

THE DISPERSION EQUATION FOR AN EXTRAORDINARY WAVE MOVING IN A PLASMA ACROSS AN EXTERNAL MAGNETIC FIELD

Yu. N. DNESTROVSKIĬ and D. P. KOSTOMAROV

Moscow State University

Submitted to JETP editor May 17, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 1527-1535 (November, 1961)

A general qualitative study has been carried out on the non-relativistic dispersion equation for extraordinary and plasma waves propagating in a plasma transversely to an external magnetic field. Frequency regions are established in which these waves can propagate without damping. An error is pointed out in the conclusion drawn by a number of authors that gaps of zero transmission can exist for waves of a given type in the vicinity of each cyclotron resonance. The laws of behavior that are established are illustrated by the results of a numerical solution of the dispersion equation.

INTRODUCTION

It is well known that two types of waves can exist in a homogeneous unbounded plasma located in a homogeneous external magnetic field H_0 . The propagation direction of these waves is perpendicular to the field H_0 . In the first place, one has a purely transverse wave with its electric vector polarized along the field H_0 (the ordinary wave) and, in the second place, one has waves in which the electric vector is polarized perpendicular to H_0 (the extraordinary and plasma waves). In our previous research,^[1] a general qualitative study was carried out on the dispersion equation for the ordinary wave. The present work is devoted to the investigation of the dispersion equation for the extraordinary and plasma waves.

The equation that connects the frequency ω with the propagation constant k has the following form:

$$D(k, \omega) = k^2 \epsilon_{11} - \omega^2 c^{-2} (\epsilon_{11} \epsilon_{22} - \epsilon_{12} \epsilon_{21}) = 0. \tag{1}$$

Here $\epsilon_{ij}(k, \omega)$ are the components of the dielectric permittivity tensor. In the non-relativistic case and for the unperturbed Maxwellian electron distribution function,

$$\epsilon_{11} = 1 + \frac{1}{4}(A + 2B + C),$$

$$\epsilon_{22} = 1 + \frac{1}{4}(A - 2B + C),$$

$$\epsilon_{12} = -\epsilon_{21} = \frac{1}{4}i(A - C);$$

$$\begin{aligned} A(k, \omega) &= C(k, -\omega) \\ &= \frac{2\omega_0^2}{\omega\omega_H} \sum_{n=-\infty}^{\infty} (\zeta_{n-1}(\mu) + \mu \zeta'_{n-1}(\mu)) \left(n - \frac{\omega}{\omega_H}\right)^{-1} \\ &= \frac{\omega_0^2}{\omega\omega_H \sin(\omega\pi/\omega_H)} \int_0^{2\pi} e^{-\mu(1-\cos\tau)} [1 - \mu(1 - \cos\tau)] \\ &\quad \times \cos\left[\left(\frac{\omega}{\omega_H} - 1\right)(\tau - \pi)\right] d\tau, \end{aligned} \tag{3}$$

$$\begin{aligned} B(k, \omega) &= \frac{2\omega_0^2}{\omega\omega_H} \mu \sum_{n=-\infty}^{\infty} \zeta'_n(\mu) \left(n - \frac{\omega}{\omega_H}\right)^{-1} \\ &= \frac{\omega_0^2 \mu}{\omega\omega_H \sin(\omega\pi/\omega_H)} \int_0^{2\pi} e^{-\mu(1-\cos\tau)} (1 - \cos\tau) \\ &\quad \times \cos\frac{\omega}{\omega_H}(\tau - \pi) d\tau, \end{aligned} \tag{4}$$

where $\omega_0 = \sqrt{4\pi N_0 e^2/m}$ is the plasma frequency, $\beta = eH_0/mc$ is the Larmor frequency, $\mu = Tk^2/m\omega_H^2$, T is the electron temperature in energy units, $\zeta_n(\mu) = e^{-\mu} I_n(\mu)$, and $I_n(\mu)$ is the Bessel function of imaginary argument.

Equation (1) was first obtained by Gross^[2] on the basis of simultaneous consideration of the non-relativistic kinetic equation for electrons and the set of Maxwell equations. It has since been studied by many authors.^[4-10] A characteristic feature of the method employed was the preliminary expansion of the equation in powers of some parameter, assumed to be small. It is usually assumed that $\mu \ll 1$. However, such an approach has a number of important weaknesses, the most

important of which are the following: 1) in replacing the transcendental equation (1) by a simplified algebraic equation, certain roots are lost; 2) the expansions in a small parameter in the vicinity of cyclotron resonances are non-uniform; therefore, the behavior of the roots of the complete and simplified equations can differ significantly in the neighborhood of the resonances. The higher cyclotron resonances are not generally taken into account in the simplified equations.

The analytic formulas thus obtained from the simplified equation have at best a limited region of application, and are seen to be invalid as a rule in the neighborhood of resonances. In attempting to draw general conclusions on the properties of Eq. (1) with the help of formulas of an appropriate type, it is easy to make a mistake. As an example, we shall point out an incorrect conclusion of a number of authors^[2,5] as to the existence (in the neighborhood of a cyclotron resonance) of gaps of zero transmission for the extraordinary and plasma waves. This conclusion is based on the incorrect use (in the neighborhood of the resonance) of formulas obtained by an expansion in terms of some small parameter. In the same way, the curves for the index of refraction of the extraordinary wave, found by Drummond^[8] as a result of numerical solution of the simplified equation, are valid only far from cyclotron resonance.

In particular, the problem of plasma waves should be considered in some detail. By taking it into account that k^2 is generally large for these waves, one can write down the following approximate equation for them:^[2,3]

$$\varepsilon_{11}(k, \omega) = 0 \quad (5)$$

(it can be shown that the transition from Eq. (1) to Eq. (5) corresponds to the assumption of a purely longitudinal character for the plasma wave). For plasma waves, $k \rightarrow \infty$ as $T \rightarrow 0$; in this case, the parameter μ is, for all frequencies, a quantity either of the order of, or larger than, unity. Therefore, the results of the study of plasma waves by expansion of Eq. (1) in powers of μ have a very limited region of application. Thus the formula for the index of refraction of the plasma wave obtained by this method in a number of works^[4-6] is qualitatively valid for $\omega_H < \omega < 2\omega_H$ and is shown to be completely invalid for $\omega > 2\omega_H$.

In the work of Bernstein,^[7] an attempt was made to show the existence of gaps of zero transmission for the plasma wave in the vicinity of each cyclotron resonance, by working with the abbreviated Eq. (5) without having recourse to other simplifying

assumptions. However, in the course of the proof, the author made the following logical error. Equation (5) can be written in the form $k^2 = F(k, \omega)$ [Eq. (48) in^[7]]. For fixed k , the function F is an alternating function of ω , negative in a certain neighborhood of the resonance. Therefore, Bernstein^[7] drew the conclusion that these regions are zero-transmission gaps for the plasma wave, inasmuch as for such frequencies, as it were, Eq. (5) cannot have real roots in k . However, the parameter k in Eq. (5) is in fact not fixed, but is a function of ω , defined by the same dispersion equation (5). The function $F(k(\omega), \omega)$ can remain positive in any arbitrary neighborhood of resonance. Thus the discussions of Bernstein prove nothing.

In the present paper, the authors aimed at carrying out a detailed qualitative study of Eqs. (1) and (5) without additional simplifying assumptions other than the non-relativistic condition. Initially, Eq. (1) is considered from the viewpoint of the determination of the propagation constant k for a given real frequency ω . Regions of the frequency ω are established in which there exist real roots $k = k(\omega)$ (the transmission region of the plasma for waves of a given type). In particular, it is shown that for $\omega_H < \omega < \omega^* < \sqrt{\omega_H^2 + \omega_0^2}$ the dispersion equation has a real root $k(\omega)$, to which corresponds a plasma wave propagating without absorption (ω^* is the point of coincidence of the roots for the extraordinary and plasma waves, located somewhat to the left of the hybrid frequency $\sqrt{\omega_H^2 + \omega_0^2}$). The established rules are illustrated by the results of a numerical solution of Eq. (1) for different values of the electron temperature and the plasma density. It is seen from a numerical calculation that the plasma roots of Eq. (1) can be formed from a solution of the abbreviated Eq. (5) with a high degree of accuracy. The results become invalid only in the vicinity of the hybrid frequency $\sqrt{\omega_H^2 + \omega_0^2}$. It is established that Eq. (1) has an infinite set of complex roots $k = k(\omega)$ in addition to the real roots. In Eq. (5) of the present work, the dispersion equation is considered as an equation which determines the frequency ω as a function of the wave number k . It is shown that there exists an infinite set of real roots $\omega = \omega(k)$, while the complex roots $\omega = \omega(k)$ are lacking in Eq. (1).

1. PRELIMINARY INVESTIGATION OF THE DISPERSION EQUATION

We shall begin our study of Eq. (1) with the study of the case in which one is required to deter-

mine the propagation constant k for a given frequency ω . We introduce the dimensionless variables: $s = k^2 c^2 / \omega^2 = N^2$ is the square of the index of refraction, $\alpha = \omega / \omega_H$, $\beta = \omega_0 / \omega_H$, $\gamma = T / mc^2$; then Eq. (1) takes the form

$$D(s, \alpha, \beta, \gamma) = s [1 + \frac{1}{4}(a + 2b + c)] - [1 + \frac{1}{2}(a + c) + \frac{1}{4}(ac - b^2)] = 0, \tag{6}$$

where $a = a(s, \alpha, \beta, \gamma)$, $b = b(s, \alpha, \beta, \gamma)$, $c = c(s, \alpha, \beta, \gamma)$ are the functions (3) and (4) in the new variables.

We shall seek roots of Eq. (6), $s = s(\alpha, \beta, \gamma)$, for all possible real values of α ($\alpha \neq n$). To begin with, we analyze the real roots (Secs. 1–3), which we do in a fashion similar to [1]. For this purpose we first compare the signs of the function D for $s = 0$ and $s = \pm \infty$, and then compare Eq. (6) and the degenerate equation corresponding to zero temperature of the electrons ($T = 0, \gamma = 0$).

1. By direct calculation, we find that

$$D(0, \alpha, \beta, \gamma) = -\left(1 - \frac{\beta^2}{\alpha(\alpha-1)}\right)\left(1 - \frac{\beta^2}{\alpha(\alpha+1)}\right). \tag{7}$$

At the same time, we can show that the behavior of the function D as $|s| \rightarrow \infty$ is determined by the asymptotic formula

$$D(s, \alpha, \beta, \gamma) \approx s - \frac{\pi\beta^4}{2\sin^2\alpha\pi} \frac{e^{-4s\alpha^2\gamma}}{(s\alpha^2\gamma)^2}. \tag{8}$$

Thus,

$$\lim_{s \rightarrow +\infty} D(s, \alpha, \beta, \gamma) = \infty, \quad \lim_{s \rightarrow -\infty} D(s, \alpha, \beta, \gamma) = -\infty.$$

Taking this into account, we can conclude that when $D(0, \alpha, \beta, \gamma) < 0$ Eq. (6) has an odd number of positive roots and an even number of negative roots and, conversely, when $D(0, \alpha, \beta, \gamma) > 0$, Eq. (6) has an even number of positive and an odd number of negative roots.

2. Substituting $\gamma = 0$ ($T = 0$) in Eq. (6), we obtain the degenerate equation

$$D(s, \alpha, \beta, 0) = s \left[1 - \frac{\beta^2}{\alpha^2 - 1}\right] - \left[1 - \frac{\beta^2}{\alpha(\alpha-1)}\right] \left[1 - \frac{\beta^2}{\alpha(\alpha+1)}\right] = 0. \tag{9}$$

We note three characteristic points of this equation, which correspond to the vanishing of each of the square brackets in (9):

$$\alpha' = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\beta^2}, \quad \alpha'' = \alpha' + 1, \quad \alpha^0 = \sqrt{1 + \beta^2} \quad (\alpha' < \alpha^0 < \alpha'', \alpha^0 > 1). \tag{10}$$

The root $s_0 = s_0(\alpha, \beta)$ of Eq. (9) is positive for $\alpha' < \alpha < \alpha^0$ and $\alpha'' < \alpha$, negative for $0 < \alpha < \alpha'$ and $\alpha^0 < \alpha < \alpha''$. As $\alpha \rightarrow \alpha^0$, we have $|s_0| \rightarrow \infty$.

Equation (6) is non-relativistic, and is therefore meaningful only at sufficiently small values of γ . As $\gamma \rightarrow 0$, the roots of Eq. (6) either approach the root of the degenerate equation or diverge to infinity. The corresponding waves are called extraordinary in the first case and plasma in the second.[6]

2. POSITIVE ROOTS OF EQUATION (6) IN THE NON-RESONANT REGION

We shall investigate the positive roots of Eq. (6) for values of α far from the integral values (non-resonant regions of frequency). In accord with (7), the function $D(0, \alpha, \beta, \gamma)$ changes sign when $\alpha = 1, \alpha = \alpha', \alpha = \alpha''$. Two cases are possible here, $\alpha' < 1$ and $\alpha' > 1$. We shall consider them separately.

1. $\alpha' < 1$ (i.e., $\omega_0^2 < 2\omega_H^2$). Comparing the data given in the first two rows of Table I, for $0 < \alpha < 1$ and $\alpha^0 < \alpha$, we see that the minimum possible number of positive roots of Eq. (6) coincides with the number of positive roots of Eq. (9). It is natural to expect that, at sufficiently small values of γ and values of α belonging to the intervals shown and far from the integral values, the transition from Eq. (9) to Eq. (6) does not lead to the appearance of new positive roots. This conclusion is supported by the results of a numerical solution of Eq. (6) (see Figs. 1–3). Thus, Eq. (6) has no positive roots for $0 < \alpha < \alpha'$ and $\alpha^0 < \alpha < \alpha''$, and has a single positive root (extraordinary wave) for $\alpha' < \alpha < 1$ and $\alpha'' < \alpha$.

We now turn to the interval $1 < \alpha < \alpha^0$. Equation (6) should have a positive root inside this range, close to the root of the degenerate equation (9) (extraordinary wave). Since the total number of positive roots of Eq. (6) is even in this interval, Eq. (6) should still have one positive root which goes to infinity as $\gamma \rightarrow 0$ (the plasma wave). The results of a numerical solution of Eq. (6) show that in the approach to the point α^0 from the left, the roots corresponding to the extraordinary and plasma waves combine. In this case, in the region between the point of coincidence of α^* and α^0 , Eq. (6) no longer has positive roots. The number of positive roots of Eq. (6) is shown in the third row of Table I. The capital letters E and P denote the wave to which the root corresponds, extraordinary or plasma. It should be noted that the data of Table I (as also of Table II) are invalid for the immediate vicinity of integral values of α , at which the formulated conclusions do not follow. The problem of the number of roots and their behavior in the resonant regions will be discussed in Sec. 3.

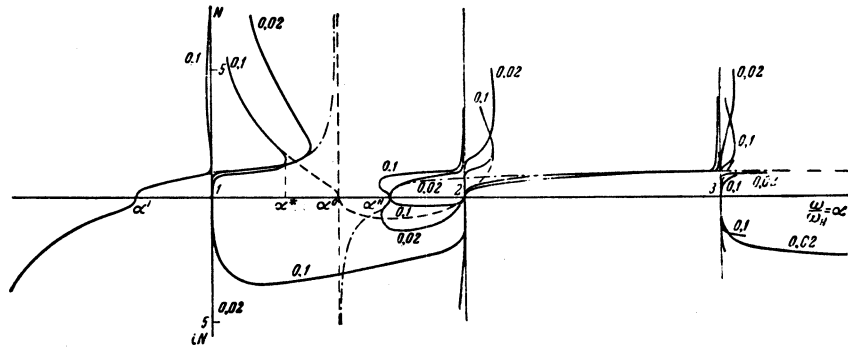


FIG. 1. Dependence of the index of refraction on the frequency for different values of γ with $\beta^2 = 1.2$.

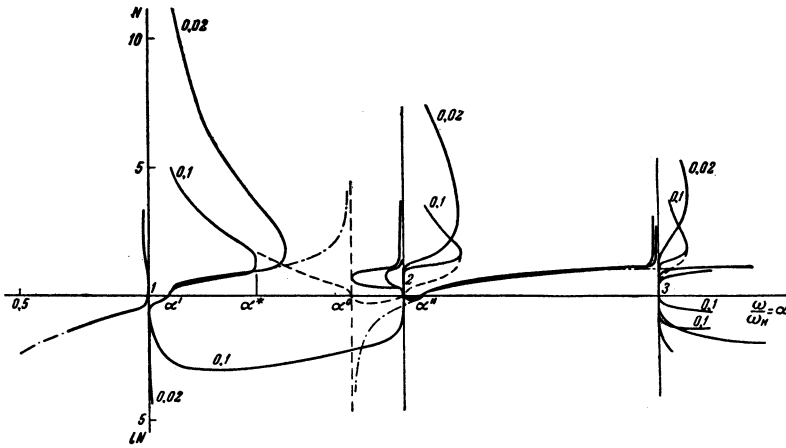


FIG. 2. Dependence of the index of refraction on the frequency for different values of γ with $\beta^2 = 2.25$.

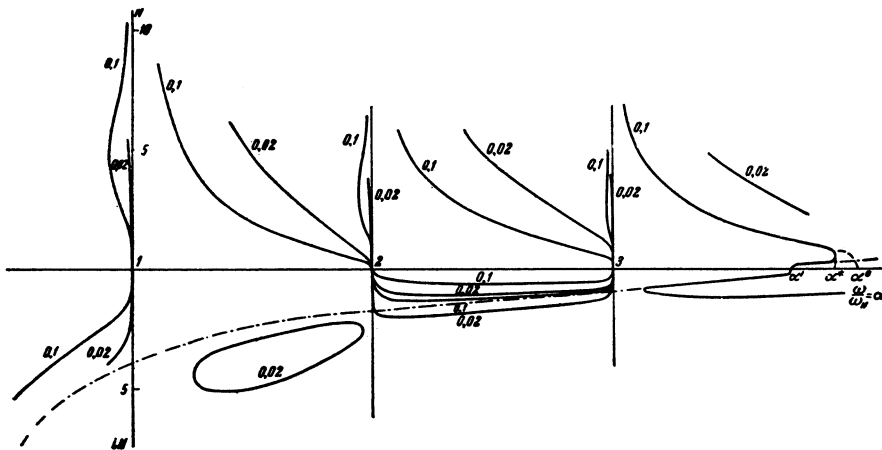


FIG. 3. Dependence of the index of refraction on the frequency for different values of γ with $\beta^2 = 17.6$.

2. $\alpha' > 1$ (i.e., $\omega_0^2 > 2\omega_H^2$). This case can be studied in a way similar to the above. The necessary data and the results of the investigation are given in Table II. As in the first case, propagation of the plasma wave is possible for $1 < \alpha < \alpha^*$ $< \alpha^0$, while only the plasma wave is propagated in the portion $1 < \alpha < \alpha'$.

The results of a numerical solution of Eq. (6) are given in Figs. 1–3. The positive and negative ordinates correspond to $N = \sqrt{s}$ and $iN = \sqrt{-s}$, respectively. The results of numerical solution of the abbreviated dispersion equation for the plasma wave (5) are shown dotted. The equation is written in corresponding dimensionless variables. The

positive root corresponding to the plasma wave and the point α^* where this root becomes equal to the root of the extraordinary wave are clearly seen in all the drawings. This point is located to the left of α^0 . Calculations that have been carried out show that the plasma wave is “quasi-longitudinal” (the ratio of the longitudinal component of the electric field to the transverse component is of the order of 10^{-3}). In this way, the excellent agreement of the results of the solution of the complete and abbreviated equations is explained everywhere except in the vicinity of the point α^* , where the “quasi-longitudinal” character becomes worse.

Table I

Range of α	$0-\alpha'$	$\alpha'-1$	$1-\alpha^*$	$\alpha^*-\alpha^0$	$\alpha^0-\alpha''$	$\alpha''<\alpha$
Parity of the number of positive roots of Eq. (6)	even	odd	even	even	even	odd
Number of positive roots of (9)	0	1	1	1	0	1
Number of positive roots of (6)	0	1,E	2,E,P	0	0	1,E

Table II

Range of α	$0-1$	$1-\alpha'$	$\alpha'-\alpha^*$	$\alpha^*-\alpha^0$	$\alpha^0-\alpha''$	$\alpha > \alpha''$
Parity of the number of positive roots of Eq. (6)	even	odd	even	even	even	odd
Number of positive roots of (9)	0	0	1	1	0	1
Number of positive roots of (6)	0	1,P	2,E,P	0	0	1,E

The temperature of the electrons T and the wave number k enter into the equation for the plasma wave (5) only in the form of the combination $\mu = Tk^2/m\omega_H^2$. Therefore, for those values of α for which the plasma root of Eq. (6) is well described by Eq. (5), the index of refraction of the plasma wave is inversely proportional to the square root of the temperature:

$$N(\alpha, \beta, \gamma)/N(\alpha, \beta, \gamma') = \sqrt{T'/T}. \tag{11}$$

This fact can be employed for the experimental determination of the electron temperature in the plasma by measurement of the phase velocity of the plasma wave. We note that $dk/d\omega < 0$ for the plasma wave, i.e., it possesses anomalous dispersion.

The negative roots of Eq. (6) can be considered in an entirely similar manner. We shall not dwell on the details of this study, but shall limit ourselves to the results of a numerical calculation, which are given in Figs. 1-3.

3. THE BEHAVIOR OF THE POSITIVE ROOTS OF EQUATION (6) IN THE NEIGHBORHOOD OF RESONANCE

The resonance points $\alpha = n$ are the singularities of the function $D(s, \alpha, \beta, \gamma)$. It can be shown that as $\alpha \rightarrow n$ the real roots of Eq. (6) should tend either to zero or to $+\infty$. Proceeding to a more detailed study of the behavior of the roots near the resonances, we first note that when $\alpha \approx n$ the principal role in Eqs. (3) and (4) will be played by two types of terms: terms having a singularity

when $\alpha = n$, and terms which vanish as $\mu \rightarrow 0$. Thus, when $\alpha \approx n$,

$$\begin{aligned} a &\approx \frac{2\beta^2}{\alpha} \left[\frac{\zeta_{n-1}(\mu) + \mu \zeta'_{n-1}(\mu)}{n-\alpha} + \frac{\zeta_0(\mu)}{1-\alpha} \right], \\ c &\approx \frac{2\beta^2}{\alpha} \left[\frac{\zeta_{n+1}(\mu) + \mu \zeta'_{n+1}(\mu)}{n-\alpha} - \frac{\zeta_0}{1+\alpha} \right], \\ b &\approx \frac{2\beta^2}{\alpha} \frac{\mu \zeta'_n(\mu)}{n-\alpha}. \end{aligned} \tag{12}$$

If $n = 1$, then the first term in the formula for a should be omitted.

Substituting (12) in (6), we get a simplified equation describing the behavior of the roots near the n -th resonance. We consider first the large positive roots. For this purpose, we replace the functions $\zeta_n(\mu)$ by their asymptotic forms as $\mu \rightarrow \infty$. As a result, the equation takes the form

$$z^2 + z - \gamma = 0, \tag{13}$$

where $z = \sqrt{2\pi}\beta^{-2}n^2(n-\alpha)\gamma^{3/2}s^{3/2}$. It then follows that on each side of the resonant point $\alpha = n$ there is a positive root that goes to infinity as $\alpha \rightarrow n$. Thus, for $\alpha < n$,

$$s \approx (\beta^2/\sqrt{2\pi}n^2(n-\alpha))^{2/3}\gamma^{-1/3}, \tag{14}$$

and for $\alpha > n$,

$$s \approx (\beta^2/\sqrt{2\pi}n^2(\alpha-n))^{2/3}/\gamma. \tag{15}$$

Inasmuch as the simplified Eq. (5) has a root with the asymptote (15), the plasma wave corresponds to this root. The extraordinary wave corresponds to the root with the asymptote (4), which is located to the left of the resonances (there are similar roots for the dispersion equation for the ordinary wave^[1]).

We now turn to the investigation of the roots that tend to zero as $\alpha \rightarrow n$. For this purpose, we expand the function $\zeta_n(\mu)$ in the simplified equation in a power series in μ and keep only the principal terms. In the consideration of the small roots, it is convenient to distinguish six cases: 1) $\alpha' < 1$: a) $n = 1$, b) $n > 1$ (in this case, $n > \alpha''$); 2) $\alpha' > 1$: a) $n = 1$, b) $1 < n < \alpha'$, c) $\alpha' < n < \alpha''$, d) $\alpha'' < n$.

We shall not stop to study these individually, but only write out the final results. For $n = 1$, a positive root exists on one side of the resonance point, which tends to zero as $\alpha \rightarrow 1$. For $\alpha' > 1$, it is located to the left of the point $\alpha = 1$, for $\alpha' < 1$, to the right. For all the remaining resonant points $\alpha = n > 1$ there exist pairs of positive roots which tend to zero as $\alpha \rightarrow n$. In this case, both roots approach the resonant point from the left if $1 < n < \alpha'$ (case 2b); both roots approach from

the right if $\alpha'' < n$ (cases 1b and 2d); one root approaches from the left, and the other from the right if $\alpha' < n < \alpha''$ (case 2c).

The total number of positive roots near resonance, may exceed that shown in Tables I and II. For example, there are three positive roots [one large (15) and two small] to the right of the resonances located in the region $\alpha'' > \alpha$ instead of the one that follows from Tables I and II. Consequently, at large distances from the resonances, the extra roots should join together and vanish, producing a pair of complex roots.

The distribution of the additional roots is as follows. To the left of each resonance, located in the region of zero-transmission of the plasma for the extraordinary wave, which was shown in Tables I and II, there is a band of frequencies where Eq. (6) has a pair of positive roots for the extraordinary wave. As $\alpha \rightarrow n - 0$ one of these approaches zero while the other (14) goes to infinity. In the same way, to the right of each resonance located in the zero-transmission region of the plasma, for the plasma wave which was shown in Tables I and II, there is a band of frequencies where Eq. (6) has a pair of positive roots for the plasma wave. As $\alpha \rightarrow n + 0$, one of these approaches zero and the other (15) infinity.

It should be particularly noted that there are always regions on both sides of an arbitrary resonance in which Eq. (6) has at least one positive root. Thus the conclusion of Gross and a number of other authors^[2,5,7] on the existence of gaps of zero transmission (for the non-relativistic dispersion equation) in the vicinity of each resonance appears to be in error. All the established laws are demonstrated in Figs. 1-3. The special features in the behavior of the negative roots in the vicinity of the resonances are also shown there.

In conclusion, we shall show that even at low electron temperatures relativistic effects begin to appear in the vicinity of the resonances. In the present paper we shall not concern ourselves with the detailed analysis of the relativistic dispersion equation obtained by Trubnikov.^[11] We only note that if α belongs to the interval $(n-1) < \alpha < n$, then the order of the relativistic effects (for example, the absorption associated with the Doppler effect) is determined by a factor of the form $\exp\{-mc^2[(n\omega_H/\omega)^2 - 1]/T\}$. Thus, for non-relativistic plasma, the relativistic effects manifest themselves primarily in the appearance of a narrow band of strong absorption to the left of each resonance (of width $\Delta\alpha \sim \gamma$). For the remaining values of α , the relativistic corrections are exponentially small.

4. COMPLEX ROOTS OF EQUATION (6)

The study of the complex roots of Eq. (6) can be carried out by the method set forth in^[1]. Applying the principle of the argument, and taking into account the asymptotic formula (8), it is easy to show that the function D has an infinite set of roots in the plane of the complex variable s . These roots are grouped into the second and third quadrants, near the imaginary axis. Each complex root of Eq. (6), together with the complex conjugate root corresponding to it, gives a quartet of complex roots of Eq. (1):

$$k = p(\omega) \pm iq(\omega), \quad k = -p(\omega) \pm iq(\omega).$$

5. DETERMINATION OF THE FREQUENCY AS A FUNCTION OF THE PROPAGATION CONSTANT FROM THE DISPERSION EQUATION (1)

We now consider the Eq. (1) from another point of view, when it is required to determine the frequency ω for a given real propagation constant k . The real roots of Eq. (1) $\omega = \omega(k)$ are found by the points of intersection of the line $N = kc/\omega_H\alpha$ with the graph of the function $N = \sqrt{s(\alpha, \beta, \gamma)}$: to each point of intersection (α_n, N_n) there corresponds a pair of roots $\omega_n = \pm\omega_H\alpha_n$. The analysis carried out in Secs. 2 and 3 shows that the number of points of intersection is infinitely great, while their abscissas α_n ($n = 1, 2, 3, \dots$) are distributed in the following fashion:

$$0 < \alpha_1 < 1, \quad n < \alpha_{2n} < \alpha_{2n+1} < n + 1 \quad (n > 1).$$

Making use of the argument principle, it is not difficult to show that Eq. (1) does not have complex roots $\omega = \omega(k)$. In proof of this assertion, one must keep in mind that the function $D(k, \omega)$ is regular at the point $\omega = 0$, has a pole of second order at the points $\omega = \pm n\omega_H$ and that there are $4n + 2$ zeros on the real axis in the plane of the complex variable s inside a circle of sufficiently small radius.

The authors express their gratitude to A. A. Chechina for help in carrying out the calculations.

¹Yu. N. Dnestrovskiĭ and D. P. Kostomarov, JETP 40, 1404 (1961), Soviet Phys. JETP 13, 986 (1961).

²E. P. Gross, Phys. Rev. 82, 232 (1951).

³G. V. Gordeev, JETP 24, 445 (1953).

⁴B. N. Gershman, JETP 24, 659 (1953).

⁵A. G. Sitenko and K. N. Stepanov, JETP 31, 642 (1956), Soviet Phys. JETP 4, 512 (1957).

⁶Gershman, Ginzburg, and Denisov, Usp. Fiz. Nauk 61, 561 (1957).

⁷J. Bernstein, Phys. Rev. 109, 10 (1958).

- ⁸J. E. Drummond, Phys. Rev. 110, 293 (1958).
⁹B. N. Gershman, JETP 37, 695 (1959), Soviet Phys. JETP 10, 497 (1960).
¹⁰B. N. Gershman, JETP 38, 912 (1960), Soviet Phys. JETP 11, 657 (1960).
¹¹B. A. Trubnikov, in the collection Fizika plazmy i problema upravlyaemykh termoyadernykh reaktsii (Plasma Physics and the Problem of Controlled Thermonuclear Reactions) Vol 3, AN SSSR, 1958, p. 104.

Translated by R. T. Beyer

259