int_L z\delta_1(z')[z'(z'-z)]^{-1}dz'. \tag{31}$$

According to Omnes

$$\Omega(s_+)=e^{\rho+i\delta_1},\qquad \Omega(s_-)=e^{\rho-i\delta_1}; \tag{32}$$

$$\rho(s)=\frac{1}{\pi}\,\mathrm{P}\int\delta_1(s')(s'-s)^{-1}ds', \tag{33}$$

or, if the integral diverges,

$$\rho(s)=\frac{1}{\pi}\,\mathrm{P}\int_a^\infty s\delta_1(s')[s'(s'-s)]^{-1}ds'. \tag{34}$$

Again, as was the case for Omnes, the equation

$$\Phi(s_+)-\Phi(s_-)=f_0(s)\sin\delta_1(t)e^{-\rho(s)} \tag{35}$$

is valid, and the general form of the solution is preserved:

$$\Phi(z)=\frac{1}{2\pi i}\int_L f_0(s')\sin\delta_1(s')\,e^{-\rho(s')}\frac{ds'}{s'-z}. \tag{36}$$

The difference between our solution and that of Omnes becomes manifest only when we actually use the singular function $f_0(s)$ in the contour integral (36) to evaluate $\Phi(z)$. Since the integrand has a pole of n-th order at the point $s'=\nu$, the integration must be performed according to Eqs. (23), (24) and (25). After substitution of these results into Eq. (28) we obtain the final formula:*

$$\varphi(s)=f_0(s)+\exp[\rho(s)$$
$$+i\delta_1(s)]\Big\{-i\Big[\sum_{l=0}^{n-1}[\sin\delta_1(\nu)e^{-\rho(\nu)}q(\nu)]^{(n-l-1)}$$
$$\times[(s-\nu)^{l+1}(n-l-1)!]^{-1}$$
$$-q(s)\sin\delta_1(s)e^{-\rho(s)}(s-\nu)^{-n}\Big]+$$
$$+\frac{1}{\pi}\,\mathrm{PP}\int_a^\infty q(s')\sin\delta_1(s')e^{-\rho(s')}\frac{ds'}{(s'-s)(s'-\nu)^n}\Big\}. \tag{38}$$

Here, as above, PP denotes taking the integral in the generalized principal value sense, as defined by Eqs. (19) and (25). Let us point out an important property of the generalized principal value: if the integrand is of the form $F(s')/(s'-\nu)^n(s'-s)$ [with $F(s')$ a regular function], i.e. if the integrand has a pole of n-th order, then the generalized principal value has no poles as a function of s. It follows from here that the solution of the integral equation (15) has a pole of the same order as the inhomogeneous function, and, as can

*After the present work was completed and sent to press the author became familiar with the work of Galanin and Grashin,[5] in which a method is given for the evaluation of integrals $\Phi(z)$, Eq. (36), in the case when the phase shift δ_l is of the form

$$y^l\sqrt{y}\cot\delta_l=\Sigma\alpha_k y^k,\qquad y=s/4\mu^2-1. \tag{37}$$

In connection with the appearance of the paper by Galanin and Grashin,[5] Sec. 4 of the present paper has been revised.

*ctg = cot.

be seen from Eq. (37), in the immediate neighborhood of the pole the coefficients of the pole term in the solution and in the inhomogeneous function are the same. It further follows directly from Eqs. (24) and (25) that already the first iteration of Eq. (15) removes the singularity. This proves the correctness of our choice of the sign of $i\epsilon$ in the inhomogeneous term of Eq. (15).

Equation (15) may be easily generalized to the case when $\varphi(s)$ falls off slowly at infinity, or tends to a constant limit. By introducing into Eq. (14) the function $\Phi(s) = \varphi_1(s)/s$ instead of the function $\varphi_1(s)$ we get

$$\Phi(s) = \frac{f_0(s)}{s} + \frac{1}{\pi}\int_a^\infty e^{-i\delta_1}[s' - s - i\varepsilon]^{-1}\sin\delta_1\,\Phi\,ds', \quad (39)$$

i.e. $\varphi_1(s) = s\Phi(s)$, where $\Phi(s)$ — the solution of Eq. (39) — is given by Eq. (38) provided we replace in the latter $q(s)$ by $q(s)/s$.

4. THE AMPLITUDE IN FIRST APPROXIMATION

For simplicity we make the following change of variables

$$x = 1 - 4\mu^2/s, \qquad x' = 1 - 4\mu^2/s'. \quad (40)$$

In this notation the solution of Eq. (14) for the electrical part of the amplitude is given in first approximation by

$$\varphi_1 = \frac{-4mGe\,(1-x)^2}{m^4\sqrt{x}\,(x-\alpha)^2}\ln\frac{1+\sqrt{x}}{1-\sqrt{x}} + \exp\left[\rho(x) + i\delta_1(x)\right]$$
$$\times\left\{-\frac{iF'(\alpha)}{(x-\alpha)} + \frac{i\left[F(x) - F(\alpha)\right]}{(x-\alpha)^2}\right.$$
$$\left. + \mathrm{PP}\frac{1}{\pi}\int_0^1\frac{F(x')\,dx'}{(x'-x)(x'-\alpha)}\right\} + Ae^{i\delta_1(x)}\left\{\cos\delta_1(x)\right.$$
$$\left. + e^{\rho(x)}\frac{1}{\pi}\,\mathrm{P}\int_0^1\sin\delta_1(x')\,e^{-\rho(x')}\frac{dx'}{x'-x}\right\}. \quad (41)$$

In Eq. (41) we have used the notation:

$$F(x) = \sin\delta_1(x)\,e^{-\rho(x)}q(x), \qquad \alpha = 1 - 4\mu^2/m^2;$$
$$q(x) = \frac{-4mGe\,(1-x)^2}{m^4\sqrt{x}}\ln\frac{1+\sqrt{x}}{1-\sqrt{x}}. \quad (42)$$

The generalized principal value integral appearing in Eq. (41) can be calculated either by the limit procedure, Eq. (19), or by expressing it in a form in which the singularities have already been removed. Using Eq. (20) we obtain

$$\mathrm{PP}\int_0^1\frac{F(x')\,dx'}{(x'-x)(x'-\alpha)^2} = \frac{1}{(x-\alpha)^2}\left[\mathrm{P}\int_{2\alpha-1}^1\frac{F(x')\,dx'}{(x'-x)}\right.$$
$$\left. - \mathrm{P}\int_0^1\frac{F(x')\,dx'}{x'-\alpha}\right] + \mathrm{P}\int_0^{2\alpha-1}\frac{F(x')\,dx'}{(x'-x)(x'-\alpha)} - \frac{1}{(x-\alpha)}$$
$$\left\{-\frac{2F(\alpha)}{1-\alpha} + \mathrm{P}\int_{2\alpha-1}^1\frac{\left[F(x') - F(\alpha)\right]dx'}{(x'-\alpha)^2}\right\}. \quad (43)$$

We note that the integrals from 2α to 1 can be neglected.

Below we give expressions for $F(x)$ and $\rho(x)$ for the representation of the P-wave $\pi\pi$ scattering phase shift $\delta_1(x)$ proposed by K. A. Ter-Martirosyan:

$$\delta_1(x) = x^{3/2}\left[\beta + \gamma x + \ldots\right]. \quad (44)$$

Carrying out calculations according to Eq. (31) we obtain

$$\rho(x) = \frac{2\beta}{\pi}\left(\frac{1}{3} + x + \frac{1}{2}x^{3/2}\ln\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)$$
$$+ \frac{2\gamma}{\pi}\left(\frac{1}{5} + \frac{x}{3} + x^2 + \frac{1}{2}x^{5/2}\ln\frac{1-\sqrt{x}}{1+\sqrt{x}}\right) + \ldots; \quad (45)$$

$$F(x) = -\sin\left[x^{3/2}(\beta + \gamma x + \ldots)\right]e^{-\rho(x)}\cdot 4mGe\,(m^4\sqrt{x})^{-1}$$
$$\times(1-x)^2\ln\frac{1+\sqrt{x}}{1-\sqrt{x}}. \quad (46)$$

For the derivative of the function $F(x)$ at the point $x = \alpha$ we get

$$F'(\alpha) = -\frac{2q(\alpha)}{1-\alpha}\sin\delta_1(\alpha)\,e^{-\rho(\alpha)}$$
$$-(1-\alpha)^2\frac{4Gem}{m^4}e^{-\rho(\alpha)}\left\{\cos\delta_1(\alpha)\left(\frac{3}{2}\beta + \frac{5}{2}\gamma\alpha + \ldots\right)\right.$$
$$\times\ln\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} + \sin\delta_1(\alpha)\left[-\frac{1}{2\alpha^{3/2}}\ln\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}} + \frac{1}{\alpha(1-\alpha)}\right]$$
$$-\sin\delta_1(\alpha)\left[\frac{2\beta}{\pi}\left(\frac{1}{\sqrt{\alpha}} + \frac{3}{4}\ln\frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}} - \frac{\sqrt{\alpha}}{2(1-\alpha)}\right)\right.$$
$$+ \frac{2\gamma}{\pi}\left(\frac{1}{3\sqrt{\alpha}} + 2\sqrt{\alpha} + \frac{5}{4}\alpha\ln\frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}\right.$$
$$\left.\left.\left. - \frac{\alpha^{3/2}}{2(1-\alpha)}\right)\right]\ln\frac{1+\sqrt{\alpha}}{1-\sqrt{\alpha}}\right\}. \quad (47)$$

The results obtained in this work are going to be tabulated on computers. In order, however, to obtain a feeling for the general character of the solution we give below the results of approximating that part of the amplitude, which is connected with radiation from the block of heavy particles.

The integrand in Eq. (41) was expanded in a power series. As a result we obtained (for the case $\gamma = 0$, $\beta < 1$, $x < 1$)

$$\varphi_A = Ae^{i\beta x^{3/2}}\left\{\cos(\beta x^{3/2}) + \left[\frac{2\beta}{\pi}\left(\frac{1}{3} + x + \frac{1}{2}x^{3/2}\ln\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right.\right.$$
$$+ \frac{4\beta^2}{\pi^2}\left(\left(\frac{1}{5} + \frac{x}{3} + x^2 + \frac{1}{2}x^{5/2}\ln\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right.$$
$$\left(\sqrt{x}\ln\frac{1+\sqrt{x}}{1-\sqrt{x}} - 1\right) \quad (48)$$
$$\left.\left. + \frac{1}{7\sqrt{x}}\ln\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)\right]\left[1 - \frac{2\beta}{\pi}\left(x + \frac{1}{2}x^{3/2}\ln\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right]^{-1}\right\}.$$

For the representation of the phase shift in the form given by Goebel,[3]

$$\operatorname{tg}\delta_1 = b^{-1}(s/4\mu^2 - 1)^{3/2} = b^{-1}y^{3/2}, \qquad y = s/4\mu^2 - 1, \quad (49)^*$$

the calculation of $e^{\rho(s)}$ has been carried out by Ishida et al.[4] We use instead the method of

*tg = tan.

Galanin and Grashin[5] to calculate directly the contour integral Φ_+ in Eq. (36). Making use of a formula from ref [5], and also of Eq. (19) when evaluating improper integrals, we obtain

$$\varphi_1 = \varphi_A + \varphi_T. \tag{50}$$

Different expressions are obtained for the "vertex" and "bremsstrahlung" parts of the amplitude φ_A and φ_T depending on the value of the quantity b (or $\rho = |b|^{2/3}$).

If $b < 0$, then

$$\frac{\varphi_A}{A} - 1 = \frac{y^2 - \rho y + \rho^2}{b - iy^{3/2}}\left[-\frac{1}{1+\rho+\rho^2} + \frac{iy^{3/2}}{y^2 - \rho y + \rho^2}\right]. \tag{51}$$

If $b > 0$, then

$$\frac{\varphi_A}{A} - 1 = \frac{1}{b - iy^{3/2}}\left[-\frac{b(1+y)}{1-\rho} + \frac{y+\rho}{1-\rho} + iy^{3/2}\right]. \tag{52}$$

For the part of the amplitude connected with internal bremsstrahlung we obtain:

If $b < 0$

$$-(\varphi_T - f_0)\frac{m_4}{4meG} = \frac{m^4}{16\mu^4}\frac{(1+y)(y^2-\rho y+\rho^2)}{b - iy^{3/2}}$$

$$\left\{i\left[\frac{q}{(z^2-\rho z+\rho^2)(z-y)} - \frac{p(\rho-2z)}{(z-y)(z^2-\rho z+\rho^2)^2}\right.\right.$$

$$-\frac{p}{(z-y)(z^2-\rho z+\rho^2)}$$

$$\left.+\frac{y}{(y-z)\sqrt{1+z}\,(y^2-y\rho+\rho^2)}\ln\frac{\sqrt{1+y}+\sqrt{y}}{\sqrt{1+y}-\sqrt{y}}\right].$$

$$-\operatorname{Im}\left[\frac{2u}{\sqrt{1+u}\,(u-y)(u-u^*)(u-z)^2}\ln\frac{\sqrt{1+u}+\sqrt{u}}{\sqrt{1+u}-\sqrt{u}}\right.$$

$$\left.-\frac{2u^*\pi}{\sqrt{1+u^*}\,(u^*-y)(u^*-u)(u^*-z)^2}\right]$$

$$-\pi\left[\frac{z}{\sqrt{1+z}\,(z^2-\rho z+\rho^2)(y-z)^2}\right.$$

$$+\frac{\rho^2 - z^2 + z(-3z^2+\rho z+\rho^2)/2}{(y-z)(1+z)\sqrt{1+z}\,(z^2-\rho z+\rho^2)^2}$$

$$\left.\left.-\frac{y}{(y-z)^2\sqrt{1+y}\,(y^2-\rho y+\rho^2)}\right]\right\}, \tag{53}$$

where we have used the following abbreviations

$$z = m^2/4\mu^2 - 1, \qquad u = \rho e^{+i\pi/3}, \qquad u^* = \rho e^{-i\pi/3},$$

$$p = \frac{z}{\sqrt{1+z}}\ln\frac{\sqrt{1+z}+\sqrt{z}}{\sqrt{1+z}-\sqrt{z}},$$

$$q = \frac{2+z}{2(1+z)\sqrt{1+z}}\ln\frac{\sqrt{1+z}+\sqrt{z}}{\sqrt{1+z}-\sqrt{z}} + \frac{\sqrt{z}}{1+z}.$$

If $b > 1$, then

$$-(\varphi_T - f_0)\frac{m^4}{4meG} = \frac{m^4}{16\mu^4}\frac{(1+y)(y+\rho)}{b-iy^{3/2}}\left\{i\left[\frac{q}{(z-y)(z+\rho)}\right.\right.$$

$$-\frac{p}{(z-y)(z+p)^2} - \frac{p}{(z+\rho)(z-y)^2}$$

$$\left.+\frac{y}{(y+\rho)(y-z)^2\sqrt{1+y}}\ln\frac{\sqrt{1+y}+\sqrt{y}}{\sqrt{1+y}-\sqrt{y}}\right]$$

$$+\frac{1}{\sqrt{\rho-1}\,(y+\rho)(\rho+z)^2}\ln\frac{\sqrt{\rho-1}+\sqrt{\rho}}{\sqrt{\rho}-\sqrt{\rho-1}}$$

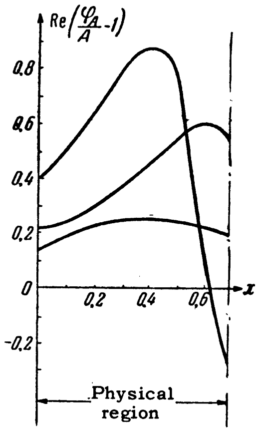

FIG. 9. Comparison of the solutions for the vertex part of the amplitude for two representations of the phase shift δ_1: lower curve – β = 0.614 (expansion in the parameter $x = 1 - 4\mu^2/s$); upper and middle curves – b equal respectively to 1.5 and 5 [expansion in the parameter $y = s/4\mu^2 - 1 = x/(1-x)$].

$$+\frac{\pi y}{(y-z)^2(y+\rho)\sqrt{1+y}} - \frac{z\pi}{\sqrt{1+z}\,(z+\rho)(y-z)^2}$$

$$\left.+\frac{\pi}{2}\frac{(z\rho - z^2 + 2\rho)}{(z+\rho)^2(1+z)^{3/2}}\frac{1}{z-y}\right\}. \tag{54}$$

The case $1 > b > 0$, which corresponds to very large phase shifts at the borders of the physical region (in excess of 70°), has not been considered. Some of the cases are shown graphically in Fig. 9.

5. AN ESTIMATE OF THE CONTRIBUTION OF THE NEGLECTED TERMS

To prove the validity of the first approximation it is necessary to make an estimate of the second approximation, in which the $K\pi$ interaction is taken into account. After averaging over s we obtain the following equation for the second order correction $\varphi_2 = \varphi - \varphi_1$, where φ is the exact solution of Eq. (8) subject to condition (14),

$$\varphi_2 = \frac{1}{\pi}\int_{4\mu^2}^{\infty}\exp\{i\delta_1(s')\}\sin\delta_1(s')\,\varphi_2(s')\left[\frac{1}{s'-s-i\varepsilon}\right.$$

$$\left.-\frac{1}{s'}\right]ds' + \Psi(s); \tag{55}$$

$$\Psi(s) \sim \frac{2}{\pi}\int_{(m+\mu)^2}^{\infty}\bar{\varphi}_0(t')\exp\{-i\delta_1'(t')\}\sin\delta_1'(t')\left[L(t',s)\right.$$

$$\left.-\frac{1}{t'}\right]dt'. \tag{56}$$

Here the function $\bar{\varphi}_0(t')$ is obtained by averaging the pole term in the amplitude $4meG(s - m^2)^{-1} \times (t - \mu^2)^{-1}$ over s at a given t.

The ratio of φ_2 to the pole term is no larger than the ratio of the function $\Psi(s)$ to the pole term. After taking into account that $|\exp\{-i\delta_1'\}\sin\delta_1'| < 1$, we get from formula (56) the following condition for the second order connection to be small compared to the pole term:

$$\frac{E_\gamma^2(E_{\gamma\,max} - E_\gamma)^{3/2}}{m^3\sqrt{m-2E_\gamma}}\ln^{-1}\left(\frac{\sqrt{m/2-E_\gamma}+\sqrt{E_{\gamma\,max}-E_\gamma}}{\sqrt{m/2-E_\gamma}-\sqrt{E_{\gamma\,max}-E_\gamma}}\right) \ll 1, \tag{57}$$

where E_γ is the photon energy. The condition (57) is certainly fulfilled for $E_\gamma = 0$; $E_\gamma = E_{\gamma max}$ or $s = 4\mu^2$; $s = m^2$.

Consequently the most unfavorable case corresponds to the midpoint $E_\gamma = E_{\gamma max}/2 = (m^2 - 4\mu^2)/2m$. At that point the expression on the left is equal to 1/80, i.e., the corrected amplitude is valid to an accuracy of $\sim 1\%$. One obtains a correction of the same order if one considers the channel with four pions. The reason why the approximation is so good has to do with the fact that we are seeking in the first place a 10% correction to the pole term. However such high accuracy presupposes a precise knowledge of the P-wave $\pi\pi$-interaction phase shift in a large region, including the singular point $s = m^2$ too. For s of the order of $20\mu^2$ the contribution of the four pion channel, which produces a branch point in the amplitude at $s = 16\mu^2$, is unimportant.

Thus, although the four pion channel does not upset things, and a solution of high accuracy can be obtained provided the P-wave phase shift is known with precision, large errors can nevertheless be introduced in the process of substituting concrete expansions of the P-wave phase shift. For this reason of the two solutions proposed above the first one should be preferred, because when the expansion proposed by Ter-Martirosyan is used many terms can be included without particular difficulties. The first of the neglected terms contains $x^{7/2}$. At $s \sim m^2$ we have $x^{7/2} \sim 1/5$. And if it should later turn out that a larger number of terms is needed to describe the P wave phase shift then our solution can be extended to cover that case without difficulty.

We gave above estimates of the contribution of the $K\pi$ channels. A 1% correction was found without taking into account the smallness of the δ_1' phase shift. In actuality the contribution from other channels with small phase shifts is smaller by another order of magnitude, i.e., is of the order of 0.1% in comparison with the pole term.

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