QUANTUM OSCILLATIONS OF THE THERMODYNAMIC QUANTITIES OF A METAL IN A MAGNETIC FIELD ACCORDING TO THE FERMI-LIQUID MODEL

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Quantum field theory methods are used to study the influence of Fermi-liquid effects on the oscillations of the thermodynamic quantities of a metal in a magnetic field. It is shown that the periods of the oscillations can be calculated by the usual scheme of I. Lifshitz and Kosevich. Deviations from the usual results appear in the amplitudes of the oscillations and are due to the change of the effective magneton of an excitation owing to the interaction between the electrons.

1. INTRODUCTION

THE theory of the oscillations of the magnetic susceptibility in metals, which was first constructed in a paper by I. Lifshitz and Kosevich,[1] is based on definite ideas about the structure of the energy levels of a metal in a magnetic field. It is assumed that these levels can be constructed on the basis of a quantization of the levels of individual excitations treated as an ideal Fermi gas. However, the electrons in a metal are a sort of Fermi liquid. The theory of the Fermi liquid, constructed by Landau[2,3] in its application to liquid He^3, shows that in such a system effects of the interaction between the excitations play a large role. The correlations that thus arise are of the order of interatomic distances. The question arises as to how the presence of these interactions between the excitations affects the various quantum phenomena in the theory of metals in magnetic fields.

We shall here treat the de Haas-van Alphen effect for the electrons in a metal on the model of the isotropic Fermi liquid. Generally speaking, the situation in the case of electrons is complicated by specific features of the Coulomb interaction. We shall assume that the long-range part of the Coulomb interaction is already screened off. The final formulas obtained below contain only the characteristics of the free-electron spectra (without a field), and therefore, in our opinion, cannot depend on this assumption. The same arguments also apply to the question of the effects of anisotropy.

The results obtained in this paper show that when the spin susceptibility is not taken into account the expression for the oscillating part of the magnetic moment can be obtained on the basis of the usual concept of a system of electrons as a gas of quasiparticles. This same result has been obtained in a paper by Luttinger[4] which has recently appeared. In that paper, however, the analysis of the Green's functions of the electrons in the magnetic field was made only in perturbation theory to terms of first order in the interaction, i.e., terms that give only a trivial renormalization of the chemical potential. Also Luttinger did not investigate the question of the effect of the paramagnetic susceptibility. The whole difference from previously known formulas arises in including the paramagnetic susceptibility, which, as is well known,[2] depends strongly on the Fermi-liquid properties of the system.

Our further study will be made by the methods of quantum theory. We shall be interested only in the quantum oscillations of all quantities. The value of the susceptibility in a weak magnetic field cannot be expressed in terms of the characteristics of the spectrum, since its diamagnetic part depends also on electrons that are located "deep" below the Fermi surface. For simplicity we shall confine ourselves to the temperature absolute zero.

2. THE ENERGY SPECTRUM

We shall begin with a study of the properties of the Green's functions of electrons in a magnetic field. As usual,[3] the Green's function G (x, x′) is defined as an average over the ground state of the system:

$$G(\mathbf{r}, \mathbf{r}'; t - t')\,\delta_{\alpha\beta} = -i\langle T(\psi_\alpha(\mathbf{r}, t)\,\psi_\beta^+(\mathbf{r}', t'))\rangle. \quad (1)$$

The particle field operators $\psi_\alpha(x)$, $\psi^+_\beta(x')$ include the dependence on the magnetic field H; we choose the vector potential $\mathbf{A}(\mathbf{r})$ of the field in the form

$$\mathbf{A}(\mathbf{r}) = \{-Hy, 0, 0\}. \tag{2}$$

The dependence of the Green's function (1) on the coordinates can be represented in the following way:

$$G(x, x') = \exp\{-i(eH/2c)(y + y')(x - x')\} \times G(\mathbf{r} - \mathbf{r}'; t - t'). \tag{3}$$

This follows from gauge invariance, since when we make a displacement of the origin of coordinates, $y \to y + b$, the operators $\psi(x)$ and $\psi^+(x)$ transform according to the law

$$\psi \to \psi e^{-ieHbx/c}, \qquad \psi^+ \to \psi^+ e^{ieHbx/c}. \tag{4}$$

We shall be dealing with $G(\mathbf{r}, \mathbf{r}'; \epsilon)$, the Fourier component of (1) with respect to the time difference $t - t'$. In the absence of a magnetic field

$$G^0(\mathbf{r} - \mathbf{r}'; \varepsilon) = \frac{1}{(2\pi)^3}\int G^0(\mathbf{p}, \varepsilon)\, e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')}\, d\mathbf{p}.$$

For small ϵ the function $G^0(\mathbf{p}, \epsilon)$ has a pole near the Fermi surface of the form[3]

$$G^0(\mathbf{p}, \varepsilon) = a/(\varepsilon - v(p - p_0) + i\delta(\varepsilon)). \tag{5}$$

The value of $\epsilon = v(p - p_0)$ determines the spectrum of the Fermi liquid.

In a magnetic field the form of the Green's function near the Fermi surface is decidedly altered owing to the quantization of the levels. We shall show, however, that the energy spectrum of the electrons in the magnetic field can be obtained from the expression (5) by the usual rules of quasiclassical quantization, as was also suggested in the paper by I. Lifshitz and Kosevich.[1]

To prove this assertion we write the Dyson equation satisfied by the Green's function $G(\mathbf{r}, \mathbf{r}'; \epsilon)$ in coordinate space:

$$\left[\varepsilon + \mu - \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2\right] G(\mathbf{r}, \mathbf{r}'; \varepsilon) - \int \Sigma(\mathbf{r}, \mathbf{r}''; \varepsilon)\, G(\mathbf{r}'', \mathbf{r}'; \varepsilon)\, d^3r'' = \delta(\mathbf{r} - \mathbf{r}'). \tag{6}$$

Here $\hat{\mathbf{p}} = -i\partial/\partial\mathbf{r}$, μ is the chemical potential of the electrons in the magnetic field, and $\Sigma(\mathbf{r}, \mathbf{r}''; \epsilon)$ is the so-called self-energy part arising from the interactions between the particles in the Fermi liquid. We shall not specify concrete forms for these interactions. The spectrum of the system is determined by the eigenvalues of the operator which appears in square brackets in Eq. (6).

In the notation of second quantization the Hamiltonian for the interaction of the electrons with the magnetic field (2) takes the form

$$H_{int} = \int \psi^+(\mathbf{r}')\left[\frac{e}{2mc}(\hat{\mathbf{p}} - \hat{\mathbf{p}}')_x + \frac{e^2Hy}{2mc^2}\right] Hy\psi(\mathbf{r})\, d^3\mathbf{r}, \tag{7}$$

where $\hat{p}_x$ and $\hat{p}_{x'}$ denote differentiation ($\hat{p}_x = -i\partial/\partial x$) with respect to the corresponding arguments in the limit $\mathbf{r} \to \mathbf{r}'$. We shall now investigate the dependence of the self-energy part $\Sigma(\mathbf{r}, \mathbf{r}'; \epsilon)$ on the magnitude of the magnetic field. According to Eq. (7) an increment of the magnetic field, $H \to H + \delta H$, is equivalent to an additional interaction Hamiltonian:

$$\delta H_{int} = \delta H \int \psi^+(\mathbf{r}')\left[\frac{e}{2mc}(\hat{\mathbf{p}} - \hat{\mathbf{p}}')_x + \frac{e^2Hy}{mc^2}\right] y\psi(\mathbf{r})\, d^3\mathbf{r}. \tag{7'}$$

Using the usual diagram technique to calculate from Eq. (7′) the change $\delta\Sigma(\mathbf{r}, \mathbf{r}'; \epsilon) = (\partial\Sigma/\partial H)\delta H$, we can assign to this quantity the diagram shown in Fig. 1. The cross denotes the operator in square brackets in the integrand in Eq. (7′); the circle denotes the complete vertex part in the magnetic field, $\Gamma_{\alpha\beta,\gamma\delta}(\xi_1, \xi_2; \xi_3, \xi_4)$.

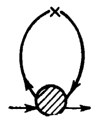

FIG. 1

The vertex part is defined as usual by the relation

$$\langle T(\psi_\alpha(\xi_1)\psi_\beta(\xi_2)\psi^+_\gamma(\xi_3)\psi^+_\delta(\xi_4))\rangle = G_{\alpha\gamma}(\xi_1, \xi_3)\, G_{\beta\delta}(\xi_2, \xi_4) - G_{\alpha\delta}(\xi_1, \xi_4)\, G_{\beta\gamma}(\xi_2, \xi_3) + i\int G_{\alpha\alpha'}(\xi_1, \xi_1')\, G_{\beta\beta'}(\xi_2, \xi_2') \times \Gamma_{\alpha'\beta',\gamma'\delta'}(\xi_1', \xi_2'; \xi_3', \xi_4')\, G_{\gamma'\gamma}(\xi_3', \xi_3)\, G_{\delta'\delta}(\xi_4', \xi_4) \times d^4\xi_1' d^4\xi_2' d^4\xi_3' d^4\xi_4'.$$

In the absence of the magnetic field $\Gamma_{\alpha\beta,\gamma\delta}(\xi_1, \xi_2; \xi_3, \xi_4)$ has Fourier components of the form $(2\pi)^4\Gamma_{\alpha\beta,\gamma\delta}(p_1, p_2; p_3, p_4) \times \delta(p_1 + p_2 - p_3 - p_4)$. As is well known,[3] the vertex part plays an important role in the theory of the Fermi liquid.

The diagram of Fig. 1 gives the following result for the derivative $\partial\Sigma/\partial H$:

$$\delta_{\alpha\beta}\frac{\partial\Sigma(\mathbf{r}, \mathbf{r}'; \varepsilon)}{\partial H} = \frac{i}{2\pi}\int d\omega' \int d^3\mathbf{r}_1 d^3\mathbf{r}_2 d^3\mathbf{l}\left[\frac{e}{2mc}(\hat{\mathbf{p}} - \hat{\mathbf{p}}')_{l_x} + \frac{e^2H}{mc^2} l_y\right] l_y \Gamma_{\alpha\gamma,\gamma\beta}(\mathbf{r}, \varepsilon; \mathbf{r}_1, \omega'; \mathbf{r}_2, \omega'; \mathbf{r}', \varepsilon)\, G(\mathbf{l}, \mathbf{r}_1; \omega') \times G(\mathbf{r}_2, \mathbf{l}'; \omega'); \qquad \hat{p}_x = -i\frac{\partial}{\partial l_x}. \tag{8}$$

It is convenient to make a transformation of this expression. To do so we note that under the infinitesimal transformation (4) $\Sigma(\mathbf{r}, \mathbf{r}'; \epsilon)$ changes by the quantity

$$- i (eH/c)\, \delta b\, (x - x')\, \Sigma(\mathbf{r}, \mathbf{r}'; \varepsilon).$$

At the same time, under the displacement of the origin, $y \to y + \delta b$, the Hamiltonian (7) receives the increment

$$\delta H_{int} = \delta b \int \psi^+(\mathbf{r}')\left[\frac{e}{2mc}(\hat{\mathbf{p}} - \hat{\mathbf{p}}')_x + \frac{e^2Hy}{mc^2}\right]_{\mathbf{r}'\to\mathbf{r}} \psi(\mathbf{r})\, d^3\mathbf{r}.$$

Then, in analogy with Eq. (8), we get

$$-\frac{ie(x-x')}{c}\delta_{\alpha\beta}\Sigma(\mathbf{r}, \mathbf{r}'; \varepsilon) = \frac{i}{2\pi}\int d\omega' \int d^3\mathbf{r}_1 d^3\mathbf{r}_2 d^3\mathbf{l}\left[\frac{e}{2mc}(\hat{\mathbf{p}} - \hat{\mathbf{p}}')_{l_x} + \frac{e^2H}{mc^2} l_y\right] \times \Gamma_{\alpha\gamma,\gamma\beta}(\mathbf{r}, \varepsilon; \mathbf{r}_1, \omega'; \mathbf{r}_2, \omega'; \mathbf{r}', \varepsilon) \times G(\mathbf{l}, \mathbf{r}_1; \omega')\, G(\mathbf{r}_2, \mathbf{l}'; \omega').$$

Combining this last relation with Eq. (8), we find

$$\frac{\partial \Sigma_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; \varepsilon)}{\partial H} = -i\frac{e(x-x')(y+y')}{2c}\Sigma_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; \varepsilon) + M_{\alpha\beta}(\mathbf{r}, \mathbf{r}', \varepsilon), \qquad (9)$$

$$M_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; \varepsilon) = \frac{i}{2\pi}\int d\omega' \int d^3\mathbf{r}_1 d^3\mathbf{r}_2 d^3\mathbf{l}\left[\frac{e}{2mc}(\hat{\mathbf{p}} - \hat{\mathbf{p}}')_{l_x} + \frac{e^2Hl_y}{mc^2}\right][l_y - (y+y')/2]\, G(\mathbf{l}, \mathbf{r}_1; \omega')\, G(\mathbf{r}_2, \mathbf{l}'; \omega') \times \Gamma_{\alpha\gamma,\gamma\beta}(\mathbf{r}, \varepsilon; \mathbf{r}_1, \omega'; \mathbf{r}_2, \omega'; \mathbf{r}', \varepsilon). \qquad (10)$$

We shall show below that the terms in $\Sigma_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; \epsilon)$ that come from $M_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; \epsilon)$ are of the order $H^{3/2}$ in the magnetic field, whereas for our purposes it is enough to know the spectrum of the system correct to terms of the order H. Neglecting the last term in Eq. (9), we get

$$\Sigma_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; \varepsilon) = \exp\left\{-\tfrac{1}{2} ieHc^{-1}(x-x')(y+y')\right\}\delta_{\alpha\beta}\Sigma^0(\mathbf{r} - \mathbf{r}'; \varepsilon), \qquad (11)$$

where $\Sigma^0(\mathbf{r} - \mathbf{r}'; \epsilon)$ is the self-energy part in the absence of the magnetic field.

The expression (11) for $\Sigma(\mathbf{r}, \mathbf{r}')$ can be written symbolically in the form

$$\Sigma(\mathbf{r}, \mathbf{r}') = \hat{\Sigma}^0(\hat{\mathbf{p}} - e\mathbf{A}/c).$$

In fact, let us apply the "Hamiltonian"

$$\hat{h} = \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 + \hat{\Sigma}^0\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)$$

to an arbitrary function $\psi(\mathbf{r})$ and go over to the $\mathbf{p}$ representation. As Zil'berman[5] has shown, in the momentum representation the operator $\hat{h}$ can be written in the following form:

$$\hat{h}\psi_\mathbf{p} = \left\{\frac{1}{2m}\left(p_x + i\frac{eH}{c}\frac{d}{dp_y}\right)^2 + \frac{1}{2m}(p_y^2 + p_z^2) + \int \Sigma^0(\mathbf{s}; \varepsilon)\exp\left[-i\left(p_x + \frac{ieH}{c}\frac{d}{dp_y}\right)s_x - ip_ys_y - ip_zs_z\right]d^3\mathbf{s}\right\}\psi_\mathbf{p}. \qquad (12)$$

The expression (12) contains definite prescriptions as to the order of the operators p_y and d/dp_y (the requirement of complete symmetry of the Hamiltonian[5]). In the isotropic model of the Fermi liquid, which we are using in this paper, the self-energy part $\Sigma^0(\mathbf{p}, \epsilon)$ is a function of $|\mathbf{p}|^2$ only. At the same time it is obvious that under the condition that $\Sigma^0(\hat{\mathbf{p}} - e\mathbf{A}/c)$ is to remain Hermitian different requirements as to the order of the operators p_y and d/dp_y in $\Sigma^0(\hat{\mathbf{p}} - e\mathbf{A}/c)$ lead to Hamiltonians (12) which differ by terms of order in the energy eigenvalues not lower than H^2. For this reason it is more convenient to choose $\Sigma^0(\hat{\mathbf{p}} - e\mathbf{A}/c)$ in the form $\Sigma^0(\hat{\mathbf{p}} - e\mathbf{A}/c) \equiv \Sigma^0(|\hat{\mathbf{p}} - e\mathbf{A}/c|^2)$.

The exact eigenvalues and eigenfunctions of the operator

$$\left(\hat{\mathbf{p}} - \frac{e\mathbf{A}}{c}\right)^2 = \left(\hat{p}_x + \frac{ieH}{c}\frac{d}{dp_y}\right)^2 + \hat{p}_y^2 + \hat{p}_z^2$$

are well known:[6]

$$\left(\hat{\mathbf{p}} - \frac{e\mathbf{A}}{c}\right)^2\psi_{\mathbf{p},n} = [p_z^2 + (2n+1)\, eH/c]\,\psi_{\mathbf{p},n},$$

$$\psi_{\mathbf{p},n} = \exp\{-icp_xp_y/eH - cp_y^2/2eH\}H_n(p_y\sqrt{c/eH}),$$

where $H_n(x)$ are the Hermite polynomials. Therefore the Green's function of a particle in a magnetic field can be written near the Fermi surface (in the $\mathbf{p}$ representation) in the following way:

$$G(\mathbf{p}, \mathbf{p}'; \varepsilon) = \sum_n \frac{\psi_n(\mathbf{p})\psi_n^*(\mathbf{p}')\,\delta(p_x - p_x')\,\delta(p_z - p_z')}{\varepsilon + \mu - (n+1/2)\, eH/mc - p_z^2/2m - \Sigma^0(p_z^2 + 2(n+1/2)\, eH/c) + i\delta(\varepsilon)}. \qquad (13)$$

According to Eq. (5), when the magnetic field $H = 0$,

$$G^0(\mathbf{p}, \mathbf{p}'; \varepsilon) = \frac{\delta(\mathbf{p} - \mathbf{p}')}{\varepsilon + \mu - p^2/2m - \Sigma^0(p^2, \varepsilon) + i\delta(\varepsilon)} = \frac{a\,\delta(\mathbf{p} - \mathbf{p}')}{\varepsilon - v(p - p_0) + i\delta(\varepsilon)}. \qquad (14)$$

Here a is a renormalization constant, and p_0 is the Fermi limiting momentum, which is determined from the equation

$$p_0^2/2m + \Sigma^0(p_0, 0) = \mu.$$

As is well known,[1] the electrons important for the de Haas-van Alphen effect are those near the Fermi surface i.e., the electrons in the region in which $(2n+1)\, eH/c + p_z^2 \approx p_0^2$. Therefore if we introduce the notation m^* for the "effective" mass ($v = p_0/m^*$), there follows directly from Eqs. (13) and (14) the following expression for the Green's function of electrons in a magnetic field near the Fermi surface:

$$G(\mathbf{p}, \mathbf{p}'; \varepsilon) = \sum_n \psi_n(\mathbf{p}) \psi_n^*(\mathbf{p}') \delta(p_x - p_x') \, \delta(p_z - p_z') G_n(p_z, \varepsilon),$$

$$G_n(p_z, \varepsilon) = a / (\varepsilon + p_0^2/2m^* - (n + 1/2)\omega^* - p_z^2/2m^* + i\delta(\varepsilon)), \quad (15)$$

where $\omega^* = eH/m^*c$. As is shown below, the constants a, m^*, and p_0 depend on the field H only through terms of order $H^{3/2}$.

The Greens' function far from the Fermi surface $[|(2n+1)eH/c + p_z^2 - p_0^2| \sim p_0^2]$ contains no sharp poles; this is due to the strong damping of the excitations in this region (for small ϵ the damping $\delta(\epsilon)$ of the excitations in Eq. (15) increases like $|\epsilon|\epsilon/\mu$). Therefore the contributions to the various quantities from these distant regions give no oscillating singularities and can be expanded in powers of the field strength.

Let us discuss the behavior of the Green's function $G(\mathbf{r}, \mathbf{r}'; \epsilon)$ in the coordinate representation. In the absence of a magnetic field

$$G^0(\mathbf{r} - \mathbf{r}'; \varepsilon) = \frac{1}{(2\pi)^3} \int G^0(\mathbf{p}) \, e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')} d^3\mathbf{p}.$$

The Green's function $G(\mathbf{R}, \epsilon)$ expresses the correlation of the electrons at different points of space. At distances large in comparison with atomic distances the only electrons that contribute are those moving like "free" electrons, i.e., those from the region near the Fermi surface. Electronic excitations far from the Fermi surface are strongly damped because of collisions with each other and contribute to the Green's function $G(\mathbf{r} - \mathbf{r}'; \epsilon)$ only at atomic distances. If in the function $G(\mathbf{r} - \mathbf{r}'; \epsilon)$ we separate out the term (5) that has a singularity near the Fermi surface,

$$G^0(\mathbf{p}, \varepsilon) = \frac{a}{\varepsilon - v(p - p_0) + i\delta(\varepsilon)} + g(\mathbf{p}, \varepsilon)$$

[so that $g(\mathbf{p}, \epsilon)$ has no singularities], it is easy to see that for small ϵ the behavior of the function $G^0(\mathbf{R}, \epsilon)$ at distances $Rp_0 \gg 1$ is given by

$$G^0(\mathbf{R}, \varepsilon) = \frac{am^*}{2\pi R} \begin{cases} \exp(ip_0R + i\varepsilon R/v), & \varepsilon > 0 \\ \exp(-ip_0R - i\varepsilon R/v), & \varepsilon < 0. \end{cases} \quad (16)$$

In the case of large ϵ the function $G^0(\mathbf{R}, \epsilon)$ is rapidly damped at distances $Rp_0 \sim \epsilon^2/\mu^2$.

Returning to the Green's function in a magnetic field, we conclude that its behavior at large distances is also determined by the singular part of the expression (15). It is easy to verify that in the coordinate representation $G(\mathbf{r}, \mathbf{r}'; \epsilon)$ (for $|\mathbf{r} - \mathbf{r}'| \gg 1/p_0$) can be written in the following form:

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}'; \varepsilon) &= \exp[-i(eH/2c)(x - x')(y + y')] \\ &\times \frac{eH}{c(2\pi)^2} \sum_n e^{-eH\rho^2/4c} L_n\left(\frac{eH}{2c}\rho^2\right) \\ &\times \int \frac{a e^{ip_z(z-z')} dp_z}{\varepsilon + p_0^2/2m^* - (n + 1/2)\omega^* - p_z^2/2m^* + i\delta(\varepsilon)} \\ &= \exp\{-i(eH/2c)(x - x')(y + y')\} \overline{G}(\mathbf{R}, \varepsilon), \end{aligned} \quad (17)$$

$L_n(x)$ are the Laguerre polynomials. [Here and in what follows we use the notation $\rho^2 = (x - x')^2 + (y - y')^2$]. As can be seen from Eq. (17), generally speaking the magnetic field causes a decided change in the character of the dependence of $G(\mathbf{r}, \mathbf{r}'; \epsilon)$ on $\mathbf{R}$. In the limit of weak magnetic field, $\omega^* \ll p_0^2/2m^*$, for large n and $\rho \ll cp_0/eH$,

$$e^{-eH\rho^2/4c} L_n(eH\rho^2/2c) \approx J_0(\sqrt{2eHn/c}\,\rho).$$

By replacing the summation over n by an integration, we would again arrive at the result (16).

A special role is played in Eq. (17) by terms with values of $(n + 1/2)\omega^*$ close to $p_0^2/2m^*$ (small p_z). Let $\Delta < \omega^*$ and $p_0^2/2m^* = \Delta + \omega^*(N_0 + 1/2)$; then each such term in Eq. (17) contributes to $G(\mathbf{R}, \epsilon)$ a small quantity of the order

$$\sim \frac{eH}{c} e^{-eH\rho^2/4c} L_{N_0}\left(\frac{eH}{2c}\rho^2\right) e^{i\Delta|z-z'|} \Delta^{-1/2}.$$

This expression is exponentially damped for $\rho \gg cp_0/eH$ and is a plane wave in its dependence on $|z - z'|$. This part of the G-function determines the oscillations of all the quantities in which we are interested.

Coming now to an estimate of the term $M(\mathbf{r}, \mathbf{r}'; \epsilon)$ dropped from Eq. (9), let us rewrite this term in a somewhat different form, by substituting Green's functions in the formula (3):

$$\begin{aligned} M_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; \varepsilon) &= \frac{i}{2\pi} \int d\omega \int d^3\mathbf{r}_1 d^3\mathbf{r}_2 d^3\mathbf{l} \left(l_y - \frac{y + y'}{2}\right) \\ &\times \exp\{-i(eH/2c)[(r_{1y} + l_y)(r_{1x} - l_x) \\ &+ (r_{2y} + l_y)(l_x - r_{2x})]\} \Gamma_{\alpha\gamma,\gamma\beta}(\mathbf{r}, \varepsilon; \mathbf{r}_1, \omega; \mathbf{r}_2, \omega; \mathbf{r}', \varepsilon) \\ &\times \Big\{ (e^2H/2mc^2) \, G(\mathbf{l} - \mathbf{r}_1, \omega) \, G(\mathbf{r}_2 - \mathbf{l}, \omega) \\ &\times \left(l_y - \frac{r_{1y} + r_{2y}}{2}\right) + \frac{ie}{2mc} \left[\frac{\partial G(\mathbf{l} - \mathbf{r}_1, \omega)}{\partial r_{1x}} G(\mathbf{r}_2 - \mathbf{l}, \omega) \right. \\ &\left. - G(\mathbf{l} - \mathbf{r}_1, \omega) \frac{\partial G(\mathbf{r}_2 - \mathbf{l}, \omega)}{\partial r_{2x}} \right] \Big\}. \end{aligned} \quad (18)$$

This expression involves integrals containing G-functions of the differences of coordinates between the various points of the diagram of Fig. 1. The region of integration in which these differences are of the order of interatomic distances are of no interest, since their contribution to Eq. (18) is of the order H (and of order H^2 in Σ).

Let the distances $|\mathbf{l} - \mathbf{r}_i|$ be large in comparison with atomic distances. In this region we can use for the Green's functions the asymptotic expression (17). In estimating the quantity (18) we shall assume that the magnetic field is weak, and set it equal to zero wherever possible. In particular, for the vertex part we use the expression for zero field. Let us rewrite this expression in the following way:

$$\Gamma_{\alpha\gamma,\gamma\beta}(\mathbf{r}, \varepsilon; \mathbf{r}_1\omega; \mathbf{r}_2, \omega; \mathbf{r}', \varepsilon) = \frac{1}{(2\pi)^9}\int \Gamma^0_{\alpha\gamma,\gamma\beta}(\mathbf{p}_1, \varepsilon; \mathbf{p}_2, \omega; \mathbf{p}_2 + \mathbf{k}, \omega; \mathbf{p}_1 - \mathbf{k}, \varepsilon) \exp[i\mathbf{p}_1(\mathbf{r}-\mathbf{r}') + i\mathbf{p}_2(\mathbf{r}_1 - \mathbf{r}_2) + i\mathbf{k}(\mathbf{r}' - \mathbf{r}_2)]\, d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{k}. \quad (19)$$

The Fourier component $\Gamma^0_{\alpha\gamma,\gamma\beta}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_3, \mathbf{p}_4)$ in Eq. (19) involves momenta p of the order of the Fermi momentum and has important variations in this region only for changes $\Delta p \sim p_0$. Besides this, in the region of integration over l in Eq. (18) where $|\mathbf{l}-\mathbf{r}|$ is large the important values of $\mathbf{k}$ in Eq. (19) are small, and therefore we can neglect the dependence of the vertex part on k.

Thus as a function of $|\mathbf{r}_2 - \mathbf{r}_1|$ and $|\mathbf{r}-\mathbf{r}'|$ the expression (19) is rapidly oscillating at atomic distances. [Close to the Fermi surface $|\mathbf{p}_1|$, $|\mathbf{p}_2| \sim p_0$ the Fourier components $\Gamma_{\alpha\gamma,\gamma\beta}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_2, \mathbf{p}_1)$ in Eq. (19) depend only on the angle between the vectors $\mathbf{p}_1$ and $\mathbf{p}_2$]. Consequently, for purposes of the integration over $\mathbf{r}_1$ and $\mathbf{r}_2$ in Eq. (18) for $|\mathbf{l}-\mathbf{r}_1| \gg 1/p_0$, for which values only the Fourier components with momenta close to the Fermi surface are important in the functions $G(\mathbf{l}-\mathbf{r}_1, \omega)$, the vertex part (19) is essentially a δ function of $\mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{r}-\mathbf{r}'$. This fact is the expression of a fundamental physical assumption of the theory of the Fermi liquid,[2] according to which the interaction between the particles is a short-range one, and all of the correlations that arise between them fall off rapidly at atomic distances. (For the electrons in a metal the Coulomb interaction is also screened off at distances of the order of atomic distances.) Nevertheless this assertion is not completely rigorous.

As has been shown by Landau,[3] in a number of cases the vertex part can have a "long-range" part. Such singularities are due to the diagrams in the vertex part which are shown in Fig. 2, a. In this diagram the squares denote irreducible vertex parts which have no singularities in the direction $1-2$. In fact, substituting in these diagrams the expressions (16) for the Green's functions and integrating over the frequencies of the internal lines, we get $(R_{12} \gg 1/p_0)$

$$\int G^0(\mathbf{R}_{12}, \omega)\, G^0(\mathbf{R}_{12}, \omega - \varepsilon)\, d\omega \sim \frac{a^2 m^{*2}\varepsilon}{(2\pi R_{12})^2} e^{i\varepsilon R_{12}/v},$$

where ϵ is the small frequency transfer. Figure 2, b shows this same diagram in the momentum representation, and from it we can see that the slowly decreasing dependence in the vertex part is due to small frequency and momentum transfers in the Fourier component $\Gamma^0(p_1, p_2; p_3, p_4)$. In Eq. (19) such a small transfer can correspond

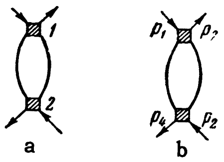

FIG. 2

either to a small value of $\mathbf{k}$ or to a small transfer $|\mathbf{p}_3 - \mathbf{p}_1| \approx |\mathbf{p}_2 - \mathbf{p}_1| \ll p_0$. Obviously in the latter case the contribution from these singularities in Eq. (19) is small, because of the smallness of the region of integration over $\mathbf{p}_1$ and $\mathbf{p}_2$. As for the small values of k, the question of these singularities arises owing to the fact that in Eq. (18) there is a factor $l_y - r_y = l_y - r_{1y} + r_{1y} - r_y$, and the quantity $|r_{1y} - r_y|$ may not be small. It is easy to see, however, that in zeroth order in the field we have by considerations of symmetry

$$\int (s_y - r_y)\, G^0(\mathbf{s}-\mathbf{r})\, G^0(\mathbf{s}-\mathbf{r})\, d^3\mathbf{s} \equiv 0$$

and therefore in Eq. (18) the factor $l_y - (y + y')/2$ can be replaced by $l_y - r_{1y}$. We shall not present the detailed proof that all of these assumptions correspond to dropping in $\mathbf{M}(\mathbf{r}, \mathbf{r}')$ terms that are of higher order in the field strength H as compared with the estimate that we get on the assumption that the vertex part is equal to the expression for this part in the absence of a magnetic field and is of short-range character.

Accordingly, in Eq. (18) we now take $|\mathbf{r}_2 - \mathbf{r}_1| \sim |\mathbf{r}_1 - \mathbf{r}| \sim 1/p_0$. We can drop the exponential factors in Eq. (18). By considerations of symmetry the term in the square brackets that contains derivatives of the Green's functions is identically zero. Therefore instead of Eq. (18) we get

$$M_{\alpha\beta}(\mathbf{r}, \mathbf{r}') \approx \frac{ie^2H}{4\pi mc^2}\int d\omega \int d^3\mathbf{r}_1 d^3\mathbf{r}_2 d^3\mathbf{l}\left(l_y - \frac{r_{1y}+r_{2y}}{2}\right)^2 \times G(\mathbf{l}-\mathbf{r}_1, \omega)\, G(\mathbf{r}_2 - \mathbf{l}, \omega)\, \Gamma^0_{\alpha\gamma,\gamma\beta} \times (\mathbf{r}, \ ; \mathbf{r}_1, \omega; \mathbf{r}_2, \omega; \mathbf{r}', \varepsilon). \quad (20)$$

According to the foregoing the quantity $\mathbf{M}(\mathbf{r}, \mathbf{r}')$ of Eq. (20) is determined by the region of integration $|\mathbf{l}-\mathbf{r}_1| \gg 1/p_0$ and $\omega \ll \mu$. For small ϵ and ω we can neglect the frequency dependence of Γ^0 in Eq. (19). We shall denote the corresponding Fourier component by $\Gamma^0_{\alpha\gamma,\gamma\beta}(\mathbf{p}_1, \mathbf{p}_2)$. Thus it follows from Eq. (20) that

$$M_{\alpha\beta}(\mathbf{r}, \mathbf{r}') \approx \frac{ie^2H}{4\pi mc^2}\left[\int \frac{d^3\mathbf{p}}{(2\pi)^6} e^{i\mathbf{p}_1(\mathbf{r}-\mathbf{r}')}\int \frac{d\Omega}{4\pi}\Gamma_{\alpha\gamma,\gamma\beta}(\mathbf{p}_1, \mathbf{p}_2)\right]\frac{1}{2} \times \int G^2(\mathbf{R}, \omega)\, \rho^2 d^3\mathbf{R}. \quad (21)$$

Proceeding to the calculation of this last integral, we substitute in it the Green's function (17):

$$\int G^2(\mathbf{R},\omega)\,\rho^2 d^3\mathbf{R} = \frac{a^2}{(2\pi)^2}\left(\frac{eH}{c}\right)^2 \iint \sum_{n,m} \frac{dp_z d\omega}{(\omega - E_n + i\delta(\omega))(\omega - E_m + i\delta(\omega))} \times \int_0^\infty \rho^2 d\rho\, e^{-eH\rho^2/4c} L_n\left(\frac{eH}{2c}\rho^2\right) L_m\left(\frac{eH}{2c}\rho^2\right),$$

$$E_n = (n + 1/2)\,\omega^* + p_z^2/2m^* - p_0^2/2m^*,$$

$$E_m = (m + 1/2)\,\omega^* + (p_z - q)^2/2m^* - p_0^2/2m^*. \qquad (21')$$

In the expression for E_m we have introduced a small quantity q. The point is that the integral in question is not uniquely defined: the expression

$$\int d\omega \int G(\mathbf{R},\omega)\, G(\mathbf{R},\omega-\omega_0)\,\rho^2 e^{iqz} d^3\mathbf{R}$$

has different limits for ω_0, $q \to 0$, depending on whether $\omega_0/q \to 0$ or $q/\omega_0 \to 0$. For the case of no magnetic field the singularities of such integrals have been studied by Landau.[3] To give a meaning to the integral (21′) we must take the limit $\omega_0/q \to 0$, which corresponds to the fact that $M(\mathbf{r}, \mathbf{r}'; \epsilon)$ is being calculated in a field that is constant in time but weakly nonuniform along the z axis. Therefore, strictly speaking, the quantity given by Eq. (21) is a definite one of the two vertex parts Γ^k and Γ^ω introduced by Landau,[3] namely Γ^k. For our estimates this is of course unimportant.

In calculating the integral of Laguerre polynomials in Eq. (21′) it is helpful to use the exact relation

$$xL_n(x) = (2n+1)\,L_n(x) - (n+1)\,L_{n+1}(x) - nL_{n-1}(x). \qquad (22)$$

Therefore m = n ± 1. Integrating Eq. (21′) over the frequencies ω, we get

$$Y = \frac{i}{2\pi}\int d\omega G^2(\mathbf{R},\omega)\,\rho^2 d^3\mathbf{R} = \frac{a^2}{2\pi^2}\sum_{n,m}\int \frac{dp_z}{E_n - E_m}\int_0^\infty xL_n(x)\,L_m(x)\,e^{-x}dx$$

($E_n > 0$, $E_m < 0$). Using Eq. (22) and performing the integration over p_z, we find after simple manipulations

$$Y = (2m^*)^{1/2}\frac{a^2}{\pi^2}\sum_n\left\{\frac{1}{2}\frac{n+1/2}{[p_0^2/2m^* - \omega^*(n+1/2)]^{1/2}} - \frac{1}{\omega^*}\sqrt{\frac{p_0^2}{2m^*} - \omega^*(n+1/2)}\right\}.$$

The summation is taken over all values of n for which the radicand is positive.

It is not hard to verify that this quantity can be represented in the following way:

$$\frac{eH}{c}Y = -\frac{\sqrt{2m^*}}{\omega^*}\frac{a^2}{\pi^2}\frac{\partial}{\partial H}\left\{\frac{eH}{c}\sum_n\sqrt{\frac{p_0^2}{2m^*} - \omega^*(n+1/2)}\right\}. \qquad (23)$$

The sum in the curly brackets has been repeatedly

FIG. 3

investigated in the literature.[7,8] Using the expression obtained by Sondheimer and Wilson,[8] which is accurate to terms of order H^2,

$$\frac{1}{2\pi^2}\frac{eH}{c}\sum_n\sqrt{\mu - \omega^*(n+1/2)} = \frac{p_0^3}{3\pi^2}\left\{1 + \frac{3}{2\pi}\left(\frac{\omega^*}{\mu}\right)^{3/2}\sum_{r=1}^\infty \frac{(-1)^r}{r^{3/2}}\sin\left(2\pi r\frac{\mu}{\omega^*} - \frac{\pi}{4}\right)\right\}$$

and substituting it in Eq. (23), we get finally

$$\frac{eH}{c}Y = -\frac{\sqrt{2m^*}}{\omega^*\pi^3}a^2\frac{\partial}{\partial H}\left\{\left(\frac{\omega^*}{\mu}\right)^{3/2}\varphi\left(\frac{\mu}{\omega^*}\right)\right\},$$

where $\varphi(x)$ is a rapidly oscillating function. Thus in actual fact the terms that correspond to $M(\mathbf{r}, \mathbf{r}'; \epsilon)$ give in Eq. (14) a correction to the energy levels of the excitations of the order $H^{3/2}$.

3. THE OSCILLATING SINGULARITIES OF THE THERMODYNAMIC FUNCTIONS

Let us now proceed to the derivation of the formula for the thermodynamic potential. In order to separate out the small terms that have an oscillating character, it is more convenient to start from the expression for the derivative $\partial N/\partial\mu$ of the particle-number density with respect to the chemical potential of the system. The particle-number density is connected in a simple way with the Green's function (1) of the system:

$$N = -iG_{\alpha\alpha}(x, x')_{x'\to x,\, t'\to t+0}.$$

The expression for the derivative $\partial N/\partial\mu$ can be obtained by the following arguments (cf. [9]). Let us place the system in a weak and slowly varying potential field $\delta U(\mathbf{r})$. Then the quantity $\mu + \delta U = \mu_0$ is conserved throughout the system. On the other hand, the interaction of the system with the field $\delta U(\mathbf{r})$ is described by the Hamiltonian

$$H_{int} = \int \psi^+(x)\,\delta U\psi(x)\,d^3\mathbf{r}.$$

According to the usual rules of the diagram technique the change of the Green's function in first order in δU can be represented by the first diagram of Fig. 3. In the limit of a δU independent of the coordinates, $\delta U = -\delta\mu$, we get

$$\frac{\partial N}{\partial\mu} = i\int G_{\alpha\gamma}(x,l)\,G_{\gamma\alpha}(l,x)\,d^4l - \int d^4\xi_1 d^4\xi_2 d^4\xi_3 d^4\xi_4 d^4l\, G_{\alpha\alpha_1}(x,\xi_1)\,G_{\beta\alpha_2}(l,\xi_2)\,G_{\alpha_3\beta}(\xi_3,l)\,G_{\alpha_4\alpha}(\xi_4,x') \times \Gamma_{\alpha_1\alpha_2,\,\alpha_3\alpha_4}(\xi_1,\xi_2;\xi_3,\xi_4)$$

or, when we go over to Fourier components with respect to the difference of the time coordinates,

$$\frac{\partial N}{\partial\mu} = \frac{i}{2\pi}\int d\omega \int G_{\alpha\gamma}(\mathbf{r}, \mathbf{l}; \omega)\, G_{\gamma\alpha}(\mathbf{l}, \mathbf{r}; \omega)\, d^3\mathbf{l}$$
$$- \frac{1}{(2\pi)^2}\iint d\omega d\omega' \int d^3s_1 d^3s_2 d^3s_3 d^3s_4 d^3\mathbf{l}\, G_{\alpha\alpha_1}(\mathbf{r}, \mathbf{s}_1; \omega)\, G_{\alpha_4\alpha}$$
$$\times (\mathbf{s}_4, \mathbf{r}; \omega)\, \Gamma_{\alpha_1\alpha_2, \alpha_3\alpha_4}(\mathbf{s}_1, \omega; \mathbf{s}_2, \omega'; \mathbf{s}_3, \omega'; \mathbf{s}_4, \omega)\, G_{\beta\alpha_2}$$
$$\times (\mathbf{l}, \mathbf{s}_2; \omega')\, G_{\alpha_3\beta}(\mathbf{s}_3, \mathbf{l}; \omega').$$

For simplicity let us first investigate the singularities of the first term in the right member of Eq. (24). These singularities are due to electrons near the Fermi surface, and therefore in this case it is sufficient to use the asymptotic expression (17) for the Green's functions. Substituting this expression in the integral and carrying out the integration, we get

$$\frac{i}{2\pi}\int d\omega \int G_{\alpha\gamma}(\mathbf{r}, \mathbf{l}; \omega)\, G_{\gamma\alpha}(\mathbf{l}, \mathbf{r}; \omega)\, d^3\mathbf{l}$$
$$= \frac{2i}{(2\pi)^3}\left(\frac{eH}{c}\right)^2 a^2 \sum_n \int d\omega \int \frac{dp_z}{(\omega - E_n + i\delta(\omega))(\omega - E_n' + i\delta(\omega))}$$

(here in $E_n' = \omega^*(n + 1/2) + (p_z - q)^2/2m^* - p_0^2/2m^*$ we have introduced a small momentum; this corresponds to the fact that in deriving Eq. (24) we started from the condition for equilibrium of the system in a field $\delta U(\mathbf{r})$ constant in time but slowly varying in space). Integrating the resulting expression first over ω, and then over p_z, we arrive at the expression

$$\frac{\sqrt{2m^*}}{2\pi^2}\left(\frac{eH}{c}\right) a^2 \sum_n \left(\frac{p_0^2}{2m^*} - \omega^*(n + 1/2)\right)^{-1/2}. \qquad (25)$$

In the sum (25) a special role is played by the values of n for which the radicand is small. These terms give the nonanalytic part of the integral under consideration. At the same time the main contribution as regards magnitude comes from values of n for which $n\omega^* \sim p_0^2/2m^*$. For a weak field the summation over the main region can be replaced by an integration, and from this we find

$$a^2 m^* p_0/\pi^2. \qquad (26)$$

(A contribution to $\partial N/\partial\mu$ of this same order of magnitude is given by the distant regions.) Let us again introduce the number N_0, the integer part of $p_0^2/2m^*$ in units ω^*: $p_0^2/2m^* = (N_0 + 1/2)\omega^* + \Delta\ (\Delta > 0)$. The terms in the sum (25) for which $(N_0 - n) \ll N_0$ are of the order of

$$(\sqrt{2m^*}/2\pi^2)(eH/c)(a^2/\sqrt{\Delta})$$

and remain small in comparison with (26) provided that

$$\Delta \gg \omega^*\left(\omega^* \Big/ \frac{p_0^2}{2m^*}\right). \qquad (27)$$

The condition (27) imposes a restriction on the study of the detailed structure of the oscillations in the immediate neighborhood of the Fermi surface. It must be remembered, however, that the actual region of interest for the de Haas-van Alphen effect just coincides with the condition (27), since in the magnetic fields that are obtainable up to the present time one begins to get a violation of the condition (27) only at the very lowest temperatures. Therefore in our later discussion of the quantum case we shall assume that the condition (27) is always satisfied.

In order to separate out the irregular part of the expression (25) we substitute $n \to N_0 - k$. Subtracting from the new sum the terms that diverge at the upper limit, we get

$$\frac{\sqrt{2m^*}}{2\pi^2}\left(\frac{eH}{c}\right) a^2 \left[\sum_0^{N_0} (\sqrt{\Delta + \omega^* k})^{-1/2} - 2\sqrt{N_0}/\sqrt{\omega^*}\right]_{N_0\to\infty}$$
$$= \frac{\sqrt{2m^*}}{2\pi^2} m^* a^2 \sqrt{\omega^*}\,\zeta\left(\frac{1}{2}, \frac{\Delta}{\omega^*}\right), \qquad (28)$$

where $\zeta(s, x)$ is the generalized Riemann zeta function[10]:

$$\zeta(s, x) = -\frac{\Gamma(1-s)}{2\pi i}\int_\infty^{(0+)} \frac{(-z)^{s-1}e^{-xz}}{1 - e^{-z}}\, dz.$$

Thus apart from regular terms of higher orders in eH/cp_0^2 the integration of the two G-functions in the loop of Fig. 4 a leads to the appearance of small terms which are rapidly oscillating functions of the ratio cp_0^2/eH.

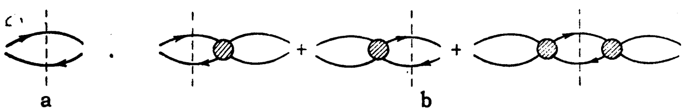

FIG. 4

Returning to Eq. (24), we see that singularities (28) in $\partial N/\partial\mu$ arise in every loop (cf. Fig. 3) in which there are two horizontal lines (with their arrows in opposite directions). Since these terms are small, we have to separate each such loop only once. In particular, the loop can belong to the diagram for a vertex part. As the result of the separation of the singular terms the situation arises which is shown graphically in Fig. 4 b. The loop from which the singularity is separated out is marked with a vertical dashed line. Since these terms are small, we can set the magnetic field strength equal to zero in all the other parts of the diagram.

In this connection we recall once again that the singularity (28) is due to the behavior at large distances $\mathbf{r} - \mathbf{l}$ of the Green's function in the loop of Fig. 4 a. Therefore, using the same procedure with the vertex part as in the derivation of Eq. (21), we get instead of Eq. (24) the following result:

$$\frac{\partial N}{\partial \mu} = \frac{\sqrt{2m^*}m^*}{2\pi^2}\sqrt{\omega^*}\zeta\left(\frac{1}{2}, \frac{\Delta}{\omega^*}\right)\Phi^2,$$

$$\Phi = a\left\{1 + \frac{i}{2(2\pi)^4}\int \Gamma^{0k}_{\alpha\gamma,\gamma\alpha}(p_1, p_2)\, G^0(p_1, \omega)\, G^0(p_2, \omega)\, d^4p\right\} \quad (29)$$

(the vector p_1 is taken on the Fermi surface). In accordance with what was said earlier, our choice between the two limits Γ^{0k} and $\Gamma^{0\omega}$ introduced by Landau[3] must be to take the limit Γ^{0k}, which means that the potential field used in the derivation of Eq. (24) is strictly independent of the time.

The connection of the renormalization factor Φ with physical quantities has been established by Pitaevskii.[9] It turns out that in the absence of a magnetic field

$$\Phi = a\left(\frac{\partial G^{-1}}{\partial \mu}\right)_{p=p_0,\ \omega=0} = \frac{p_0}{m^*}\frac{dp_0}{d\mu}. \quad (29')$$

Substituting this result in the expression obtained earlier, we get

$$\frac{\partial N}{\partial \mu} = \frac{1}{\sqrt{2m^*}\,4\pi^2}\sqrt{\omega^*}\left(\frac{dp_0^2}{d\mu}\right)^2\zeta\left(\frac{1}{2}, \frac{\Delta}{\omega^*}\right). \quad (30)$$

We must calculate the thermodynamic quantities N and Ω (the potential) in terms of the variables μ and V. According to Eqs. (14) and (23), apart from terms of the order $H^{3/2}$ the quantity $p_0(\mu)$ which appears in Eq. (29) and also in Eq. (25) is a function of the chemical potential as defined on the Fermi surface in the absence of a magnetic field. Taking for the oscillating part of Ω only the double integral of the rapidly oscillating terms $\zeta(1/2, \Delta/\omega^*) = \varphi(cp_0^2/eH)$, we get

$$\delta\Omega_{osc} = -\frac{4m^{*2}\omega^{*5/2}}{3\sqrt{2m^*}\,\pi^2}\zeta\left(-\frac{3}{2}, \frac{\Delta}{\omega^*}\right)$$

$$= \frac{m^{*3/2}\omega^{*5/2}}{4\pi^4}\sum_1^{\infty} r^{-5/2}\cos\left(2\pi r\frac{\Delta}{\omega^*} - \frac{\pi}{4}\right),$$

that is, an expression which agrees with the results of I. Lifshitz and Kosevich[1] (in the isotropic model).

4. THE EFFECT OF THE SPIN ON THE OSCILLATIONS

In the scheme that has been expounded it is easy to include also the interaction of the magnetic field with the spin magnetic moment of the electron. The paramagnetic susceptibility of a Fermi liquid has been calculated by Landau.[2] In a metal, however, one cannot separate the paramagnetic part of the susceptibility from the diamagnetic part. At the same time the latter, as we have already said, is due to all of the electrons, and not just to those near the Fermi surface, and therefore cannot be expressed in terms of the characteristics of the spectrum. Owing to this we shall concern ourselves here only with the effect of the spin susceptibility on the quantum oscillations of the thermodynamic quantities.

Let us find how the Green's function (15) changes its form when we include in the Hamiltonian (7) the additional term

$$-\beta\int \psi^+(\mathbf{r})(\sigma\mathbf{H})\psi(\mathbf{r})\,d^3\mathbf{r}.$$

To do this we again consider the derivative $\partial\Sigma_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; \epsilon)/\partial H$. Using the results obtained earlier, we get instead of Eq. (9)

$$\partial\Sigma_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; \varepsilon)/\partial H = -(ie/2c)(x - x')(y + y')\,\Sigma_{\alpha\beta}(\mathbf{r}, \mathbf{r}'; \varepsilon)$$
$$+ \frac{i}{2\pi}\beta\int d\omega\int d^3\mathbf{r}_1 d^3\mathbf{r}_2 d^3\mathbf{l}\,\Gamma_{\alpha\rho,\tau\beta}(\mathbf{r}, \varepsilon; \mathbf{r}_1, \omega; \mathbf{r}_2, \omega; \mathbf{r}', \varepsilon)$$
$$\times G_{\lambda\rho}(\mathbf{l}, \mathbf{r}_1; \omega)\, G_{\tau\varkappa}(\mathbf{r}_2, \mathbf{l}; \omega)\,(\sigma\mathbf{n})_{\varkappa\lambda} \quad (31)$$

(n is a vector in the direction of the field). It is not hard to verify that in the last term we can set H = 0 from the very beginning. Substituting here the expression (19), we get

$$\frac{1}{(2\pi)^6}\int e^{i\mathbf{p}_1(\mathbf{r}-\mathbf{r}')}d^3\mathbf{p}_1\,\frac{i}{2\pi}\int d\omega d^3\mathbf{p}_2\Gamma^k_{\alpha\rho,\tau\beta}(\mathbf{p}_1, \varepsilon; \mathbf{p}_2, \omega; \mathbf{p}_2, \omega; \mathbf{p}_1, \varepsilon)$$
$$\times G^2(\mathbf{p}_2, \omega)(\sigma\mathbf{n})_{\rho\tau}.$$

Because the free functions are isotropic and because for momenta $\mathbf{p}_1$ close to the Fermi surface the integral in brackets (sic) is a slowly varying function of $|\mathbf{p}_1|$, we can write the second term in the right member of Eq. (31) in the form

$$A\,(\sigma\mathbf{n})_{\alpha\beta}\,\delta(\mathbf{r} - \mathbf{r}'), \quad (32)$$

where A is a constant. It follows from Eqs. (31) and (32) that as before the Green's function is of the form (15), but the energy of the excitations is compounded of the orbital part $\omega^*(n + 1/2) + p_z^2/2m^*$ and the spin part $-\xi\boldsymbol{\sigma}\cdot\mathbf{H}$, where ξ is the "effective" magnetic moment of the electron spin: $\xi = a(\beta + A)$. The connection between ξ and the magnitude of the paramagnetic susceptibility is given by relations obtained by Landau.[2]

We can now proceed to the calculation of $\partial N/\partial\mu$. Here the calculations are of a nature quite analogous to that of those done in the preceding section. We first concern ourselves with the calculation of the first term in the formula (24):

$$\frac{i}{2\pi}\int d\omega d^3\mathbf{l}\ \mathrm{Sp}\ \hat{G}(\mathbf{r}, \mathbf{l}; \omega)\,\hat{G}(\mathbf{l}, \mathbf{r}; \omega)$$
$$= \frac{i}{(2\pi)^3}\left(\frac{eH}{c}\right)^2 a^2\sum_n\iint d\omega dp_z\left[(\omega - E_n)^2 + \frac{1}{4}\xi^2H^2\right]$$
$$\times [(\omega - E_n + \xi H/2 + i\delta)(\omega - E_n - \xi H/2 + i\delta)$$
$$\times (\omega - E_n' - \xi H/2 + i\delta)(\omega - E_n' + \xi H/2 + i\delta)]^{-1}.$$

As before, we first integrate this expression over

$d\omega$. Here we note that the integral vanishes both in the case in which all of the poles of the integrand are on one side of the axis of ω, and also in the case in which there are two poles above the axis and two below. The integral is different from zero only in the region of $E_n = \pm\xi H/2$. Defining the integral in a suitable way as a k-limit, we finally get for the first term

$$\frac{1}{4\pi^2}\sqrt{2m^*}\left(\frac{eH}{c}\right)a^2\sum_n\left\{[p_0^2/2m^* - \omega^*(n+1/2) - \xi H/2]^{-1/2} + \left[\frac{p_0^2}{2m^*} - \omega^*\left(n+\frac{1}{2}\right) + \xi H/2\right]^{-1/2}\right\}, \tag{33}$$

that is, the only difference from the formulas written before is that the single sum over n is replaced by the average of two terms with $p_0^2/2m^* \to p_0^2/2m^* \pm \xi H/2$.

Let us now recall that in the second term of Eq. (24) the singularity (33) can be separated out in three ways, as shown in Fig. 4 b. Since the terms involving the magnetic field are small, we can set the field equal to zero in all quantities except the two Green's functions marked with the dashed line in the diagram. In the isotropic model the integral is

$$\int\Gamma_{\alpha\rho,\rho\beta}(p_1,p_2;p_2,p_1)G(p_2)\,G(p_2)d^4p_2 = \frac{1}{2}\delta_{\alpha\beta}\int\Gamma_{\gamma\rho,\rho\gamma}(p_1,p_2;p_2,p_1)\,G^2(p_2)\,d^4p_2.$$

When this fact is used it is not hard to show that the sum of all of the loops of Fig. 4 b gives a singularity of the form (33) multiplied by the renormalization factors (29). Thus instead of Eq. (30) we get as the final result for $\partial N/\partial\mu$

$$\frac{\partial N}{\partial\mu} = \frac{\sqrt{\omega^*}}{8\pi^2\sqrt{2m^*}}\left(\frac{dp_0^2}{d\mu}\right)^2\left\{\zeta\left(\frac{1}{2},\frac{\Delta}{\omega^*}+\frac{\xi H}{2\omega^*}\right)+\zeta\left(\frac{1}{2},\frac{\Delta}{\omega^*}-\frac{\xi H}{2\omega^*}\right)\right\},$$

and for the oscillating part of the thermodynamic potential

$$\sigma\Omega_{osc} = -\frac{2m^{*2}\omega^{*5/2}}{3\pi^2\sqrt{2m^*}}\left\{\zeta\left(-\frac{3}{2},\frac{\Delta+\xi H/2}{\omega^*}\right) + \zeta\left(-\frac{3}{2},\frac{\Delta-\xi H/2}{\omega^*}\right)\right\}. \tag{34}$$

The oscillating parts of all the thermodynamic quantities are usually stated in the form of series of harmonics.[1] For example, by going from Eq. (34) to the oscillating part of the free energy, we get for the magnetic moment the expression

$$M_{osc} = -\frac{m^{*3/2}(\beta^* H)^{1/2}\mu}{2\pi^3\hbar^3 H}\sum_1^\infty\frac{(-1)^r}{r^{3/2}}\cos\left(\frac{\xi}{\beta^*}\pi r\right)\times\sin\left(\pi r\frac{cp_0^2}{e\hbar H}-\frac{\pi}{4}\right),$$

where $\beta^* = e\hbar/m^*c$, and ξ can be connected with the paramagnetic susceptibility χ and with the coefficient γ in the linear term in the heat capacity per unit volume $c = \gamma T$, if we use the results obtained in Landau's paper[2]:

$$\xi/\beta^* = 4\pi^2\chi/3\beta\beta^*\gamma. \tag{36}$$

In this last formula $\beta = e\hbar/mc$ is the magnetic moment of the free electron. Although the factor with the cosine has indeed been written previously, it was then assumed that the spin energy of the electron was equal to $-\beta\boldsymbol{\sigma}\cdot\mathbf{H}$; that is, the Fermi-liquid properties were not taken into account, and the result of this was that the argument of the cosine was written $\pi rm^*/m$.

We also mention that according to Landau's results[2]

$$\frac{1}{\chi} = \beta^{-2}\left(\bar{\zeta}+\frac{4\pi^2}{3\gamma}\right),\quad \bar{\zeta} = \frac{1}{4\pi}\int\zeta(\Omega)\,d\Omega.$$

$\bar{\zeta}$ is the integral of the spin part of the function f[2] of the Fermi liquid:

$$f_{\sigma\sigma'}(\mathbf{p},\mathbf{p}') = f(\mathbf{p},\mathbf{p}') + \zeta(\mathbf{p},\mathbf{p}')(\boldsymbol{\sigma}\boldsymbol{\sigma}')$$

(all of the notations are taken from the paper of Abrikosov and Khalatnikov[11]). Therefore from this one could get an estimate of the size of the Fermi-liquid effects in a metal. In particular, the sign of $\bar{\zeta}$ is very interesting, since for $\zeta > 0$ there could exist in a metal spin oscillations of the type of zeroth sound.[2,11]

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