THE THEORY OF ELECTROMAGNETIC FLUCTUATIONS IN A PLASMA

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We consider electromagnetic fluctuations in a non-equilibrium plasma both when there are no strong fields present and in the case when the plasma is in a strong magnetic field.

1. One can assume that at the present time the theory of thermal fluctuations in a plasma in thermodynamic equilibrium has already been developed.* The difference—which is essentially not one of principle—between the theory of electromagnetic fluctuations of a plasma and the one given, for instance, in the book by Landau and Lifshitz[2] consists in the use of the complex dielectric-constant tensor with account of spatial dispersion. Such a theory of fluctuations can, however, not be used for an application to a rarefied plasma in which the collisions are extremely rare and which can thus stay in a thermodynamic non-equilibrium state for a long time. The present paper is devoted to a consideration of fluctuations in such a non-equilibrium plasma.† We then obtain expressions for the fluctuations by straightforward calculation, by regarding the plasma as a system of weakly interacting particles.

2. We introduce the Fourier components of the electrical field operator

$$\hat{\mathbf{E}}(\omega, \mathbf{k}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \int d\mathbf{r}\, \hat{\mathbf{E}}(\mathbf{r}, t)\, e^{i\omega t - i\mathbf{k}\mathbf{r}}. \quad (1)$$

To find expressions for the fluctuations we must know the quantum-mechanical average of the following operator

$$\tfrac{1}{2}[\hat{E}_j(\omega, \mathbf{k})\hat{E}_i(\omega', \mathbf{k}') + \hat{E}_i(\omega', \mathbf{k}')\hat{E}_j(\omega, \mathbf{k})]. \quad (2)$$

It is clear that the average value of such an operator can at once be written down if we know the explicit form of the electrical field operators. In the case of interest to us, of weakly interacting particles, one can easily obtain the explicit form of the field operators. To do this one must take into account the fact that the form of the matrix elements of the field operators is determined by the matrix elements of the particle current density when the particles make a transition from one state to another.[3]

We first consider a case when there is no strong field in the plasma. The states of the particles in such a plasma can be described by plane waves and the transition matrix element of the current density of particles of the kind a, which corresponds to a transition from a state n to a state m, is of the form

$$e_a \boldsymbol{\alpha}_{mn} \exp\{i\hbar^{-1}[(\mathbf{p}_n - \mathbf{p}_m, \mathbf{r}) - (E_n - E_m)t]\}, \quad (3)$$

where α are the Dirac matrices. We can then write down for the Fourier component of the electrical field operator (1) the equation

$$\{(\omega/c)^2\varepsilon_{ij}(\omega, \mathbf{k}) - k^2\delta_{ij} + k_i k_j\}\langle m|\hat{E}_j(\omega, \mathbf{k})|n\rangle = -\frac{4\pi i\omega}{c} e_a \alpha^i_{mn} \delta(\hbar\mathbf{k} - \mathbf{p}_n + \mathbf{p}_m)\, \delta\left(\omega - \frac{E_n - E_m}{\hbar}\right), \quad (4)$$

where ϵ_{ij} is the dielectric-constant tensor.

We can then immediately write down an expression for the matrix element of the field corresponding to a transition of a particle from a state n to a state m:

$$\langle m|\hat{E}^i(\omega, \mathbf{k})|n\rangle = -4\pi i\omega c^{-1} e_a A^{-1}_{ij}(\omega, \mathbf{k})\, \alpha^j_{mn} \times \delta\left(\mathbf{k} - \frac{\mathbf{p}_n - \mathbf{p}_m}{\hbar}\right)\delta\left(\omega - \frac{E_n - E_m}{\hbar}\right), \quad (5)$$

$$A_{ij}(\omega, \mathbf{k}) = (\omega/c)^2\varepsilon_{ij}(\omega, \mathbf{k}) - k^2\delta_{ij} + k_i k_j. \quad (6)$$

Equation (5) enables us easily to evaluate the expectation value of the operator (2). Using the fact that the number of particles of kind a per unit volume is equal to N_a and that the probability that the a-th particle has a momentum $\mathbf{p}_a$ [and accordingly an energy $E(\mathbf{p}_a)$] is determined by the distribution functions $f_a(\mathbf{p}_a)$, we have

$$\overline{\tfrac{1}{2}[\hat{E}_j(\omega, \mathbf{k})\hat{E}_i(\omega', \mathbf{k}') + \hat{E}_i(\omega', \mathbf{k}')\hat{E}_j(\omega, \mathbf{k})]} = \delta(\omega + \omega')\,\delta(\mathbf{k} + \mathbf{k}')\,(E_i E_j)_{\omega, \mathbf{k}}, \quad (7)$$

*A survey of such a theory and the corresponding bibliography are given in the book by Rukhadze and the present author.[1]

†The theory of non-equilibrium electrical fluctuations was considered by L. V. Keldysh—to whom the author is grateful for information—in connection with another problem.

where the bar indicates a quantum-mechanical average, and

$$(E_iE_j)_{\omega,\mathbf{k}} = \frac{1}{4}\sum_a\left(\frac{4\pi e_a\omega}{c}\right)^2 A_{il}^{-1}(\omega,\mathbf{k})\,A_{jr}^{*-1}(\omega,\mathbf{k})\,N_a$$
$$\times\int d\mathbf{p}_a f_a(\mathbf{p}_a)\left\{\delta\left[\omega-\frac{1}{\hbar}\{E(\mathbf{p}_a+\hbar\mathbf{k})-E(\mathbf{p}_a)\}\right]\right.$$
$$\times\left(-\delta_{lr}\frac{[E(\mathbf{p}_a+\hbar\mathbf{k})-E(\mathbf{p}_a)]^2-c^2\hbar^2k^2}{2E(\mathbf{p}_a)E(\mathbf{p}_a+\hbar\mathbf{k})}+\frac{c^2p_a^r(\mathbf{p}_a+\hbar\mathbf{k})^l}{E(\mathbf{p}_a)E(\mathbf{p}_a+\hbar\mathbf{k})}\right.$$
$$\left.+\frac{c^2p_a^l(\mathbf{p}_a+\hbar\mathbf{k})^r}{E(\mathbf{p}_a)E(\mathbf{p}_a+\hbar\mathbf{k})}\right)+\delta\left[\omega+\frac{1}{\hbar}\{E(\mathbf{p}_a-\hbar\mathbf{k})-E(\mathbf{p}_a)\}\right]$$
$$\times\left(-\delta_{lr}\frac{[E(\mathbf{p}_a-\hbar\mathbf{k})-E(\mathbf{p}_a)]^2-c^2\hbar^2k^2}{2E(\mathbf{p}_a)E(\mathbf{p}_a-\hbar\mathbf{k})}\right.$$
$$\left.\left.+\frac{c^2p_a^r(\mathbf{p}_a-\hbar\mathbf{k})^l}{E(\mathbf{p}_a)E(\mathbf{p}_a-\hbar\mathbf{k})}+\frac{c^2p_a^l(\mathbf{p}_a-\hbar\mathbf{k})^r}{E(\mathbf{p}_a)E(\mathbf{p}_a-\hbar\mathbf{k})}\right)\right\}. \tag{8}$$

The spectral functions for the vectors of the magnetic and the electrical induction **B** and **D** are connected with $(E_iE_k)_{\omega,\mathbf{k}}$ by the relation

$$(B_iB_j)_{\omega,\mathbf{k}} = c^2\omega^{-2}k_rk_se_{irl}e_{jst}(E_lE_t)_{\omega,\mathbf{k}};$$
$$(D_iD_j)_{\omega,\mathbf{k}} = \varepsilon_{ir}(\omega,\mathbf{k})\,\varepsilon_{jl}^*(\omega,\mathbf{k})\,(E_rE_l)_{\omega,\mathbf{k}}$$

In the classical limit $\hbar = 0$

$$(E_iE_j)_{\omega,\mathbf{k}} = \sum_a\left(\frac{4\pi e_a\omega}{c}\right)^2 A_{il}^{-1}(\omega,\mathbf{k})\,A_{jr}^{*-1}(\omega,\mathbf{k})$$
$$\times N_a\int d\mathbf{p}_a f_a(\mathbf{p}_a)\,\frac{v_a^lv_a^r}{c^2}\,\delta(\omega-\mathbf{k}\mathbf{v}_a), \tag{9}$$

where $\mathbf{v}_a$ is the velocity of the a-th particle.

We point out that the spectral formula for the fluctuations of the Lorentz force acting upon particle a is equal to*

$$\left(e_a^2\left(\mathbf{E}+\frac{1}{c}[\mathbf{v}_a\mathbf{B}]\right)_i\left(\mathbf{E}+\frac{1}{c}[\mathbf{v}_a\mathbf{B}]\right)_j\right)_{\omega,\mathbf{k}} = \sum_b\left(\frac{4\pi e_ae_b\omega}{c^2}\right)^2$$
$$\times\int d\mathbf{p}_bN_bf_b\delta(\omega-\mathbf{k}\mathbf{v}_b)\,v_b^lv_b^r\left(\delta_{is}\left[1-\frac{\mathbf{k}\mathbf{v}_a}{\omega}\right]\right.$$
$$\left.+\frac{k_iv_a^s}{\omega}\right)A_{sl}^{-1}(\omega,\mathbf{k})A_{tr}^{*-1}(\omega,\mathbf{k})\left(\delta_{jt}\left[1-\frac{\mathbf{k}\mathbf{v}_a}{\omega}\right]+\frac{k_jv_a^t}{\omega}\right).$$

This last formula can be used to write down the diffusion coefficient in the transport equation for charged particles.

In the particular case of particles distributed isotropically over the plasma, when the dielectric-constant tensor is of the form

$$\varepsilon_{ij}(\omega,\mathbf{k}) = (\delta_{ij}-k_ik_j/k^2)\,\varepsilon^{tr}(\omega,k)+k_ik_jk^{-2}\varepsilon^l(\omega,k), \tag{10}$$

and therefore

$$A_{ij}^{-1}(\omega,\mathbf{k}) = \frac{\delta_{ij}-k_ik_j/k^2}{(\omega/c)^2\varepsilon^{tr}-k^2}+\frac{c^2}{\omega^2\varepsilon^l}\frac{k_ik_j}{k^2}, \tag{11}$$

Equation (9) becomes of the form

*$[\mathbf{v}_a\mathbf{B}] = \mathbf{v}_a\times\mathbf{B}$.

$$(E_iE_j)_{\omega,\mathbf{k}} = \sum_a\left(\frac{4\pi e_a}{k^2}\right)^2\int d\mathbf{p}_aN_af(\mathbf{p}_a)\delta(\omega-\mathbf{k}\mathbf{v}_a)$$
$$\times\left\{\frac{k_ik_j}{|\varepsilon^l(\omega,k)|^2}+\frac{1}{2}\frac{\omega^2[\mathbf{k}\mathbf{v}_a]^2(k^2\delta_{ij}-k_ik_j)}{|\omega^2\varepsilon^{tr}(\omega,k)-c^2k^2|^2}\right\}. \tag{12}$$

In the case of an equilibrium particle distribution, Eq. (12) and, of course, also Eqs. (8) and (9), go over into the formulae following from the theory of thermal fluctuations.[1]

3. We turn now to the case of a plasma in a strong magnetic field $\mathbf{B}_0$. We note first that it is clear from what we have already said that the fluctuations of the electrical field can be written in the form*

$$\frac{1}{2}\overline{[\hat{E}_j(\omega,\mathbf{k})\,\hat{E}_i(\omega',\mathbf{k}')+\hat{E}_i(\omega',\mathbf{k}')\,\hat{E}_j(\omega,\mathbf{k})]} = -(4\pi/c^2)^2\omega\omega'$$
$$\times A_{il}^{-1}(\omega',\mathbf{k}')\,A_{jr}^{-1}(\omega,\mathbf{k})\,I_{rl}(\omega,\mathbf{k};\omega',\mathbf{k}'). \tag{13}$$

On the right hand side stands the correlation function of the transition current

$$I_{rl}(\omega,\mathbf{k};\omega',\mathbf{k}')$$
$$= \frac{1}{2}\overline{[\hat{j}_r(\omega,\mathbf{k})\,\hat{j}_l(\omega',\mathbf{k}')+\hat{j}_l(\omega',\mathbf{k}')\,\hat{j}_r(\omega,\mathbf{k})]},$$

and we now turn to evaluate this. We note in passing that one should not confuse the transition-current correlation function with the induced-current correlation function, which can easily be found by using the field equations provided, naturally, that the electrical field correlation function, for instance, is known.

We restrict ourselves to a non-relativistic analysis. Moreover, we shall not take spin effects into account. For the evaluation of the matrix elements we use the Landau representation, and we use thus for the complete set the value p_n^z of the component of the momentum parallel to $\mathbf{B}_0$, the energy pertaining to the transverse motion $E^{\perp}(l_n) = \hbar\Omega_a(l_n+\frac{1}{2})$ (here $\Omega_a = |e_a|B_0/\mu_ac$), and finally $y_{0n} = -cp_n^x/e_aB_0$, the projection of the center of the Larmor orbit upon the y axis. We have then for the matrix element of the Fourier component of the transition current[4,5]

$$\langle m|\hat{\mathbf{j}}_a(\omega,\mathbf{k})|n\rangle = \delta(\omega-[l_n-l_m]\,\Omega_a-p_n^zk_z/\mu_a+\hbar k_z^2/2\mu_a)$$
$$\times\delta\left(k_z-\frac{p_n^z-p_m^z}{\hbar}\right)\delta\left(k_x+\frac{e_aB_0}{\hbar c}[y_{0n}-y_{0m}]\right)$$
$$\times\exp\left\{-\frac{1}{2}ik_y(y_{0m}+y_{0n})\right\}\langle l_m|\hat{\mathbf{j}}_a(\mathbf{k})|l_n\rangle, \tag{14}$$

*We note that without taking further conditions, say boundary conditions, into account, Eq. (13) and also Eqs. (5), (8), (9), and (12) can become meaningless in the region where the plasma is transparent, i.e., under conditions where the determinant of the tensor $A_{ij}(\omega,\mathbf{k})$ tends to zero. This latter fact corresponds to the circumstance that in the transparency region the non-equilibrium fluctuations of the electromagnetic field are not determined by the states of the charges. This observation arose from a conversation with M. A. Leontovich to whom I should like to express my gratitude.

where

$$\langle l_m|\hat{j}^x_a(\mathbf{k})|l_n\rangle = |e_a|\,\Omega_a i\frac{\partial}{\partial k_y}\langle l_m|e^{-ik_y y}|l_n\rangle,$$

$$\langle l_m|\hat{j}^y_a(\mathbf{k})|l_n\rangle = i\hbar k_x\frac{|e_a|}{2\mu_a}\langle l_m|e^{-ik_y y}|l_n\rangle + ie_a\sqrt{\hbar\Omega_a/2\mu_a}$$

$$\times\{\sqrt{l_m}\langle l_m-1|e^{-ik_y y}|l_n\rangle - \sqrt{l_n}\langle l_m|e^{-ik_y y}|l_n-1\rangle\}, \quad (15)$$

$$\langle l_m|\hat{j}^z_a(\mathbf{k})|l_n\rangle = \frac{e_a}{\mu_a}\left(p^z_n - \frac{1}{2}\hbar k_z\right)\langle l_m|e^{-ik_y y}|l_n\rangle,$$

$$\langle l_m|e^{-ik_y y}|l_n\rangle = l!\,(l_n!\,l_m!)^{-1/2}e^{-X/2}L_l^{|l_m-l_n|}(X)$$

$$\times\left\{\sqrt{\frac{c\hbar}{2|e_a|B_0}}\left[k_x\operatorname{sign}e_a(l_n-l_m)-ik_y\right]\right\}^{|l_m-l_n|}. \quad (16)$$

In the last equation $l = \min(l_m, l_n)$, $X = c\hbar k_\perp^2/2|e_a|B_0$, $k_\perp^2 = k_x^2 + k_y^2$, and

$$L_s^r(X) = \sum_{t=0}^{\infty}\binom{n+r}{n-t}\frac{1}{t!}(-X)^t$$

is a Laguerre polynomial.

Equations (14) to (16) enable us to determine the transition-current correlation function. We get thus, for the case of a spatially uniform distribution for which the density matrix $f_a(p_z, n)$ is independent of y_0

$$I_{ij}(\omega, \mathbf{k}; \omega', \mathbf{k}')$$

$$= \delta(\omega+\omega')\,\delta(\mathbf{k}+\mathbf{k}')\int dp_z\sum_{n=0}^{\infty}N_a f_a(p_z, n)\,2\pi\frac{\hbar|e_a|B_0}{c}$$

$$\times\sum_{m=0}^{\infty}\Big\{\delta\Big(\omega-[m-n]\,\Omega_a - \frac{p^z_n k_z}{\mu_a} - \frac{\hbar k_z^2}{2\mu_a}\Big)\langle n|\hat{j}^i_a(\mathbf{k})|m\rangle\langle m|\hat{j}^j_a(-\mathbf{k})|n\rangle$$

$$+\,\delta\Big(\omega+[m-n]\,\Omega_a - \frac{p^z_n k_z}{\mu_a} + \frac{\hbar k_z^2}{2\mu_a}\Big)\langle n|\hat{j}^j_a(-\mathbf{k})|m\rangle\langle m|\hat{j}^i_a(\mathbf{k})|n\rangle\Big\}. \quad (17)$$

This equation determines the fluctuations of the random transition current and, combined with Eq. (13), also the electromagnetic field fluctuations.

The fluctuation formulae can appreciably be simplified in the classical limit $\hbar = 0$. We can in that case use the asymptotic representation of the Laguerre polynomials and we have

$$\langle l_m|\hat{j}^z_a(\mathbf{k})|l_n\rangle = e_a v_z(-1)^{l_m-l_n}J_{|l_m-l_n|}\left(\frac{k_\perp v_\perp}{\Omega_a}\right)$$

$$\times\left[i(k_y/k_\perp)+(k_x/k_\perp)\operatorname{sign}e_a(l_m-l_n)\right]^{|l_m-l_n|}, \quad (18)$$

$$\langle l_m|\hat{j}^x_a(\mathbf{k})|l_n\rangle = |e_a|\,\Omega_a(-1)^{l_m-l_n}i\frac{\partial}{\partial k_y}\Big\{J_{|l_m-l_n|}\left(\frac{k_\perp v_\perp}{\Omega_a}\right)$$

$$\times\left[i(k_y/k_\perp)+(k_x/k_\perp)\operatorname{sign}e_a(l_m-l_n)\right]^{|l_m-l_n|}\Big\}, \quad (19)$$

$$\langle l_m|\hat{j}^y_a(\mathbf{k})|l_n\rangle = \frac{i}{2}e_a v_\perp(-1)^{l_m-l_n}\Big\{\Big[i\frac{k_y}{k_\perp} + \frac{k_x}{k_\perp}\operatorname{sign}e_a(l_m-l_n+1)\Big]^{|l_m-l_n+1|}$$

$$\times J_{|l_m-l_n+1|}\left(\frac{k_\perp v_\perp}{\Omega_a}\right) - J_{|l_m-l_n-1|}\left(\frac{k_\perp v_\perp}{\Omega_a}\right)$$

$$\times\Big[i\frac{k_y}{k_\perp}+\frac{k_x}{k_\perp}\operatorname{sign}e_a(l_m-l_n-1)\Big]^{|l_m-l_n-1|}\Big\}, \quad (20)$$

where $v_z = p_n^z/\mu_a$, $v_\perp = (2E_n^\perp/\mu_a)^{1/2}$, $J_l(z)$ —a Bessel function.

Bearing in mind the fact that the matrix elements (18) and (19) do not change when we interchange the states m and n, while the matrix element (20) changes sign, we can use these equations to write down an equation which determines the fluctuations of the random currents:

$$I_{ij}(\omega, \mathbf{k}; \omega', \mathbf{k}') = \delta(\omega+\omega')\,\delta(k_z+k_z')\,\delta(k_x+k_x')\sum_a e_a^2 N_a$$

$$\times\int_{-\infty}^{+\infty}dp_z\int_0^{\infty}p_\perp dp_\perp\int_{-\infty}^{+\infty}dy_0 f_a(p_z, p_\perp, y_0)\sum_{l=-\infty}^{+\infty}\delta(\omega+l\Omega_a$$

$$-k_z v_z)\exp\{-i(k_y+k_y')y_0\}F_i^{(1)}(k_x, k_y)F_j^{(2)}(k_x, k_y'), \quad (21)$$

$$F_x^{(1)}(k_x, k_y) = F_x^{(2)}(k_x, k_y) = v_\perp(-1)^l\Big[i\frac{k_y}{k_\perp} + \frac{k_x}{k_\perp}\operatorname{sign}(le_a)\Big]^{|l|}\Big\{i\frac{k_y}{k_\perp}J'_{|l|}\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\operatorname{sign}e_a$$

$$-\frac{k_x}{k_\perp}\frac{\Omega_a}{k_\perp v_\perp}lJ_{|l|}\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\Big\}, \quad (22)$$

$$F_y^{(1)}(k_x, k_y) = F_y^{(2)}(k_x,k_y)(-1)$$

$$= iv_\perp(-1)^l\frac{1}{2}\Big\{J_{|l+1|}\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\Big[i\frac{k_y}{k_\perp} + \frac{k_x}{k_\perp}\operatorname{sign}(e_a[l+1])\Big]^{|l+1|}$$

$$-\Big[i\frac{k_y}{k_\perp}+\frac{k_x}{k_\perp}\operatorname{sign}(e_a[l-1])\Big]^{|l-1|}J_{|l-1|}\left(\frac{k_\perp v_\perp}{\Omega_a}\right)\Big\}, \quad (23)$$

$$F_z^{(1)}(k_x, k_y) = F_z^{(2)}(k_x, k_y) = v_z(-1)^l J_{|l|}(k_\perp v_\perp/\Omega_a)$$

$$\times\Big[i\frac{k_y}{k_\perp}+\frac{k_x}{k_\perp}\operatorname{sign}(e_a l)\Big]^{|l|}. \quad (24)$$

In the particular case of distributions which are independent of y_0, the projection of the center of the Larmor orbit on the y axis, Eq. (21) becomes of the form $(k_y = 0)$

$$I_{ij}(\omega, \mathbf{k}; \omega', \mathbf{k}') = \delta(\omega+\omega')\,\delta(\mathbf{k}+\mathbf{k}')\sum_a N_a e_a^2$$

$$\times 2\pi\int_{-\infty}^{+\infty}dp_z\int_0^{\infty}p_\perp dp_\perp f_a(p_z, p_\perp)\sum_{l=-\infty}^{+\infty}\delta(\omega-l\Omega_a-k_z v_z)\pi_{ij}, \quad (25)$$

where the tensor π_{ij} has the following components

$$\pi_{xx} = v_\perp^2 [l(\Omega_a / k_\perp v_\perp) J_l(k_\perp v_\perp / \Omega_a)]^2,$$
$$\pi_{yy} = v_\perp^2 [J_l'(k_\perp v_\perp / \Omega_a)]^2, \qquad \pi_{zz} = v_z^2 J_l^2(k_\perp v_\perp / \Omega_a),$$
$$\pi_{xy} = -\pi_{yx} = i v_\perp^2 l J_l(k_\perp v_\perp / \Omega_a) J_l'(k_\perp v_\perp / \Omega_a)(\Omega_a / k_\perp v_\perp)\operatorname{sign} e_a,$$
$$\pi_{xz} = \pi_{zx} = v_\perp v_z l J_l^2(k_\perp v_\perp / \Omega_a)(\Omega_a / k_\perp v_\perp)\operatorname{sign} k_x,$$
$$\pi_{zy} = -\pi_{yz} = i v_\perp v_z J_l(k_\perp v_\perp / \Omega_a) J_l'(k_\perp v_\perp / \Omega_a)\operatorname{sign}(e_a k_x). \tag{26}$$

This tensor is exactly the same as the one arising in the theory of the dielectric susceptibility of a plasma in a constant magnetic field.[1]

Equations (25) and (26) combined with (13) enable us to determine the classical fluctuations of the electromagnetic field in a plasma in a strong magnetic field. Stepanov and Kitsenko[6] have given general formulae for the dielectric-constant tensor of a plasma with particle-distribution functions of the form $f(p_z, p_\perp)$. As an application of our equations we obtain an expression for the fluctuations of the Coulomb field. We can then take

$$A_{il}^{-1}(\omega, \mathbf{k}) = \frac{k_i k_l}{k^2} \frac{c^2}{\omega^2} \frac{k^2}{k_s k_r \varepsilon_{sr}(\omega, \mathbf{k})}. \tag{27}$$

In that case*

$$(E_i E_j)_{\omega, \mathbf{k}} = \sum_a \frac{(4\pi e_a)^2 N_a k_i k_j}{|k_s k_r \varepsilon_{sr}(\omega, \mathbf{k})|^2} \times 2\pi \int_{-\infty}^{+\infty} dp_z \int_0^{\infty} dp_\perp p_\perp f_a(p_z, p_\perp) \sum_{l=-\infty}^{+\infty} \delta(\omega - l\Omega_a - k_z v_z) J_l^2\left(\frac{k_\perp v_\perp}{\Omega_a}\right). \tag{28}$$

We point out at this juncture that an expression similar to the one occurring on the right-hand side of Eq. (28) occurred earlier when the collision integral for charged particles in a strong magnetic field was written down.[7,4] It is thus possible to express such a collision integral directly in terms of the fluctuations of the electromagnetic field.

We note in conclusion that the equations obtained here can also be used in the case where the particle-distribution function in the plasma depends on the coordinates and the time. A correct description of the fluctuations will then, clearly, be given for frequencies and wave numbers larger than the characteristic frequencies and wave numbers which determine the change in the non-equilibrium distribution function.

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*The sum over l in Eq. (28) can also be written in the form of the following integral:

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} dx J_0\left(2 \frac{k_\perp v_\perp}{\Omega_a} \sin \frac{\Omega_a x}{2}\right) \cos\{x(\omega - k_z v_z)\}.$$

APPENDIX

The author has shown[8] that when there are no strong fields in the plasma the collision integral for charged particles is of the form

$$J_a = \frac{\partial}{\partial p_a^i}\left(D_{ij} \frac{\partial f_a}{\partial p_a^j}\right) - \frac{\partial}{\partial p_a^i}(A_i f_a), \tag{A.1}$$

where

$$D_{ij} = \sum_b N_b \int d\mathbf{p}_b f_b \int \frac{d\mathbf{k}}{(2\pi)^3} \left(\frac{4\pi e_a e_b}{c^2}\right)^2 \pi k_i k_j \delta(\mathbf{k}\mathbf{v}_a - \mathbf{k}\mathbf{v}_b) \,|\, v_a^r A_{rl}^{-1}(\mathbf{k}\mathbf{v}_a, \mathbf{k}) v_b^l \,|^2,$$

$$A_i = \sum_b N_b \int d\mathbf{p}_b \frac{\partial f_b}{\partial p_b^j} \int \frac{d\mathbf{k}}{(2\pi)^3} \left(\frac{4\pi e_a e_b}{c^2}\right)^2 \pi k_i k_j \delta(\mathbf{k}\mathbf{v}_a - \mathbf{k}\mathbf{v}_b) \,|\, v_a^r A_{rl}^{-1}(\mathbf{k}\mathbf{v}_a, \mathbf{k}) v_b^l \,|^2.$$

It is of interest to express the diffusion coefficients D_{ij} and the friction coefficient A_i of Eq. (A.1) in terms of quantities which determine the electromagnetic properties of the plasma. To do this we note that

$$\left(e_a^2 \left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_a \mathbf{B}]\right)_i \left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_a \mathbf{B}]\right)_j\right)_{\mathbf{k}\mathbf{v}_a, \mathbf{k}} = \sum_b \left(\frac{4\pi e_a e_b}{c^2}\right)^2 \times N_b \int d\mathbf{p}_b f_b \delta(\mathbf{k}\mathbf{v}_a - \mathbf{k}\mathbf{v}_b) k_i k_j \,|\, v_a^r A_{rl}^{-1}(\mathbf{k}\mathbf{v}_a, \mathbf{k}) v_b^l \,|^2.$$

Using this relation we get

$$D_{ij} = \int \frac{d\mathbf{k}}{(2\pi)^3} \pi \left(e_a^2 \left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_a \mathbf{B}]\right)_i \left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_a \mathbf{B}]\right)_j\right)_{\mathbf{k}\mathbf{v}_a, \mathbf{k}} \tag{A.2}$$

We note further that when there are no strong fields the following relation holds[1] for the complex dielectric constant

$$\varepsilon_{ij}(\omega, \mathbf{k}) - \varepsilon_{ji}^*(\omega, \mathbf{k}) = -2\pi i \sum_a \frac{4\pi e_a^2}{\omega^2} N_a \int d\mathbf{p}_a \left(\mathbf{k} \frac{\partial f_a}{\partial \mathbf{p}_a}\right) v_a^i v_a^j \delta(\omega - \mathbf{k}\mathbf{v}_a),$$

which enables us to write the friction coefficient in the form

$$A_i = \frac{i}{2} \int \frac{d\mathbf{k}}{(2\pi)^3} k_i \{\varepsilon_{rt}(\mathbf{k}\mathbf{v}_a, \mathbf{k}) - \varepsilon_{tr}^*(\mathbf{k}\mathbf{v}_a, \mathbf{k})\} \times 4\pi e_a^2 \left(\frac{\mathbf{k}\mathbf{v}_a}{c^2}\right)^2 v_a^l A_{lr}^{-1}(\mathbf{k}\mathbf{v}_a, \mathbf{k}) v_a^s A_{st}^{*-1}(\mathbf{k}\mathbf{v}_a, \mathbf{k}). \tag{A.3}$$

We denote by $\mathbf{E}^{(a)}(\omega, \mathbf{k})$ the Fourier component of the electrical field produced in the plasma by particle a. It is clear that in our case such a component differs from zero only for $\omega = \mathbf{k} \cdot \mathbf{v}_a$ and is of the form

$$E_i^{(a)}(\mathbf{k}\mathbf{v}_a, \mathbf{k}) = -(4\pi i e_a / c^2)\, \mathbf{k}\mathbf{v}_a A_{ij}^{-1}(\mathbf{k}\mathbf{v}_a, \mathbf{k}) v_j^a.$$

Taking this formula into account we can write Eq. (A.3) in the form

$$A_i = \frac{i}{2}\int \frac{d\mathbf{k}}{(2\pi)^3}\,\frac{k_i}{4\pi}\,\{\varepsilon_{rt}(\mathbf{kv}_a, \mathbf{k}) - \varepsilon_{tr}^*(\mathbf{kv}_a, \mathbf{k})\}\,E_r^{(a)}(\mathbf{kv}_a, \mathbf{k})\,E_t^{(a)*}(\mathbf{kv}_a, \mathbf{k}). \qquad (A.4)$$

Bearing in mind that the energy dissipated per unit time by the field $\mathbf{E}^{(a)}$ is equal to[1]

$$\frac{i}{2}\int \frac{d\mathbf{k}}{(2\pi)^3}\,\frac{\mathbf{kv}_a}{4\pi}\,\{\varepsilon_{rt}^*(\mathbf{kv}_a, \mathbf{k}) - \varepsilon_{tr}(\mathbf{kv}_a, \mathbf{k})\}\,E_r^{(a)}(\mathbf{kv}_a, \mathbf{k})\,E_t^{(a)*}(\mathbf{kv}_a, \mathbf{k}),$$

we can say that Eq. (A.4) determines the force braking the particle a or, which comes to the same, the change in its momentum per unit time when we take the origin of the electromagnetic field accompanying the motion of the particle a into account.

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