A NEW TREATMENT OF THE GRAVITATIONAL FIELD

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As is well known, the requirement of gauge invariance of the equations for arbitrary spin induces a vector field, where the vector character of the field is prescribed by the locality of the transformation. Similarly, we can derive in as natural a manner a gravitational interaction in terms of a tensor field, by assuming that the parameters of the Lorentz group depend on the coordinates. It can be shown that the known affinity coefficients can be obtained in the case of a spinor field. Gravitational coupling to a scalar field for the Klein-Gordon equation is given by the ordinary affinity coefficients.

SAKURAI[1] has recently proposed an interesting method of introducing vector boson fields. To each conservation law there corresponds a phase transformation which induces a vector field, in the same way as charge conservation, connected with invariance under the transformation $\psi' = \psi e^{i\Lambda(x)}$, leads to the introduction of the electromagnetic field described by the gauge invariant vector potential $A'_\mu = A_\mu + \partial\Lambda/\partial x^\mu$. The isotopic conservation laws (baryon charge, isospin, strangeness) correspond to invariance with respect to definite phase transformations. In contrast to the usual treatment, the phase transformation parameter is in this case not considered constant, but depends on the coordinates in such a way that it has differing values only in points separated by space-like distances. In other words, we impose the natural requirement that the phase transformations in such points be independent of one another.

Since phase transformations with coordinate-dependent parameters do not any more commute with derivative operators, they will give rise to additional terms in the field equations which cannot have any physical meaning. One therefore introduces new vector fields which transform in such a way under the gauge transformation as to cancel these additional terms. This is completely analogous to the procedure one uses in the case of the electromagnetic field. We should like to call attention to the fact that, by analogous considerations, one can introduce not only vector fields, but also tensor fields with zero rest mass (in the case of an exact conservation law), and, in particular, a field with spin 2.

This is a natural way of introducing a field which is in many respects equivalent to the gravitational field and yet follows from considerations which are different from the usual ones of Einstein. We must start with a generalized Lorentz transformation the parameters of which, while always satisfying the condition

$$a_{\mu\nu}a^{\nu\pi} = \delta_\mu^\pi, \tag{1}$$

are not constant, but depend on the coordinates in such a way that for all time-like vectors a^α $(a^\alpha a_\alpha < 0)$

$$a^\alpha \, \partial a_{\mu\nu} / \partial x^\alpha = 0. \tag{2}$$

For space-like a^α the left-hand side of (2) is arbitrary. As a result, the components of the Minkowski metric tensor $g_{\mu\nu}$, viewed locally, will depend in a definite way on the coordinates, conserving everywhere the Galilean values $g^{(0)}_{\alpha\beta}(x)$ (orthogonal system rotating around spatial directions). Since the metric tensor depends on the coordinates, the differentiations in the field equations must be replaced by covariant differentiations which give rise to the appearance of an additional term involving a Christoffel symbol with a $g_{\mu\nu}$ of the particular form mentioned above. At the same time the Lorentz transformation can be given the form of a phase transformation, which emphasizes the analogy to the introduction of other boson fields even more (see below).

For clarity, let us first consider the simplest case of the equation for a scalar field φ, which can be written in covariant form:

$$g^{\alpha\beta} \frac{\partial^2 \varphi}{x\partial^\alpha \, \partial x^\beta} + \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g} g^{\alpha\beta}}{\partial x^\alpha} \frac{\partial \varphi}{\partial x^\beta} - m^2 \varphi = 0. \tag{3}$$

In our case the metric tensor $g_{\alpha\beta}$ in (3) must have the Galilean values everywhere, and its derivative in space-like directions must be different from zero:

$$g^{\alpha\beta}(x) = g^{(0)\,\alpha\beta}(x);$$
$$\sqrt{-g} = 1, \quad a^{\nu}\,\partial g^{\alpha\beta}(x)/\partial x^{\nu} = 0, \quad a^{\nu}a_{\nu} < 0. \tag{4}$$

Using (4) and writing $g^{\alpha\beta}(x) = g^{\alpha\beta}(0) + h^{\alpha\beta}(x)$, we find from (3) that a Lorentz transformation of the above-mentioned type takes the free scalar field equation into the form

$$g^{(0)\,\alpha\beta}\frac{\partial^2\varphi}{\partial x^{\alpha}\,\partial x^{\beta}} + \frac{\partial h^{\alpha\beta}(x)}{\partial x^{\alpha}}\frac{\partial\varphi}{\partial x^{\beta}} - m^2\varphi = 0. \tag{5}$$

In order to remove the additional term involving $\partial h^{\alpha\beta}/\partial x^{\alpha}$, which does not have any physical meaning by itself, we must, according to the basic novel idea, introduce a field with the corresponding gauge group transformations. This field will evidently be related to the field φ in the same way as the linearized gravitational field and must, by gauge invariance, have a vanishing mass.

In the general case of a field with arbitrary spin we can start with the equation

$$\Gamma_i\partial_i\Psi + im\Psi = 0. \tag{6}$$

We give the Lorentz transformation the form of a phase transformation by associating with it the matrix

$$l_{ij} = \exp\left[\frac{i}{2}\,\varepsilon_{sp}h_l^s h_m^p\,(g_{im}\delta_{lj} - g_{il}\delta_{mj})\right], \tag{7}$$

where h_l^s are the unit vectors of the orthogonal coordinates and ϵ_{sp} are the group parameters. Let us now assume that the h_m^l are not constant, but are functions of the coordinates in the sense explained above. Then the transformations (7) will not take Eq. (6) into itself but into

$$\Gamma_i(\partial_i + S\partial_i S^{-1})\,\Psi + im\Psi = 0, \tag{8}$$

where

$$l_{ik}S\Gamma_k S^{-1} = \Gamma_l, \qquad S = \exp\,[i\varepsilon_{ik}h_s^i h_p^k I_{ps}], \tag{9}$$

where the I_{ps} are the infinitesimal generators of the representation of the Lorentz group. In the general case we must therefore also introduce a compensating field Φ_σ which transforms under gauge transformations as

$$\Phi'_\sigma = l_\sigma^\alpha(\Phi_\alpha + S\partial_\alpha S^{-1}). \tag{10}$$

Using the following formula for the derivative of an operator exponential [2]

$$\frac{d}{dt}e^{A(t)} = e^{A(t)}\int_0^1 e^{-sA(t)}A'(t)\,e^{sA(t)}\,ds \tag{11}$$

and the relations of the Lie algebra, we can write the expression for $S\partial_\alpha S^{-1}$ in the form

$$S\partial_\alpha S^{-1} = I_{ml}\frac{\partial\varepsilon_{sl}}{\partial x_\alpha}\int_0^1 \exp\{it\,(\varepsilon_{ms}\delta_{pl} + \varepsilon_{lp}\delta_{sm} + \varepsilon_{sl}\delta_{pm} + \varepsilon_{pm}\delta_{sl})\}\,dt. \tag{12}$$

Here we have made use of the formula [3]

$$e^{\varepsilon_i\chi_i}\chi_k e^{-\varepsilon_i\chi_i} = \langle k|\,e^{\varepsilon_i b^i}\,|\alpha\rangle\,\chi_\alpha, \tag{13}$$

where $\langle k\,|\,b^i\,|\,\alpha\rangle = |\,c_{ik}^{\alpha}\,|$, and the c_{ik}^{α} are the structure coefficients of the associated Lie algebra χ_i. If (13) is substituted in the relation $[\chi_i\chi_j] = c_{ij}^{\alpha}\chi_\alpha$ for the Lie algebra, it leads to the well known Jacobi identity.

In the particular case of the Dirac equation,

$$\Gamma_\alpha = \gamma_\alpha. \tag{14}$$

From this we find the known affinity coefficients obtained earlier [4,5] by covariant differentiation of the spinors. In the case of the Klein-Gordon equation we obtain the expression quoted earlier.

We note that one can take account of the nonlinear character of the gravitational field with the help of interaction terms in the equations for the new field itself, which must be introduced owing to the presence of derivative terms in the equations.

Finally, we call attention to the fact that the conservation law for isotopic spin and, possibly, baryon number are not exact, [6,7] which induces a nonvanishing rest mass in the corresponding boson fields of Sakurai. In analogy, the presence of gauge noninvariant terms (of the type of the cosmological term) in the gravitational field equations can be related to the violation of Lorentz invariance at small distances of the order $\lambda^{-1/2}$.

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223