THEORY OF REACTIONS INVOLVING THE FORMATION OF THREE PARTICLES NEAR THRESHOLD. THE τ DECAY

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It is shown that for an arbitrary reaction involving the formation of three particles one can separate (with an accuracy to terms which are linear in energy) the long-range interaction contribution, which is proportional to the pair-interaction amplitudes and cannot be expanded into a series in the above-threshold momenta. A separation of this type allows us to determine the scattering amplitude for unstable particles at zero energy by analyzing the reactions in which they are produced. The reactions $K^+ \to 2\pi^+ + \pi^-$ and $2\pi^0 + \pi^+$ are considered in detail.

1. INTRODUCTION

REACTIONS involving formation of three particles are at present the only practical means of studying interactions between unstable particles. However, it is often impossible to extract information on the amplitudes of interactions of unstable particles from data on the energy and angular distributions in the reactions by which they are produced, owing to the complex character of the production and subsequent three-particle interaction. The only exceptions are cases of sharply pronounced resonances, so that experimental investigations frequently reduce to searches for such resonances.

One of the most important theoretical problems in this field is to ascertain whether it is possible to gain unambiguous information on the amplitude of pair interactions from an analysis of reactions in which three particles are produced. This question is the subject of many papers, most noteworthy of which is that of Chew and Low,[1] who proposed to determine the pair interaction amplitude by analytic continuation of the amplitude of particle production in the momentum transfer variable.

Several methods have been proposed[2-4] for determining the zero-energy amplitude of scattering on stable particles from an analysis of the reactions near threshold. It was shown earlier[2] that the correlation between the momenta of the produced particles depends essentially on the interaction in the final state and can serve as a means of determining the scattering amplitudes.

We shall show in the present paper that it follows even from the results of [2] that the energy distribution of the produced particles also depends appreciably on the scattering amplitudes of the particle pairs.

In view of the presence of interaction in the final state, it becomes impossible to expand the reaction probability in the momenta of the produced particles and to retain only a few terms at low energy. The probability of the reaction depends in essential fashion on the ratios of the particle momenta to their possible maximum value at specified full energy. Accurate to quantities of order $(kr_0)^2$, where k^2 is the mean square of the momentum of the produced particles and r_0 is the interaction radius, this dependence can be determined and the reaction probability expressed in terms of the pair-interaction amplitudes at zero energy and a small number of constant parameters [formulas (5) and (6)]. The additional parameters are due to the interaction at small distances and in the p state of relative motion. The result obtained allows us in principle to determine the pair interaction amplitudes from the momentum distribution of the particles in their production reaction.

In Sec. 3 we rederive the results of [2] by analyzing the analytic properties of the three-particle production amplitude. This derivation is simple and clear.

In Sec. 4 we obtain the distribution over the momenta of the pions produced in the τ and τ' decays ($K^+ \to 2\pi^+ + \pi^-$, $K^+ \to 2\pi^0 + \pi^+$) as a function of the pion-pion scattering amplitudes, a_2 and a_0, and one unknown constant parameter in each reaction [formulas (18), (19), (16) and (13)]. The dependence of the probabilities of both decays on the relative-motion energies of similarly charged pions is calculated [formula (21) and Fig. 6], as is the dependence on their energies in the K^+ rest system [formula (22) and Fig. 7, the analog of the angular distribution].

The best way of obtaining the scattering amplitude is to study the dependence of the decay probability either on the energy difference in any of the decays or on the relative-motion energy in the $K^+ \to 2\pi^0 + \pi^+$ decay.

2. FORMULATION OF RESULTS

The main result of [2] is that the wave function of three spinless particles [masses m_1, m_2, m_3, momenta p_1, p_2, p_3, and relative pair momenta $\mathbf{k}_{12}$, $\mathbf{k}_{13}$, $\mathbf{k}_{23}$ – $\psi_{k_{12},k_{13},k_{23}}(r_1, r_2, r_3)$] has at low energies ($k_{il}r_0 \ll 1$ where r_0 is the interaction radius) in the region $\rho_{12} \sim \rho_{13} \sim \rho_{23} \sim r_0$ ($\rho_{il} = |\mathbf{r}_i - \mathbf{r}_l|$), accurate to terms of order $k_{il}^2 r_0^2$, the form

$$\psi_{k_{12}, k_{13}, k_{23}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = C(k_{12}, k_{13}, k_{23})\, \psi_{0,0,0}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) + \psi' \text{ (1)}, \tag{1}$$

where $\psi_{0,0,0}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ is the exact wave function of the three particles at zero energy. $C(k_{12}, k_{13}, k_{23})$ is a standard factor, which depends only on k_{il} and on the zero-energy scattering amplitudes of the particle pairs, a_{il}, which will be written out below. ψ' tends to zero as $k_{li} \to 0$ and contains the dependence on the cosines of the angles between $\mathbf{p}_3$ and $\mathbf{k}_{12}$, $\mathbf{p}_2$ and $\mathbf{k}_{13}$, and $\mathbf{p}_1$ and $\mathbf{k}_{23}$, raised to a power not higher than the first.

Formula (1) follows almost directly from simple considerations, connected with the penetrability of the centrifugal barrier.[2] The factor $C(k_{12}, k_{13}, k_{23})$ takes into account the interaction between particles in configurations such that the distance between two particles is on the order of r_0 and the third is outside the force radius. It is natural for the contribution from the interaction in such configurations to be expressed in terms of the particle-pair scattering amplitudes.

If there are no interactions in such configurations, particles with non-vanishing relative angular momenta can penetrate the region $\rho_{12} \sim \rho_{13} \sim \rho_{23} \sim r_0$ only by overcoming the centrifugal barrier, the penetrability of which is $k_{12}^\Lambda p_3^L$, $k_{13}^\Lambda p_2^L$, or $k_2^\Lambda p_1^L$, where Λ is the angular momentum of the particle pair and L the momentum of the third particle about the center of gravity of the first two. Therefore, for example in the case of a state with zero total momentum, the only states possible (accurate to terms quadratic in the momenta) are those with $L = \Lambda = 1$, and consequently states where the dependence on the angles between the momenta contains the first powers of the cosine.

This is precisely why the function ψ', which contains the contribution of the interactions in configurations other than those indicated above, depends only on the first powers of the cosines. We shall henceforth confine ourselves to states with zero total momentum. In this case, obviously, the most general expression for ψ' has the form

$$\psi' = k_{12}^2\psi_3' + k_{13}^2\psi_2' + k_{23}^2\psi_1', \tag{2}$$

where ψ_i' is independent of the particle momenta.

An expression for $C(k_{12}, k_{13}, k_{23})$ is obtained directly from formulas (26) and (32) of [2]:

$$\begin{aligned} C(k_{12}, k_{13}, k_{23}) = {} & 1 - ik_{12}a_{12} - ik_{13}a_{13} - ik_{23}a_{23} \\ & + a_{12}a_{13}[J_1(k_{12}) + J_1(k_{13})] + a_{12}a_{23}[J_2(k_{12}) \\ & + J_2(k_{23})] + a_{13}a_{23}[J_3(k_{13}) + J_3(k_{23})], \end{aligned} \tag{3}$$

$J_i(k_{il})$ are standard real functions of order k_{il}^2, calculated in [2] [formula (31)] and given below.

Formula (3) differs from the sum of (26) and (32) in [2] in that summation is carried over l and two terms dependent on k_{12} are added. In [2] these terms were left out, for only the dependence on the angle between k_{12} and p_3 was of interest. Addition of these two terms to $C(k_{12}, k_{13}, k_{23})$ ensures independence of ψ' of the higher powers of the cosines of the angles between any of the directions listed above.

The physical meaning of the individual terms in (3) is exceedingly simple. The terms linear in k_{il} correspond to a single account of the interaction. Terms containing $J_i(k_{il})$ are due to two successive particle interactions. These two types of terms can be obtained from the Feynman diagrams shown in Figs. 1 and 2 respectively.[5]

With the aid of (1), (2), and (3) it is easy to calculate the amplitude of a reaction producing three particles with zero total momentum, accurate to terms quadratic in the momenta:

$$\begin{aligned} M(k_{12}, k_{13}, k_{22}) = M_0\{ & C^*(k_{12}, k_{13}, k_{23}) + \alpha_1 k_{23}^2 \\ & + \alpha_2 k_{13}^2 + \alpha_3 k_{12}^2\}, \end{aligned} \tag{4}$$

where M_0 is the amplitude at zero energy. The reaction cross section, averaged over all the orientations of the plane containing the momenta of the resultant particles relative to the momentum of the incident particle, has therefore the form

$$\begin{aligned} d\sigma = M_0^2\{ & 1 + 2a_{12}a_{13}[k_{12}k_{13} + J_1(k_{12}) + J_1(k_{13})] \\ & + 2a_{12}a_{23}[k_{12}k_{23} + J_2(k_{23}) + J_2(k_{12})] + 2a_{13}a_{23}[k_{13}k_{23} \\ & + J_3(k_{23}) + J_{13}(k_{13})] + \alpha_3 k_{12}^2 + \alpha_1 k_{23}^2 + \alpha_2 k_{13}^2\}\, d\Gamma. \end{aligned} \tag{5}$$

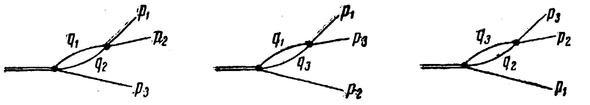

FIG. 1

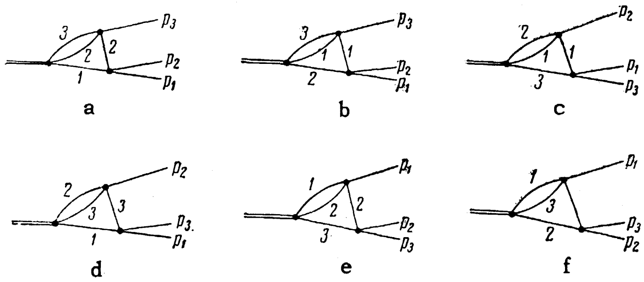

FIG. 2

Terms such as k_{12}^2, k_{13}^2, or k_{23}^2 in $|C(k_{12}, k_{13}, k_{23})|^2$ result in overdetermination of the coefficients α_i, which are not calculated anyway.

It follows from Eq. (5) that a detailed study of the distribution of the momenta of the produced particles will in principle yield the pair interaction amplitudes, since the contribution due to the interaction in the final state depends in a complicated manner on k_{il}, and can therefore be separated from the simple terms such as $\alpha_1 k_{23}^2$ etc.

In conclusion, let us write out an explicit expression for $J_i(k_{il})$, obtained by simple transformation from formula (31) of [2]:*

$$J_i(k_{il}) = I_i(x_{il});$$

$$I_i(x) = -2E\sqrt{\frac{m_1m_2m_3}{m_1+m_2+m_3}}\frac{2x\arccos x}{\pi\sqrt{1-x^2}}$$

$$\times\left[\beta_i + \frac{1}{3}x^2(1-4\beta_i)\right];$$

$$\mu_{il} = \frac{m_im_l}{m_i+m_l},$$

$$\beta_1 = m_1(m_1+m_2+m_3)/(m_1+m_2)(m_1+m_3),$$

$$x_{il} = k_{il}/\sqrt{2\mu_{il}E}, \tag{6}$$

where E is the kinetic energy of the produced particles.

Expression (5) for $d\sigma$ has the following interesting property. If we examine it as a function, say, of $E_{12} = k_{12}^2/2\mu_{12}$ it has at first glance a root singularity at $E_{12} = 0$, owing to the terms $k_{12}k_{13}$, and $k_{12}k_{23}$ ($k_{12} = \sqrt{2\mu_{12}E_{12}}$). However, $J_1(k_{12})$ and $J_2(k_{12})$ also have singularities at $E_{12} = 0$:

$$J_1(k_{12}) \approx -k_{12}\sqrt{2\mu_3E}\,\beta_1, \quad J_2(k_{12}) \approx -k_{12}\sqrt{2\mu_3E}\,\beta_2;$$
$$1/\mu_3 = 1/m_3 + 1/(m_1+m_2) \tag{7}$$

*A misprint in [2] has been corrected in Eq. (6).

and in addition

$$k_{13} = \frac{m_3}{m_1+m_3}k_{12} - \beta_1 p_3, \qquad k_{23} = \frac{m_3}{m_2+m_3}k_{12} - \beta_2 p_3. \tag{8}$$

When $k_{12} = 0$ we have $p_3 = \sqrt{2\mu_3E}$, since $k_{12}^2/2\mu_{12} + p_3^2/2\mu_3 = E$, and consequently the singular parts in $K_{12}k_{13} + J_1(k_{12})$ and $k_{12}k_{23} + J_2(k_{12})$ cancel out, and $d\sigma$ does not have a root singularity at $E_{12} = 0$. The situation is obviously the same when $E_{13} = 0$ and $E_{23} = 0$. The situation here is similar to that in electrodynamics in the infrared region, where singularities of the second-approximation diagram [the analog of $J_1(k_{12})$] cancel the singularity of the square of the first approximation (the analog of $k_{12}k_{13}$).

3. ANALYTIC PROPERTIES OF THE REACTION AMPLITUDE NEAR THRESHOLD

In the preceding section we leaned exclusively on the results of [2]. In the present section we re-derive these results by using only the analytic properties of the reaction amplitude. We first follow closely Dyatlov's paper.[5] Consider the amplitude of the process corresponding to the diagram shown in Fig. 3a as a function of the invariants s_{15}, s_{34} and s_{12}, s_{13}, s_{23} near threshold:

$$s_{54} = s_{12} + s_{13} + s_{23} - m_1^2 - m_2^2 - m_3^2$$
$$\approx (m_1 + m_2 + m_3)^2 + 2(m_1 + m_2 + m_3)E,$$
$$E \to 0. \tag{9}$$

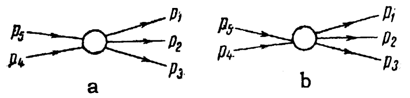

FIG. 3

At sufficiently small E the range of variation of the invariants s_{ik} tends to zero; therefore, if the amplitude has no singularities with respect to any of the invariants near their threshold values, it can be expanded in powers of the deviations from these threshold values. Actually, as a function of the invariants s_{12}, s_{13}, s_{23}, and s_{45}, the amplitude has singularities precisely at the threshold values. These singularities correspond to the diagrams shown in Figs. 1 and 4. Generally speaking, the amplitude has no singularities in s_{15} and s_{34} near threshold and can be expanded in a series. Such an expansion results in terms of the type[5] $\mathbf{p}_i^2$ and $\mathbf{p}_5 \cdot \mathbf{p}_i$ or $\mathbf{p}_4 \cdot \mathbf{p}_i$. In the zero-order term of this expansion we should set s_{15} and s_{43} equal to their threshold values, i.e., we should consider the diagram of Fig. 3b instead of 3a.

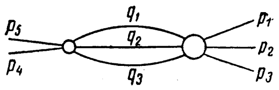

FIG. 4

As was explained in detail in [5], in the first approximation in the above-threshold momenta we can write the amplitude corresponding to the diagram 3b in the form

$$M = M_0(1 + ik_{12}a_{12} + ik_{13}a_{13} + ik_{23}a_{23}),$$
$$k_{il}^2 = [s_{il} - (m_i + m_l)^2]\,\mu_{il} / (m_i + m_l). \qquad (10)$$

As already mentioned, this expression corresponds to an account of the diagrams shown in Fig. 1. We must determine the form of the second-order terms.

We shall show that the amplitude expansion cannot contain terms of the type $k_{12}k_{13}$. Terms of this type have root singularities when $s_{12} = (m_1 + m_2)^2$ and $s_{13} = (m_1 + m_3)^2$ simultaneously, whereas there are no diagrams with such a property. Actually, according to Landau, to find the coefficient of k_{12} it is necessary to integrate over the lines q_1 and q_2 of the diagram of Fig. 1. At the same time (see also [5]) we can neglect the momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ in the remaining parts of the diagram. Under these conditions the diagram as a whole is independent of k_{13} and k_{23}, and therefore cannot contain terms of the indicated type.

Consequently, all that we can add at first glance is an expression of the type

$$\alpha_1 k_{23}^2 + \alpha_2 k_{13}^2 + \alpha_3 k_{12}^2. \qquad (11)$$

This would be correct were the amplitude not to have at $E = 0$ the three-particle singularity shown in Fig. 4. In the presence of this singularity we can have a large number of terms of equal order, such as $\sqrt{E}\, k_{12}$, $k_{12}^2/\sqrt{E}$, ...

Let us examine the three-particle singularity in greater detail. The behavior of the amplitude near this singularity is determined by the integral corresponding to the diagram of Fig. 4, which in the nonrelativistic approximation has the form

$$\int d^3q_1\, d^3q_2\, d^3q_3\, \delta(q_1 + q_3 + q_3) \frac{M(q_1q_2q_3)}{E - E(q_1) - E(q_2) - E(q_3)}$$
$$\times A(q_1q_2q_3 \mid p_1p_2p_3). \qquad (12)$$

If M and A are finite near $E = 0$, we find that the contribution of the three-particle singularity is of the order $E^2 \ln E$, and therefore cannot be significant in terms quadratic in the momenta, i.e., linear in the energy.

Thus, the three-particle singularity can contribute to the terms of interest to us only if $M(q_1q_2q_3)$ or $A(q_1q_2q_3 \mid p_1p_2p_3)$ become infinite in the region of integration. The amplitude $M(q_1q_2q_3)$ cannot have poles in the physical region. The amplitude $A(q_1q_2q_3 \mid p_1p_2p_3)$ of the three-particle interaction can have in the physical region poles corresponding to diagrams similar to those shown in Fig. 5.

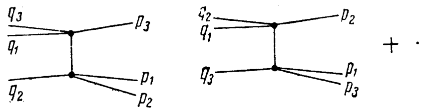

FIG. 5

If we substitute these pole terms in (12) we find that the contribution of the three-particle singularity is of the order $E \ln E$ or $k_{il}^2 \ln E$, i.e., it must be taken into account.

We thus conclude that along with terms written out in (11), it is necessary to take into account the diagrams of Fig. 4, in which A is described by diagrams of Fig. 5. Obviously, these are none other than the diagrams shown in Fig. 2.

4. τ DECAY

In this section we apply the results of the preceding sections to an analysis of the reactions

$$K^+ \to 2\pi^+ + \pi^-, \qquad K^+ \to 2\pi^0 + \pi^+.$$

We shall denote by M^- and M^+ the amplitudes of the first and second reactions, respectively. Particles with like charges will be denoted by the indices 1 and 2.

Unlike [2] we do not confine ourselves to the dependence on the angle between $\mathbf{p}_3$ and $\mathbf{k}_{12}$. Furthermore, the Clebsch-Gordan coefficients in [2] have been incorrectly calculated and are corrected here.

We note first that only one of the three unknown parameters in (4) and (5), namely α_1, α_2, and α_3, remains. By virtue of the symmetry of the amplitude relative to the momenta of particles with like charges, $\alpha_1 = \alpha_2$. By virtue of

$$k_{12}^2 + k_{13}^2 + k_{23}^2 = \tfrac{3}{2}\varkappa^2, \qquad \varkappa^2 = \mu\,[M_K - 3\mu], \qquad (13)$$

the quantity $k_{13}^2 + k_{23}^2$ is expressed in terms of k_{12}^2 and κ^2. The constant term $3\alpha_1\kappa^2/2$ is not essential, since it influences only the normalization of the energy distribution.

The essential difference between the reactions considered here and those described in the earlier sections is that, along with simple scattering, charge exchanges $\pi^+ + \pi^- \rightleftarrows \pi^0 + \pi^0$ are possible in the final state. It is therefore convenient to introduce in place of a_{il} the quantities a_0 and a_2 —the scattering lengths in states with isotopic spins T = 0 and T = 2. In terms of a_0 and a_2, we have for the scattering amplitudes a_s^{++}, a_s^{+-}, a_s^{00}, a_s^{0+}, and a_s^{0-} the following expressions

$$a_s^{++} = a_s^{0+} = a_s^{-0} = a_2,\quad a_s^{+-} = \tfrac{1}{3}(2a_0 + a_2),$$

$$a_s^{00} = \tfrac{1}{3}(2a_2 + a_0).$$

The upper signs obviously denote the charges of the scattered particles: the charge-exchange amplitude is $a_e = (a_2 - a_0)/3$.

It is easy to write out in terms of these amplitudes the contribution to the matrix elements M^- and M^+ from the diagrams shown in Figs. 1 and 2. The contribution from the diagram of Fig. 1 to the matrix element M^- has the form

$$[ik_{12}a_2 + i(k_{13} + k_{23})\tfrac{1}{3}(2a_0 + a_2)]M_0^- + i(k_{13} + k_{23})\tfrac{1}{3}(a_2 - a_0)M_0^+, \qquad (14a)$$

and the contribution to the matrix element M^+ is

$$[ik_{12}\tfrac{1}{3}(2a_2 + a_0) + i(k_{13} + k_{23})a_2]M_0^+ + ik_{12}\tfrac{2}{3}(a_2 - a_0)M_0^-. \qquad (14b)$$

The contribution to M^- from diagrams 2a and 2b is

$$2J(k_{12})[\tfrac{1}{3}a_2(2a_0 + a_2)M_0^- + \tfrac{1}{3}a_2(a_2 - a_0)M_0^+];$$

that from diagrams 2c and 2d is

$$[J(k_{13}) + J(k_{23})][\tfrac{1}{3}(2a_0 + a_2)a_2M_0^- + \tfrac{1}{3}(a_2 - a_0)a_2M_0^+] \qquad (15a)$$

and that from diagrams 2e and 2f is

$$J(k_{13}) + J(k_{23})][\tfrac{1}{3}(2a_0 + a_2)\tfrac{1}{3}(2a_0 + a_2)M_0^- + \tfrac{1}{3}(2a_0 + a_2)\tfrac{1}{3}(a_2 - a_0)M^+ + \tfrac{1}{3}(a_2 - a_0)a_2M_0^+].$$

The contributions from the corresponding diagrams to M^+ are

$$2J(k_{12})\{\tfrac{1}{3}(2a_2 + a_0)a_2M_0^+ + \tfrac{1}{3}(a_2 - a_0)a_2M_0^- + \tfrac{1}{3}(a_2 - a_0)\tfrac{1}{3}(2a_0 + a_2)M_0^- + \tfrac{1}{9}(a_2 - a_0)^2M_0^+\},$$

$$(J(k_{13}) + J(k_{23}))\{a_2\tfrac{1}{3}(2a_2 + a_0)M_0^+ + a_2\tfrac{2}{3}(a_2 - a_0)M_0^-\},$$

$$(J(k_{13}) + J(k_{23}))a_2^2M_0^+, \qquad (15b)$$

where $M_0^{\pm}$ are the matrix elements at zero energy.

The functions $J(k_{il})$ have been defined in (6). In our case (for equal masses) they are all equal to

$$J(k_{il}) = I(x_{il}) = -\frac{\sqrt{3}}{\pi}\varkappa^2\frac{x_{il}\arccos x_{il}}{(1 - x_{il}^2)^{1/2}}\left(1 - \frac{8}{9}x_{il}^2\right),\quad x_{il} = \frac{k_{il}}{\varkappa}. \qquad (16)$$

Collecting terms from (14) and (15) and recognizing that the matrix elements can contain terms in the form αk_{12}^2, we obtain

$$M^- = M_0^-\{1 + ik_{12}a_2 + i(k_{13} + k_{23})\tfrac{1}{3}[2a_0 + a_2 + \rho(a_2 - a_0)] + 2J(k_{12})a_2\tfrac{1}{3}[2a_0 + a_2 + \rho(a_2 - a_0)] + (J(k_{13}) + J(k_{23}))[\tfrac{1}{3}(2a_0 + a_2) + a_2] \times \tfrac{1}{3}[2a_0 + a_2 + \rho(a_2 - a_0)] + \tfrac{1}{3}(a_2 - a_0)a_2\rho] + \alpha_-k_{12}^2\}; \qquad (17a)$$

$$M^+ = M_0^+\{1 + ik_{12}\tfrac{1}{3}[2a_2 + a_0 + 2\rho^{-1}(a_2 - a_0)] + i(k_{13} + k_{23})a_2 + [J(k_{13}) + J(k_{23})] \times [\tfrac{1}{3}a_2(2a_2 + a_0 + 2\rho^{-1}(a_2 - a_0)) + a_2^2] + 2J(k_{12}) \times a_2[\tfrac{1}{3}(2a_2 + a_0) + \tfrac{2}{3}\rho^{-1}(a_2 - a_0)] + 2J(k_{12})\tfrac{1}{9}(a_2 - a_0)^2(1 - 2\rho^{-1}) + \alpha_+k_{12}^2\},$$

$$\rho = M_0^+/M_0^-. \qquad (17b)$$

Squaring (17a) and (17b) and taking account of the fact that conservation of combined parity calls for real matrix elements M_0^- and M_0^+, we find that the probabilities of the decays $K^+ \to 2\pi^+ + \pi^-$ and $K^+ \to 2\pi^0 + \pi^+$ are respectively equal to

$$dW^{(-)} = |M_0^-|^2\{1 + \beta_1[k_{12}(k_{13} + k_{23}) + 2J(k_{12}) + J(k_{13}) + J(k_{23})] + \beta_2[k_{13}k_{23} + J(k_{13}) + J(k_{23})] + \beta_3[J(k_{13}) + J(k_{23})] + \alpha_-k_{12}^2\}d\Gamma, \qquad (18a)$$

$$dW^{(+)} = |M_0^+|^2\{1 + \gamma_1[k_{12}(k_{13} + k_{23}) + 2J(k_{12}) + J(k_{13}) + J(k_{23})] + \gamma_2[k_{13}k_{23} + J(k_{13}) + J(k_{23})] + \gamma_3 2J(k_{12}) + \alpha_+k_{12}^2\}d\Gamma; \qquad (18b)$$

$$\beta_1 = \tfrac{2}{3}a_2(2a_0 + a_2 + \rho(a_2 - a_0)),$$

$$\gamma_1 = \tfrac{2}{3}a_2[2a_2 + a_0 + 2\rho^{-1}(a_2 - a_0)],$$

$$\beta_2 = \tfrac{2}{9}[2a_0 + a_2 + \rho(a_2 - a_0)]^2,\qquad \gamma_2 = 2a_2^2,$$

$$\beta_3 = \tfrac{2}{9}\rho(a_2 - a_0)^2(2 - \rho),\quad \gamma_3 = -\tfrac{2}{9}\tfrac{1}{\rho}(a_2 - a_0)^2(2 - \rho). \qquad (19)$$

We note that (18a) and (18b) differ in structure from (5). In particular, the terms proportional to β_3 and γ_3 have root singularities for $E_{13} = 0$, $E_{23} = 0$, and $E_{12} = 0$ respectively, and these singularities do not cancel. This is a consequence of the possibility of going from one channel to the other via charge exchange, and is a phenomenon of the same type as discussed in [3,4].

Formulas (18) and (19) determine the dependence of the probabilities of both decays on all the variables and make it possible, in principle, to determine the amplitudes a_2 and a_0 from their experimental analysis, in spite of the fact that the parameters α_- and α_+ are not expressed in terms of the scattering amplitudes.

It is most natural to regard $dW^{(-)}$ and $dW^{(+)}$ as a function of the orthogonal coordinates on a Dalitz diagram. It is convenient to choose as one of the variables $x^2 = k_{12}^2/k_m^2$ — the ratio of the square of the relative momentum of like charged pions to its maximum value. k_{12}^2 is simply related to the customarily used variable ϵ, the energy of the third pion:

$$\varepsilon = (M_K^2 + 3\mu^2 - 4k_{12}^2)/2M_K,$$

$$k_m^2 = \frac{1}{4}(M_K^2 - 2\mu M_K - 3\mu^2) \approx \varkappa^2. \qquad (20)$$

For the second variable we choose z, the difference between the energies of like charged pions divided by the maximum value of this difference.

For convenience in the experimental analysis, it is useful to have decay-probability formulas integrated over one of the variables. Integrating with respect to z, we obtain the so-called energy distribution

$$dW^{(-)}(x) = |M_0^-|^2\{1 + \varkappa^2[\beta_1 F_1(x)$$

$$+ \beta_2 F_2(x) + \beta_2 F_3(x) + \alpha_- x^2]\}\, d\Gamma_{x^2}, \qquad (21a)$$

$$dW^{(+)}(x) = |M_0^+|^2\{1 + \varkappa^2[\gamma_1 F_1(x)$$

$$+ \gamma_2 F_2(x) + \gamma_3 2J(x) + \alpha_+ x^2]\}\, d\Gamma_{x^2}. \qquad (21b)$$

The functions $F_1(x)$, $F_2(x)$, $F_3(x)$, and $J(x)$ are plotted in Fig. 6 and are written out in the Appendix. In the calculation of $F_i(x)$ we have left out terms of the form $a + bx^2$, since they enter either into the normalization or in $\alpha_\pm$. As can be seen from Fig. 6, the curves for F_1, F_2, and F_3 differ little from straight lines and it is therefore little likely that it will be possible to determine a_0 and a_2 from an analysis of the energy distribution in the reaction $K^+ \rightarrow 2\pi^+ + \pi^-$ only.

It follows, however, from the same diagram that owing to the presence of the term $2\kappa^2\gamma_3 J(x)$ the energy distribution in the reaction $K^+ \rightarrow 2\pi^0 + \pi^+$ differs appreciably from $\alpha_+ x^2$, and can therefore be used to determine the charge-exchange amplitude. The effect of this term on the energy distribution is also anomalously exaggerated by the fact that $\rho = 1/4$, and consequently

$$\gamma_3 = -\frac{16}{9}(a_2 - a_0)^2.$$

It is clear from all the foregoing that the cor-

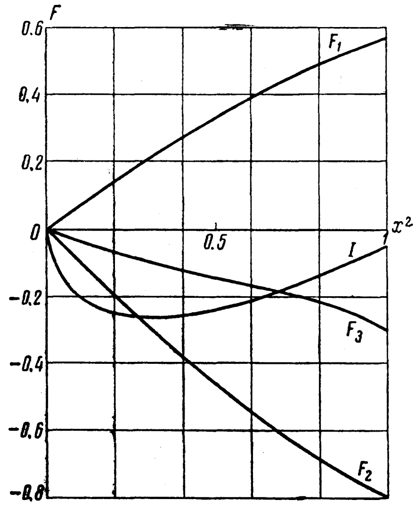

FIG. 6

rections to the energy distribution, calculated in [6,7], cannot be successful since they are reduced to a calculation of first-order diagrams (Fig. 1), which in many cases are almost completely cancelled by the diagrams shown in Fig. 2.

Integrating (18a) and (18b) with respect to x^2, we obtain an analog of the angular distribution

$$dW^{(-)}(z) = |M_0^-|^2\{1 + \varkappa^2[\beta_1\varphi_1(z) + \beta_2\varphi_2(z)$$

$$+ \beta_3\varphi_3(z)]\}\, d\Gamma_z, \qquad (22a)$$

$$dW^{(+)}(z) = |M_0^+|^2\{1 + \varkappa^2[\gamma_1\varphi_1(z) + \gamma_2\varphi_2(z)$$

$$+ \gamma_3\varphi_4(z)]\}\, d\Gamma_z. \qquad (22b)$$

The functions $\varphi_1(z)$, $\varphi_2(z)$, $\varphi_3(z)$, and $\varphi_4(z)$ are shown in Fig. 7 and are given in the Appendix. A study of the dependence of $dW^{(-)}$ and $dW^{(+)}$ on z can serve as a means of determining a_2 and a_0, but statistics on the order of 10^4 events are needed for this case.

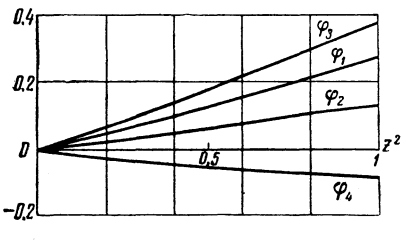

FIG. 7

In conclusion, I am deeply grateful to L. D. Landau and I. T. Dyatlov for useful discussions.

APPENDIX

$$F_1(x) = \int_{z_1}^{z_2} k_{12}(k_{13}+k_{23})\frac{dz}{4\varkappa^2 x\sqrt{1-x^2}} + 2I(x) + F_3(x) = 2I(x) + F_3(x) + \begin{cases} \frac{\sqrt{3}x}{\sqrt{1-x^2}}\left(1-\frac{8}{9}x^2\right) & x^2<\frac{3}{4} \\ 1 & x^2>\frac{3}{4}, \end{cases} \quad \text{(A1)}$$

$$F_2(x) = F_3(x) + \int_{z_1}^{z_2} k_{13}k_{23}\frac{dz}{4\varkappa^2 x\sqrt{1-x^2}} = F_3(x) + \frac{(3-2x^2)^2}{16\sqrt{3}x\sqrt{1-x^2}}\arcsin\frac{2x\sqrt{3(1-x^2)}}{3-2x^2} + \frac{1}{8}|3-2x^2|, \quad \text{(A2)}$$

$$F_3(x) = \int_{z_1}^{z_2}[I(x_{13})+I(x_{23})]\frac{dz}{4\varkappa^2 x\sqrt{1-x^2}} = \begin{cases} -\frac{2}{9}\frac{\arcsin x}{x\sqrt{1-x^2}}\left[1+\frac{\sqrt{3}x^2}{\pi}(3-4x^2)\right]+\frac{1}{6}(3-4x^2) & x^2<\frac{3}{4} \\ -\frac{2}{9}\frac{2\arccos x}{x\sqrt{1-x^2}}\left[1+\frac{\sqrt{3}x^2}{2\pi}(4x^2-3)\right]+\frac{1}{3}(4x^2-3) & x^2>\frac{3}{4}. \end{cases} \quad \text{(A3)}$$

$I(x)$ is given by Eq. (16) in the text, and $z_{1,2} = \mp 2x\sqrt{1-x^2}$.

$$\varphi_1(z) = \int k_{12}(k_{13}+k_{23})\frac{dx^2}{\varkappa^2\sqrt{1-z^2}} + \varphi_3(z) + \varphi_4(z) = \varphi_3(z) + \varphi_4(z) + \frac{3}{16}\frac{(\sqrt{3}+z)^2}{\sqrt{2(1-z^2)}}\arccos\frac{1+\sqrt{3}z}{(\sqrt{3}+z)} + \frac{1}{16}(1+\sqrt{3}z)$$

$$+\begin{cases} \frac{3}{16}\frac{(\sqrt{3}-z)^2}{\sqrt{2(1-z^2)}}\arccos\sqrt{\frac{1}{3}} + \frac{1}{16\sqrt{1-z^2}}(1+2\sqrt{3}z-5z^2), & z<\frac{\sqrt{3}}{2}, \\ \frac{3}{16}\frac{(\sqrt{3}-z)^2}{\sqrt{2(1-z^2)}}\arccos\frac{\sqrt{3}z-1}{(\sqrt{3}-z)} + \frac{\sqrt{3}z-1}{16}, & z>\frac{\sqrt{3}}{2}, \end{cases} \quad \text{(A4)}$$

$$\varphi_2(z) = \varphi_3(z) + \int k_{13}k_{23}\frac{dx^2}{\varkappa^2\sqrt{1-z^2}} = \varphi_3(z) + \frac{3}{8}\begin{cases} \frac{1}{\sqrt{1-z^2}}\left[\frac{2}{3}(2-z^2) - \frac{z^2}{2}\operatorname{arcsh}\frac{4}{3}\right], & z<\frac{\sqrt{3}}{2} \\ \frac{5}{3} - \frac{z^2}{2\sqrt{1-z^2}}\operatorname{arcsh}\frac{2\sqrt{1-z^2}}{z^2}, & z>\frac{\sqrt{3}}{2}, \end{cases} \quad \text{(A5)*}$$

$$\varphi_3(z) = \int (I(x_{13})+I(x_{23}))\frac{dx^2}{\varkappa^2\sqrt{1-z^2}} = -\frac{1}{3\sqrt{3}}\frac{\arccos z}{\sqrt{1-z^2}}\left[1+\frac{(\sqrt{3}-2z)(1+\sqrt{3}z)}{\pi}\right]$$

$$-\frac{1}{3\sqrt{3}}\begin{cases} \frac{\pi-\arccos z}{\sqrt{1-z^2}}\left[1+\frac{(\sqrt{3}+2z)(1-\sqrt{3}z)}{\pi}\right], & z<\frac{\sqrt{3}}{2}, \\ \frac{5\arccos z}{\sqrt{1-z^2}}\left[1-\frac{(\sqrt{3}+2z)(1-\sqrt{3}z)}{5\pi}\right] - \sqrt{3}-2z, & z>\frac{\sqrt{3}}{2}, \end{cases} \quad \text{(A6)}$$

$$\varphi_4(z) = \int I(x)\frac{dx^2}{\varkappa^2\sqrt{1-z^2}} = -\frac{1}{2\sqrt{3}}\frac{\arccos z}{\sqrt{1-z^2}}\left(1+\frac{2z}{3\pi}\right) - \frac{z}{3\sqrt{3}};$$

$$\frac{1}{2}(1-\sqrt{1-z^2}) < x^2 < \frac{1}{2}(1+\sqrt{1-z^2}), \qquad z>0. \quad \text{(A7)}$$

[1] G. F. Chew and F. E. Low, Phys. Rev. **113**, 1380 (1959).

[2] V. N. Gribov, Nucl. Phys. **5**, 633 (1958).

[3] A. A. Ansel'm and V. N. Gribov, JETP **36**, 1890 (1959) and **37**, 501 (1959), Soviet Phys. JETP **9**, 1345 (1959) and **10**, 354 (1960).

[4] V. N. Gribov, JETP **38**, 553 (1960), Soviet Phys. JETP **11**, 400 (1960).

[5] I. T. Dyatlov, JETP **37**, 1330 (1959), Soviet Phys. JETP **10**, 947 (1960).

[6] N. N. Khuri and S. B. Treiman, Phys. Rev. **119**, 1115 (1960).

[7] R. E. Sawyer and K. G. Wale, Phys. Rev. **119**, 1426 (1960).

*Arcsh = $\sinh^{-1}$.

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211