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MINIMUM NUMBER OF PARTIAL WAVES IN REACTIONS IN WHICH THERE ARE SEVERAL PARTICLES IN THE FINAL STATE

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A universal inequality is derived for the minimum number of partial waves involved in a reaction in which more than two particles occur in the final state.

1. INTRODUCTION

THE minimum number L_{min} of partial waves occurring in collisions of particles at high energies has been determined by several authors. Rarita and Schwed[1] have shown how to calculate L_{min} for elastic scattering, given a knowledge of the total cross section of the interaction. Recently Grishin and Ogievetskii[2] have derived an inequality which is very effective in the determination of the minimum number of partial waves in two-particle reactions, if one knows the total elastic cross section and the differential cross section for certain angles.

As is well known, most high-energy collision processes are multiple processes in which more than two particles appear in the final state. The question arises as to how to determine the minimum number of partial waves in such collisions. A knowledge of L_{min} is important for the processing of experimental results, since L_{min} is connected with the minimum interaction radius.

In the present paper the inequality obtained by Grishin and Ogievetskii for two-particle reactions is extended to the case of reactions in which more than two particles appear in the final state. It involves the angular distribution of one of the particles in the final state, the total cross section for the given channel, and L_{min}. In Sec. 2 we discuss in detail the choice of the independent variables for the description of the final states of a three-particle system. In Sec. 3 the inequality is obtained for the case of particles without spin ($0 + 0 \rightarrow 0 + 0 + 0$). In Sec. 4 it is shown that the inequality obtained in Sec. 3 can be carried over without change to the cases

$$0 + 1/2 \rightarrow 0 + 0 + 1/2, \quad 1/2 + 1/2 \rightarrow 0 + 1/2 + 1/2,$$
$$1/2 + 1/2 \rightarrow 0 + 0 + 0, \quad 0 + 0 \rightarrow 0 + 1/2 + 1/2.$$

In Sec. 5 the inequality for the case of three particles in the final state is extended to the case in which there are n particles in the final state; in Sec. 6 applications of this inequality are discussed.

2. KINEMATICS OF THE THREE-PARTICLE SYSTEM

To describe the states of a three-particle system we shall introduce three other vectors instead of the momentum vectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ of the particles. As the first we take the momentum of the center of mass, $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$; in the center-of-mass system of the three particles (which we shall hereafter call simply the 3c system), $\mathbf{P} = 0$. As the second we choose $\mathbf{P}_{3c}$ ($|\mathbf{P}_{3c}|$, Ω_{3c}), the momentum in the 3c system of that one of the three particles which can be identified experimentally (for example, the recoil nucleon, or K meson, or hyperon). In the system in which the other two particles, taken as a whole, are at rest (we shall call it the 2c system), these two particles move in opposite directions, so that as the third vector we can take the direction Ω_{2c} of the relative momentum of the two particles in the 2c system and the energy $\mathfrak{M}_{2c}$ of these particles in the 2c system.

This choice of the independent variables has a number of advantages. First, as is shown in the Appendix, the integration over the phase space can be separated into two parts:

$$\iiint \frac{d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3}{8 E_1 E_2 E_3} \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{P}_i)\, \delta(E_1 + E_2 + E_3 - E_i) = \int d\mathfrak{M}_{2c}^2 G(\mathfrak{M}_{2c}^2, \mathfrak{M}_{3c}^2) \iint d\Omega_{2c}\, d\Omega_{3c}. \quad (1)$$

Here $\mathfrak{M}_{3c}$ is the total energy of the three particles in the 3c system, and $\mathbf{P}_i$ and E_i are the momentum and energy in the initial state. The function $G(\mathfrak{M}_{2c}^2, \mathfrak{M}_{3c}^2)$ does not depend on the angles, and the limits of the integrations over $d\Omega_{2c}$ and $d\Omega_{3c}$ do not depend on each other. Thus the integrations over the angles and the energies are separated.

Second, starting from the work of Chou Kuang-chao and Shirokov,[3] one can show that the total angular momentum of the three particles can be obtained in the usual way by compounding l_{2c}, the relative angular momentum of the two particles in the 2c system, and l_{3c}, the angular momentum of the identified particle relative to the two other particles in the 3c system.

3. INTERACTIONS OF THE TYPE $0 + 0 \to 0 + 0 + 0$

Let us first consider the simplest case, in which all of the particles are spinless ($0 + 0 \to 0 + 0 + 0$). Let us choose the variables as indicated in Section 2; then the general form of the amplitude for the process is

$$F = \sum R_L^{l_{2c}, l_{3c}} Y^*_{L, M}(\Omega_i) C_{L, M}^{l_{2c}, \mu_{2c}; l_{3c}, \mu_{3c}} Y_{l_{2c} \mu_{2c}}(\Omega_{2c}) \times Y_{l_{3c}, \mu_{3c}}(\Omega_{3c}). \quad (2)$$

Here the summation is taken over all quantum numbers. The function $R_L^{l_{2c}, l_{3c}}$ depends on the total angular momentum L of the three-particle system, and on l_{2c} and l_{3c}, the angular momenta mentioned at the end of the preceding section. The other arguments of the function $R_L^{l_{2c}, l_{3c}}$ are invariant under Lorentz transformations. $Y_{l,m}$ are spherical harmonics; Ω_i is the unit vector in the direction of the momentum of the incident particle in the 3c system; $C^{\cdots}_{\cdot\cdot}$ are Clebsch-Gordon coefficients.

If we choose the z axis in the direction Ω_{3c}, Eq. (2) can be written in the form

$$F = \sum R_L^{l_{2c} l_{3c}} \left(\frac{2l_{3c}+1}{4\pi}\right)^{1/2} C_{L, M}^{l_{2c}, M; l_{3c}, 0} Y_{l_{2c}, M}(\Omega_{2c}) Y^*_{L, M}(\Omega_i). \quad (3)$$

The angular distribution of the identified particle is of the form

$$\sigma(\theta) = \int G(\mathfrak{M}^2_{2c}, \mathfrak{M}^2_{3c}) d\mathfrak{M}^2_{2c} \int |F|^2 d\Omega_{2c}$$
$$= \sum_{Ml_{2c}} \int \Big| \sum_{Ll_{3c}} R_L^{l_{2c} l_{3c}} \left(\frac{2l_{3c}+1}{4\pi}\right)^{1/2} C_{L, M}^{l_{2c}, M; l_{3c}, 0} Y^*_{L, M}(\theta) \Big|^2 \times G(\mathfrak{M}^2_{2c}, \mathfrak{M}^2_{3c}) d\mathfrak{M}^2_{2c}, \quad (4)$$

and the cross section for the process is given by the formula

$$\sigma_3 = \int \sigma(\theta) d\Omega = \sum_{LMl_{2c}} \int G(\mathfrak{M}^2_{2c}, \mathfrak{M}^2_{3c}) d\mathfrak{M}^2_{2c} \Big| \sum_{l_{3c}} \left(\frac{2l_{3c}+1}{4\pi}\right)^{1/2} \times R_L^{l_{2c} l_{3c}} C_{L, M}^{l_{2c}, M; l_{3c}, 0} \Big|^2. \quad (5)$$

Let us assume that in the expressions (4) and (5) we can confine ourselves to a finite number L_{min} of partial waves. Then by using the Cauchy inequality

$$\sum |A_i B_i|^2 \leqslant \sum |A_i|^2 \sum |B_i|^2,$$

we get

$$\sigma(\theta) \leqslant \sum_{Ml_{2c}} \int \sum_L \Big| \sum_{l_{3c}} R_L^{l_{2c} l_{3c}} \left(\frac{2l_{3c}+1}{4\pi}\right)^{1/2} C_{L, M}^{l_{2c}, M; l_{3c}, 0} \Big|^2 \times \sum_L |Y_{L, M}(\theta)|^2 G(\mathfrak{M}^2_{2c}, \mathfrak{M}^2_{3c}) d\mathfrak{M}^2_{2c}$$
$$\leqslant \sum_{LMl_{2c}} \int \Big| \sum_{l_{3c}} R_L^{l_{2c} l_{3c}} \left(\frac{2l_{3c}+1}{4\pi}\right)^{1/2} C_{L, M}^{l_{2c}, M; l_{3c}, 0} \Big|^2 \times G(\mathfrak{M}^2_{2c}, \mathfrak{M}^2_{3c}) d\mathfrak{M}^2_{2c} \sum_{LM} |Y_{L, M}(\theta)|^2 = \sigma_3 \sum_{L=0}^{L_{min}} \frac{2L+1}{4\pi} \quad (6)$$

or

$$4\pi\sigma(\theta) / \sigma_3 \leqslant (L_{min} + 1)^2. \quad (7)$$

It must be noted that after the summation over M the right member of Eq. (6) already does not depend on θ; this is unlike the case of the other two particles, for which there is no such summation over M and the right side of the inequality depends on θ.[2]

4. INTERACTIONS OF THE TYPE $0 + \frac{1}{2} \to 0 + 0 + \frac{1}{2}$

Let us consider the case $0 + \frac{1}{2} \to 0 + 0 + \frac{1}{2}$. The amplitude for this process is given by the expression

$$F = \sum R_{JLL'}^{l_{2c} l_{3c}} C_{J,M}^{L', M-\beta; 1/2, \beta} C_{L',M-\beta}^{l_{3c}, 0; l_{2c}, M-\beta} C_{J, M}^{L, M-\alpha; 1/2, \alpha} \times \left(\frac{2l_{3c}+1}{4\pi}\right)^{1/2} Y_{l_{2c}, M-\beta}(\Omega_{2c}) Y^*_{L, M}(\Omega_i); \quad (8)$$

here the summation is over J, L, L′, M, l_{2c}, and l_{3c}, where J is the total angular momentum of the three-particle system in the 3c system, L and L′ are the total orbital angular momenta, and α and β are the spin projections of the particles in the initial and final states in the 3c system. The other symbols are the same as in the preceding section.

The angular distribution of the identified particle takes the form

$$\sigma(\theta) = \sum_{\alpha\beta} \int G(\mathfrak{M}^2_{2c}, \mathfrak{M}^2_{3c}) d\mathfrak{M}^2_{2c} \int |F|^2 d\Omega_{2c}$$
$$= \sum_{\substack{\alpha, \beta \\ l_{2c}, M}} \int \Big| \sum_{\substack{L, L', J \\ l_{3c}}} R_{J, L, L'}^{l_{2c}, l_{3c}} C_{J, M}^{L', M-\beta; 1/2, \beta} C_{L', M-\beta}^{l_{3c}, 0; l_{2c}, M-\beta} \times C_{J, M}^{L, M-\alpha; 1/2, \alpha} \left(\frac{2l_{3c}+1}{4\pi}\right)^{1/2} Y^*_{L, M}(\theta) \Big|^2 G(\mathfrak{M}^2_{2c}, \mathfrak{M}^2_{3c}) d\mathfrak{M}^2_{2c}. \quad (9)$$

Using the Cauchy inequality and making calculations like those given in Sec. 3, we get the inequality

$$4\pi\sigma(\theta) / \sigma_3 \leqslant (L_{min} + 1)^2, \quad (10)$$

where

$$\sigma_3 = \int \sigma(\theta) d\Omega. \quad (11)$$

The inequality (10) is of the same form as for the spinless case. It is easy to show that this same result can also be obtained for the cases

$$1/2 + 1/2 \to 0 + 1/2 + 1/2, \qquad 1/2 + 1/2 \to 0 + 0 + 0,$$
$$0 + 0 \to 0 + 1/2 + 1/2$$

5. INTERACTIONS IN WHICH THERE ARE n PARTICLES IN THE FINAL STATE

Let us first consider the case $0 + 0 \to \overbrace{0 + \ldots + 0}^{n}$. In choosing the variables for this case we can take first the momentum $\mathbf{P}$ of the center of mass of the n particles. In the center-of-mass system of these n particles (the nc system), $\mathbf{P} = 0$. Next, $\mathbf{P}_{nc}(|\mathbf{P}_{nc}|, \Omega_{nc})$ is the momentum of the identified particle in the nc system. In the system in which the $n-1$ particles are at rest taken as a whole [the $(n-1)c$ system], it is convenient to take as the variables $\mathfrak{M}_{(n-1)c}$, the total energy of the $n-1$ particles in the $(n-1)c$ system, and $\Omega_{(n-1)c}$, the unit vector in the direction of the relative momentum $\mathbf{P}_{(n-1)c}$ of one of these $n-1$ particles and the other $n-2$ particles, taken as a whole, in the $(n-1)c$ system. The remaining variables are chosen analogously, and are written $\mathfrak{M}^2_{(n-2)c}$, $\Omega_{(n-2)c}, \ldots, \mathfrak{M}^2_{2c}, \Omega_{2c}$. The advantages of this choice of the independent variables have been indicated in Section 2.

The integration over the phase space is of the form

$$\int \frac{d\mathbf{p}_1 d\mathbf{p}_2 \ldots d\mathbf{p}_n}{2^n E_1 E_2 \ldots E_n} \delta(\mathbf{p}_1 + \mathbf{p}_2 + \ldots + \mathbf{p}_n - \mathbf{p}_i)\, \delta(E_1 + \ldots$$
$$+ E_n - E_i) = \int G(\mathfrak{M}^2_{2c}, \mathfrak{M}^2_{3c}, \ldots, \mathfrak{M}^2_{nc})\, d\mathfrak{M}^2_{2c} \ldots d\mathfrak{M}^2_{(n-1)c}$$
$$\times \int d\Omega_{2c} \ldots d\Omega_{nc}, \tag{12}$$

where G is a function which depends only on the energies and not on the angles.

By means of the set of independent variables that has been indicated one can reduce the amplitude for the process to the form

$$F_n = \sum R^{l_{2c}, l_{3c}, \ldots, l_{nc}}_{L, L_{3c}, \ldots, L_{(n-1)c}} C^{l_{2c}, \mu_{2c}; l_{3c}, \mu_{3c}}_{L_{3c}, M_{3c}} C^{L_{3c}, M_{3c}; l_{4c}, \mu_{4c}}_{L_{4c}, M_{4c}} \ldots$$
$$\times C^{L_{(n-1)c}, M_{(n-1)c}; l_{nc}, \mu_{nc}}_{L, M} Y^*_{L, M}(\Omega_i)\, Y_{l_{2c}, \mu_{2c}}(\Omega_{2c}) \ldots$$
$$\times Y_{l_{nc}, \mu_{nc}}(\Omega_n), \tag{13}$$

where L and M are the total angular momentum of all n particles in the nc system and the z component of this angular momentum; l_{ic} and μ_{ic} are the relative angular momentum and z component of angular momentum that correspond to the momentum $\mathbf{p}_{ic}$ in the ic system; L_{ic} and M_{ic} are the total angular momentum and its z component in the ic system; the summation is taken over all the angular momenta and z components.

Let us take the z axis along Ω_{nc}; then the angular distribution of the identified particle and the total cross section can be written in the forms

$$\sigma_n(\theta) = \int |F_n|^2 G(\mathfrak{M}^2_{2c}, \ldots, \mathfrak{M}^2_{nc})\, d\mathfrak{M}^2_{2c} \ldots d\mathfrak{M}^2_{(n-1)c}\, d\Omega_{2c} \ldots$$
$$\times d\Omega_{(n-1)c}, \tag{14}$$

$$\sigma_n = \int \sigma_n(\theta)\, d\Omega. \tag{15}$$

Using the Cauchy inequality, we get from Eqs. (14) and (15)

$$4\pi\sigma_n(\theta)/\sigma_n \leqslant (L_{min} + 1)^2. \tag{16}$$

This inequality is analogous to the one obtained in the case of three particles without spin. For the general cases in which several particles have spins different from zero it can be shown in just the same way that the result is the same as in the case of spinless particles.

6. DISCUSSION

The inequality (16) is useful for the determination of L_{min}. It enables us to calculate L_{min} if we know the total cross section for a process in which a definite number of particles emerge in the final state and the angular distribution of one identified particle (K meson, hyperon, recoil nucleon, or antibaryon) in the c.m.s.

In view of the fact that the right member of Eq. (16) does not depend on θ, the inequality is to be written in the form

$$4\pi(\sigma_n(\theta))_{max}/\sigma_n \leqslant (L_{min} + 1)^2. \tag{17}$$

It must be noted that if we take the z axis in the direction Ω_i, then between the angular distribution of one identified particle out of the j particles in the jc system

$$\sigma_n(\theta_j) = \int |F_n|^2 G(\mathfrak{M}^2_{2c}, \ldots, \mathfrak{M}^2_{nc})\, d\mathfrak{M}^2_{2c} \ldots d\mathfrak{M}^2_{(n-1)c}$$
$$\times d\Omega_{2c} \ldots d\Omega_{(j-1)c}\, d\Omega_{(j+1)c} \ldots d\Omega_{nc} \tag{18}$$

and the cross section for the process we have the relation

$$4\pi\sigma_n(\theta_j)/\sigma_n \leqslant [(l_i)_{min} + 1]^2. \tag{19}$$

The inequality (19) makes it possible to determine the minimum number of partial waves essential for the description of the subsystem jc.

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APPENDIX*

For the case of an n-particle process the integral over phase space is of the form

$$I = \int \frac{d\mathbf{p}_1 \ldots d\mathbf{p}_n}{2^n E_1 E_2 \ldots E_n} \delta(\mathbf{p}_1 + \mathbf{p}_2 + \ldots + \mathbf{p}_n - \mathbf{p}_i)\,\delta(E_1 + \ldots + E_n - E_i). \quad (A.1)$$

It can be written in the explicitly invariant form

$$I = \int d^4p_1 \ldots d^4p_n \delta(p_1^2 + m_1^2) \ldots \delta(p_1^2 + m_n^2)\,\delta^4(p_1 + p_2 + \ldots + p_n - p_i). \quad (A.2)$$

After the transformation

$$p_1 + p_2 = k_2, \quad p_1 - p_2 = 2q_2$$

we get

$$I = \int d^4k_2\, d^4q_2\, d^4p_3 \ldots d^4p_n \delta\left(\left(\frac{k_2}{2} + q_2\right)^2 + m_1^2\right)\delta\left(\left(\frac{k_2}{2} - q_2\right)^2 + m_2^2\right)\delta(p_3^2 + m_3^2) \ldots \delta(p_n^2 + m_n^2) \times \delta^4(k_2 + p_3 + \ldots + p_n - p_i). \quad (A.3)$$

Noting that

$$d^4q_2 = \tfrac{1}{2}\sqrt{q_2^2 + q_{20}^2}\, dq_{20}\, dq_2^2\, d\Omega_2,$$

we can perform the integration over d^4q_2 in the 2c system, in which

$$k_{2c}^2 = -k_{2c0}^2, \quad (k_{2c}q_{2c}) = -k_{2c0}q_{2c0}$$

and Eq. (A.3) then takes the form

$$I = \left(\frac{1}{2}\right)^3 \int d\mathfrak{M}_{2c}^2\, d\Omega_{2c}\, d^4k_{2c}\, d^4p_3 \ldots d^4p_n \left(1 - 2\frac{m_1^2 + m_2^2}{\mathfrak{M}_{2c}^2} + \left(\frac{m_1^2 - m_2^2}{\mathfrak{M}_{2c}^2}\right)^2\right)^{1/2} \delta(k_{2c}^2 + \mathfrak{M}_{2c}^2)\,\delta(p_3^2 + m_3^2) \ldots \delta(p_n^2 + m_n^2) \times \delta^4(k_{2c} + p_3 + \ldots + p_n - p_i),$$

where

$$\mathfrak{M}_{2c}^2/4 = q_{2c}^2 + (m_1^2 + m_2^2)/2.$$

It is easy to see that $\mathfrak{M}_{2c}$ is the total energy of the two particles in the 2c system.

Continuing the indicated procedure, i.e., setting

$$k_i + p_{i+1} = k_{i+1}, \quad k_i - p_{i+1} = 2q_{i+1}, \quad i = 3, 4, \ldots, n-1$$

and integrating over d^4q_{i+1} in the (i + 1)c system, we finally obtain the integral over phase space in the form

$$I = \int G(\mathfrak{M}_{2c}^2, \ldots, \mathfrak{M}_{nc}^2)\, d\Omega_{2c} \ldots d\Omega_{nc}\, d\mathfrak{M}_{2c}^2 \ldots d\mathfrak{M}_{(n-1)c}^2 \quad (A.4)$$

($\mathfrak{M}_{ic}$ is the total energy of particle i in the ic system), where

$$\tfrac{1}{4}\mathfrak{M}_{ic}^2 = q_{ic}^2 + \tfrac{1}{2}(\mathfrak{M}_{(i-1)c}^2 + m_i^2), \quad \mathfrak{M}_{1c} = m_1,$$

$$G(\mathfrak{M}_{2c}^2, \ldots, \mathfrak{M}_{nc}^2) = \left(\frac{1}{2}\right)^{(n-1)} \prod_{i=2}^{n} \left[1 - 2\frac{\mathfrak{M}_{(i-1)c}^2 + m_i^2}{\mathfrak{M}_{ic}^2} + \left(\frac{\mathfrak{M}_{(i-1)c}^2 - m_i^2}{\mathfrak{M}_{ic}^2}\right)^2\right]^{1/2}.$$

*An analogous method of integration over phase space has been proposed earlier by Kopylov[4] for the calculation of statistical weights and distributions in theories of multiple production.

[1] W. Rarita and P. Schwed, Phys. Rev. 112, 271 (1958).

[2] V. G. Grishin and V. I. Ogievetskii, Nuclear Phys. 18, 516 (1960).

[3] Chou Kuang-Chao and M. I. Shirokov, JETP 34, 1230 (1958), Soviet Phys. JETP 7, 851 (1958).

[4] G. I. Kopylov, JETP 39, 1091 (1960), Soviet Phys. JETP 12, 761 (1961).

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