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THE EQUATION OF STATE AT ULTRAHIGH DENSITIES AND ITS RELATIVISTIC LIMITATIONS

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The most rigid equation of state compatible with the requirements of relativity theory is $p = \epsilon \sim n^2$, $D \rightarrow c$, where p is the pressure, ϵ the volume density of energy, n the density of baryons, D the speed of sound, and c the speed of light. This differs from the previously proposed asymptotic behavior $3p = \epsilon \sim n^{4/3}$, $D \rightarrow 3^{-1/2}c$. The case of interaction of the baryons through a vector field is considered and it is shown (both by considering the interaction of pairs of baryons and by using the stress tensor of the field) how in this case the equation $p = \epsilon \sim n^2$ is realized and how the transition to the equation $3p = \epsilon$ occurs as the mass of the field quanta goes to zero.

1. INTRODUCTION

In connection with the problem of the last stage of the evolution of heavy stars—gravitational collapse—there is now intensified discussion of the question of the equation of state of matter at ultrahigh densities.[1-4] Attempts are being made to perfect the idea of a neutron condensation, which was first put forward by Landau,[5] on one hand by taking into account the various elementary particles, and on the other by taking into account the nuclear interaction between nucleons (and other baryons). Here use is sometimes made of the approximation of a rigid repulsion of nucleons, which leads to an infinite pressure at a finite density. It is obvious that near such a state the speed of sound D would exceed the speed of light, $D > c$. The rigid repulsion is in obvious contradiction with the theory of relativity, and its use in discussing the asymptotic behavior of the equation of state makes no sense, even in case the rigid-repulsion model does give satisfactory numerical agreement for the usual range of nuclear densities. What are the actual limitations imposed by relativity on the law of repulsion and on the asymptotic behavior of the equation of state?

It is generally assumed[6] that already from the special theory of relativity there follows the inequality $3p \leq \epsilon$, where p is the pressure and ϵ the energy density, and ϵ includes the rest masses of the particles. The grounds advanced for this are that for the electromagnetic field $3p = \epsilon$ and for free noninteracting particles with non-vanishing rest masses $3p < \epsilon$. We shall construct below an example of a relativistically invariant theory in which $3p > \epsilon$ is possible and in the limit $p = \epsilon$. An example of this kind is a classical vector field with a mass, interacting with stationary classical point charges.

In Sec. 2 the field equations are formulated and the interaction energy of the charges is found as a function of the density of the charges and of the pressure; then in the limit of large density $p \rightarrow \epsilon$ (Sec. 3). The same result is obtained in a more formal way by considering the stress tensor T_{ik} of the vector field (Sec. 4).

If the energy density ϵ has a power-law dependence on the charge density n (the density of the particles that are sources of the field), $\epsilon = an^\nu$, then the energy and pressure of one particle are

$$\varepsilon_1 = An^{\nu-1}, \qquad p = -d\varepsilon_1/d(1/n) = (\nu-1)an^\nu = (\nu-1)\varepsilon.$$

Thus the asymptotic behavior $3p = \epsilon$ corresponds to $\nu = 4/3$, whereas our asymptotic behavior $p = \epsilon$ corresponds to $\nu = 2$, $p = \epsilon = an^2$.

Finally, the speed of sound is given by the formula (cf. [7])

$$D^2 = c^2 \partial p/\partial\varepsilon,$$

so that for $3p = \epsilon$ we have $D = 3^{-1/2}c$, whereas our asymptotic behavior gives in the limit $D = c$; from this it can be seen that the equation of state obtained from the model of the vector field is the most rigid one possible. A higher ratio $p/\epsilon > 1$ and a higher power $\nu > 2$ are impossible in principle, since a relativistic theory cannot give $D > c$.

When the quantity that plays the role of the mass of the quanta of the vector field goes to zero one gets the well known result which holds for the electromagnetic field, $3p \leq \epsilon$ (Sec. 5).

The main purpose of the present work is to bring out the possibility in principle of a violation of the previously proposed relation $3p \leq \epsilon$.

The choice of the vector field with a mass has been influenced by a paper by Kobzarev and Okun',[8] which develops the theory of the interaction of baryons through a field of heavy neutral vector mesons (vectons). If this theory is confirmed, then at ultrahigh densities (exceeding by a large factor the density of nucleons in nuclei) the pressure will be mainly due to the repulsion of the baryons ($p = an^2$) and not to their Fermi energy ($p_F = Bn^{4/3}$) (Sec. 6). The question of which baryons, and how many kinds of baryons, are to be regarded as elementary particles[9] will then have no effect on the asymptotic behavior of the equation of state.

2. THE FIELD EQUATIONS

Let us take the Lagrangian density in the form

$$L = -\frac{1}{16\pi} F_{ik}^2 - \frac{1}{8\pi}\mu^2 A_k^2, \qquad S_f = i\int L d^4 x, \tag{2.1}$$

$$\partial A_k / \partial x_k = 0, \qquad F_{ik} = \partial A_k / \partial x_i - \partial A_i / \partial x_k. \tag{2.2}$$

We everywhere set $c = 1$; the metric used is $A_k^2 = \mathbf{A}^2 + A_4^2$, $A_4 = iA_0 = i\varphi$. In the quantum theory the mass m of the field quanta is expressed in terms of the constant μ: $m = \mu\hbar$.

We must add to S_f the terms corresponding to the motion of the charges and their interaction with the field:

$$S_p = -M\int ds, \qquad S_i = g\int A_k dx_k, \tag{2.3}$$

where M is the mass of the charges (baryons) and g is their charge.

Varying A, we get the field equations

$$\frac{\partial F_{ik}}{\partial x_k} = -\sum_k \frac{\partial^2 A_i}{\partial x_k^2} = -\mu^2 A_i + 4\pi j_i, \tag{2.4}$$

and varying the trajectories of the particles we get the equations of motion of the particles. These latter do not differ from the equations of motion of particles of charge g in an electromagnetic field F_{ik}.

For a point charge at rest at the origin j_k is

$$j_4 = i\delta(\mathbf{x}), \qquad \mathbf{j} = 0$$

and Eq. (2.4) has the solution

$$\varphi = ge^{-\mu r}/r, \qquad \mathbf{A} = 0. \tag{2.5}$$

3. THE INTERACTION OF THE CHARGES AND THE EQUATION OF STATE

Two charges at rest repel each other with the force

$$|\mathbf{f}_{12}| = -g^2 \frac{d}{dr_{12}}(e^{-\mu r_{12}}/r_{12}). \tag{3.1}$$

The interaction energy of the two charges is

$$g\varphi_{r_1}(r_2) = g^2 e^{\mu r_{12}}/r_{12}. \tag{3.2}$$

Here the action of its own potential on a given charge is obviously included in the mass M of the charge.

In classical theory with quadratic L and linear equations there is no limitation on the application of the principle of superposition. Let us consider a system composed of a large number of charges. Its total energy is

$$E = \sum M_r + \frac{g^2}{2}\sum_{s\neq t}\frac{1}{r_{st}}e^{-\mu r_{st}}. \tag{3.3}$$

If the average density of charges is n and we assume that $n^{-1/3} < \mu^{-1}$, we find as the energy of one charge

$$E_1 = M + \frac{g^2 n}{2}\int e^{-\mu r}\frac{dv}{r} = M + \frac{2\pi g^2 n}{\mu^2}. \tag{3.4}$$

From this we find the energy density

$$\varepsilon = nE_1 = Mn + 2\pi g^2 n^2/\mu^2 \tag{3.5}$$

and the pressure

$$p = -\partial E_1/\partial(1/n) = 2\pi g^2 n^2/\mu^2. \tag{3.6}$$

It can be seen from Eqs. (5) and (6) that in the limit of large n we indeed have $p \to \epsilon$.

The pressure could also have been found from the virial theorem

$$\begin{aligned} 3pV &= \sum \mathbf{r}_s \mathbf{f}_s = \sum_{s\neq t} \mathbf{r}_s \mathbf{f}_{st} = \frac{1}{2}\sum_{s\neq t} \mathbf{r}_{st}\mathbf{f}_{st} \\ &= \frac{g^2}{2}\sum_{s\neq t} r_{st}e^{-\mu r_{st}}(1+\mu r_{st})r_{st}^{-2} \\ &= nV\frac{g^2 n}{2}\int\frac{e^{-\mu r}}{r}(1+\mu r)\,dv = 6\pi g^2 n^2 V/\mu^2; \end{aligned} \tag{3.7}$$

the result naturally agrees with Eq. (3.6).

The increase of E_1 with the density n and the law $p \sim n^2$ are due not to the decrease of the distance to the nearest neighbor, but to the increase of the number of neighbors at a given constant distance $\sim 1/\mu$, which plays the most important part in the integrals (3.4) and (3.7).

Let us assume that the mass of the vector meson is much smaller than that of the baryon, and that the coupling constant is in a definite range of values:

$$m < M, \quad \hbar c\,(m/M)^2 < g^2 < \hbar c\,(M/m). \tag{3.8}$$

Then it is easy to verify that the state in which we are interested, with $3p > \epsilon$, is attained at a density at which both the characteristic length $1/\mu$ and the distance to the nearest neighbor $n^{-1/3}$ are larger than the classical baryon radius g^2/Mc^2 and larger than the baryon Compton wavelength $\hbar/Mc$.*

Consequently the conclusion that states with $3p > \epsilon$ are possible is not due to the extrapolation of the theory to a region in which there are doubts as to its applicability (concerning the potential in the region in which we are interested see the end of Section 6). The interaction law (3.1), (3.2), which has led to the equation of state (3.5), (3.6), was not chosen arbitrarily, but comes from the relativistically invariant field theory with the Lagrangian (2.1).

We remind the reader that the purpose of this paper is to settle the question of the logical possibility of the inequality $3p > \epsilon$ in a relativistic theory; the question of the actual existence of the neutral vector field remains open.

4. THE STRESS TENSOR

The stress tensor, whose diagonal components are $T_{44} = -\epsilon$, $T_{xx} = T_{yy} = T_{zz} = p$ ([6], p. 108), is obtained from L by the formula

$$T_{ik} = L\delta_{ik} - \frac{\partial A_l}{\partial x_i}\frac{\partial L}{\partial(\partial A_l/\partial x_k)}. \tag{4.1}$$

As in the case of the electromagnetic field, to symmetrize this tensor we subtract from it the quantity

$$\frac{1}{4\pi}\frac{\partial}{\partial x_l}(A_i F_{kl}).$$

According to the field equations (3.4), in the absence of charges (cf. [6], p. 103)

$$\frac{\partial}{\partial x_l}(A_i F_{kl}) = F_{kl}\frac{\partial A_i}{\partial x_l} + A_i\frac{\partial F_{kl}}{\partial x_l} = F_{kl}\frac{\partial A_i}{\partial x_l} - \mu^2 A_i A_k. \tag{4.2}$$

From this we finally get the following expressions:

$$\varepsilon = -T_{44} = [\mathbf{E}^2 + \mathbf{H}^2 + \mu^2(\mathbf{A}^2 + \varphi^2)/8\pi]/8\pi, \tag{4.3}$$

$$3p = T_{xx} + T_{yy} + T_{zz} = (\mathbf{E}^2 + \mathbf{H}^2)/8\pi + \mu^2(3\varphi^2 - \mathbf{A}^2)/8\pi. \tag{4.4}$$

For a system of stationary charges distributed with uniform density n the field equations give

$$-\Delta\varphi = -\mu^2\varphi + 4\pi g n, \tag{4.5}$$

*The inequalities (3.8) at the same time assure the validity of the condition $1/\mu > n^{-1/3}$, which is necessary for the replacement of the sum (3.3) by the integral (3.4).

from which we have for a system of large dimensions ($|\Delta\varphi| \ll \mu^2\varphi$)

$$\varphi = 4\pi g n/\mu^2, \quad \varepsilon = 2\pi g^2 n^2/\mu^2, \quad 3p = 6\pi g^2 n^2/\mu^2. \tag{4.6}$$

Adding to the field energy density the energy density coming from the rest mass of the charges, $\epsilon_p = Mn$ (the charges do not contribute to the pressure), we get again the expressions (3.5) and (3.6) and the result

$$p \to \varepsilon, \quad 3p > \varepsilon \quad \text{for} \quad n > \mu^2 M/4\pi g^2, \varphi > M.$$

In (4.5) the system of point charges with the density $\Sigma g\delta(r - r_i)$ has been replaced by a continuous and uniform charge density. We have thus lost the singularities $\varphi \sim (r - r_i)^{-1}$, $|E| \sim (r - r_i)^{-2}$ near the individual charges. These singularities should indeed not be taken into account, since the corresponding energy density has been included in the experimental rest mass of the particles (charges), and the contribution to the pressure is compensated by internal forces, which in classical theory secure the existence of elementary charges.

Let us consider the field in a region free from charges. From (4.3) and (4.4) we find

$$\varepsilon - 3p = \mu^2(\mathbf{A}^2 - \varphi^2)/4\pi. \tag{4.7}$$

We try to find the potentials in the form of a combination of plane waves

$$\mathbf{A}_k(x, t) = \Sigma \mathbf{a}_k e^{ikx - i\omega t}, \quad \varphi_k(x, t) = \Sigma\varphi_k e^{ikx - i\omega t}. \tag{4.8}$$

From the field equations we get the relation

$$\omega^2 = k^2 + \mu^2, \tag{4.9}$$

and from the supplementary condition (2.2) the relation

$$\omega_k\varphi_k = \mathbf{a}_k\mathbf{k}, \tag{4.10}$$

from which it follows that $|\varphi_k| < |\mathbf{a}_k|$, and consequently, according to Eq. (4.7), $\epsilon > 3p$ for such a field.

Thus the free vector field with a mass actually gives $\epsilon > 3p$, in accordance with the picture of heavy field quanta with spin 1, nonvanishing rest mass, and speed of motion less than c. But the relation $\epsilon > 3p$ can be violated for a field of charges. What is the cause of this difference?

It must be remembered that the Lagrangian of the vector field involves not three (the number $2s + 1$ of components of the spin $s = 1$), but four components of the potential, so that the content of the theory is not exhausted by the concept of heavy particles with spin 1. The fourth component just describes the static repulsion. Electrodynamics also is not exhaustively described by the transverse field quanta, but has also the longitudinal Coulomb field. In this connection we may remark that the present theory of the weak interaction can be formulated as the interaction of the fermion current with a vector meson field. Furthermore, the theory includes 0—0 transitions in β decay, which could not be understood from the point of view of the emission by the nucleus of a meson with spin 1 and subsequent decay of this meson into e and ν. Here also the fourth component of the vector meson field comes into action.[10] In electrodynamics $\epsilon \to 3p$ for $\epsilon \to \infty$, both for the free quanta and for the Coulomb interaction. In the theory we are now considering, with the term $\mu^2 A^2$ in L for the free quanta, we naturally have $\epsilon > 3p$, but for the analog of the Coulomb interaction $\epsilon < 3p$; we only have to remember that we cannot confine ourselves to the consideration of the free vector-field quanta alone.

5. THE TRANSITION TO ELECTRODYNAMICS

The transition to the case $\mu = 0$, i. e., to ordinary electrodynamics, is not entirely trivial, since the expressions for ϵ and p, Eqs. (3.4)—(3.6) have the quantity μ^2 in the denominator. The solution of the paradox is that these formulas are valid only for $\mu > 1/R$, where R is the dimensions of the system, and that the equations change their form before μ reaches zero.

The physical peculiarity of the system in question is that the system is not neutral; there is a charge density, which is everywhere of the same sign. With the Coulomb interaction ($\mu = 0$) the energy of such a system cannot be written as $V\epsilon(n)$. In an infinite system with a finite charge density the energy density diverges in the Coulomb case. Let us consider a finite system of charges. In such a system we must prescribe a pressure to retain the charges. According to the virial theorem we get [the notation is as in Eq. (3.7)]

$$3\int p\,dv = 3\bar{p}V = \sum_s \mathbf{r}_s \mathbf{f}_s = \frac{1}{2}\sum \mathbf{r}_{st}\mathbf{f}_{st}. \qquad (5.1)$$

But for the Coulomb potential

$$\mathbf{r}_{st}\mathbf{f}_{st} = e^2/r_{st} = u_{st}, \qquad (5.2)$$

so that

$$3\int p\,dV = 3\bar{p}V = E_{es} = \bar{\varepsilon}_{es}V, \qquad 3\bar{p} = \bar{\varepsilon}_{es}, \qquad (5.3)$$

where the index es denotes the electrostatic part of the energy (the energy density). Recalling also the contribution to ϵ from the rest masses of the charges, we get for the Coulomb field $3\bar{p} < \bar{\epsilon}$, in agreement with[6].

In the argument that led to Eq. (4.6) we cannot let μ go to zero, since Eq. (4.5) for the potential has the solution (4.6) only so long as $|\Delta\varphi| \ll |\mu^2\varphi|$. In order of magnitude, $\Delta\varphi = -\varphi/R^2$, where R is the dimensions of the system. For $\mu < 1/R$, the solution of Eq. (4.5) will be of the form

$$\varphi \sim R^2 gn, \qquad \varepsilon_\varphi \sim \mu^2\varphi^2 \sim \mu^2 R^4 g^2 n^2, \qquad (5.4)$$

where ϵ_φ is the contribution to ϵ from the term $\mu^2\varphi^2$ [cf. Eq. (4.3)]; ϵ_φ goes to zero as it should for $\mu \to 0$, but only after μ has become smaller than 1/R. On the other hand, for $\mu < 1/R$ the contribution to ϵ from $\mathbf{E}^2$ becomes finite, whereas for $\mu \gg 1/R$ this quantity was proportional to the surface, and not to the volume of the system. The term in $\mathbf{E}^2$ occurs with the same coefficient in ϵ and 3p in the forms (4.3) and (4.4), so that again for the field (electrodynamical) part $\epsilon = 3p$.

6. ON THE PRACTICALITY OF THE STATIONARY-CHARGE MODEL

Is the model we have considered, for which we can have $\epsilon < 3p$, a mechanically possible model, a stable one? What could be expected under the actual conditions of an ultradense gas, with quantum phenomena taken into account?

According to Eq. (3.6) the pressure is proportional to n^2, and consequently $\partial p/\partial n > 0$. This sign assures the stability of the system against macroscopic fluctuations of the density n for a prescribed $\bar{n}$ in the volume. On the microscopic scale, according to the field equations (2.4), at the point where the i-th particle is located the potential $\varphi_{(i)}$ produced by all the other particles satisfies the equation

$$\Delta\varphi_{(i)} = \mu^2\varphi_{(i)}, \qquad (6.1)$$

and since $\varphi_{(i)} > 0$ and grad $\varphi_{(i)} = 0$ by considerations of symmetry, $\varphi_{(i)}$ has a minimum, which corresponds to stable equilibrium of the i-th particle, if this particle is at a site of a regular lattice with all the other sites occupied by the other particles.

As we know from Earnshaw's theorem, in the case of the Coulomb interaction a system of charges does not have a stable configuration: the charges enclosed in a given volume will concentrate themselves on the walls of the volume. This property of the system is changed, however, when Coulomb's law is replaced by the potential $e^{-\mu r}/r$.

According to Kobzarev and Okun',[8] we may take for quantum estimates

$$g^2/\hbar = 1, \qquad m = \hbar\mu = M/2, \qquad (6.2)$$

with g assumed the same for all three elementary baryons (n, p, Λ in the scheme of Sakata and Okun'). Then the value of the density at which $3p = \epsilon$ is reached is

$$n_c = \mu^2 M/4\pi g^2 = M^3c^3/16\pi\hbar^3, \qquad (6.3)$$

which corresponds to the nearest-neighbor distance

$$r_{c(ik)} = 4\hbar/Mc = 2/\mu = 0{,}8 \text{ fermi} \qquad (6.4)$$

The value of n_c is twenty times the nuclear density that corresponds to the known expression $R = 1.2A^{1/3}$ f for the radius of a heavy nucleus. At $n \approx n_c$, however, we can still not expect that the formulas will apply, because the density is not large enough for us to regard the nucleons as "crushed" and quit giving separate consideration to other baryons and π and K mesons.

The law

$$\varepsilon = 2aN + aN^2, \qquad p = aN^2;$$

$$N = n/n_c, \qquad a = \tfrac{1}{2} n_c Mc^2 = M^4c^5/100\hbar^3 \qquad (6.4)$$

at best applies for $N > 10$, i.e., just in the region which, in Salpeter's opinion,[4] is impossible because of the "incompressibility" of the hard cores of the nucleons.

Let us estimate the quantum corrections. On the assumption of three types of independent particles (cf. [9]) the energy of the free Fermi gas can be approximated by the expression

$$\varepsilon = 2aN\sqrt{1 + 0.2N^{2/3}} \to 0.9\,aN^{4/3}, \qquad N \gg 1, \qquad (6.5)$$

which replaces the term 2aN in the expression (6.4). The effect of the interaction on the quantum kinetic energy of ultradense matter can be estimated by considering the zero-point energy of the Debye spectrum of the matter with the speed of sound equal to c, the density ϵ/c^2, and 3n independent vibrations per unit volume. We get

$$\varepsilon_d = 1.0\,\hbar c n^{4/3} \approx 0.9\,aN^{4/3}. \qquad (6.6)$$

Although in the region in which we are interested, the potential $g\varphi$ exceeds the rest mass of the particles (charges), we may suppose that as usual the vacuum polarization depends on the fields (**E**, **H**), and not on the potentials, since the equations for motion of particles and pair production are not changed by the addition of the term μ^2A^2 to L. In the system considered the fields do not increase with increase of n. Finally, the quantum motion of the baryons, even with speeds $\sim c$, does not change the charge density they produce, which is involved in the equation for φ. Thus on the assumptions of Kobzarev and Okun' about the role of the vector meson field as the basis of the strong interaction we can evidently expect that the asymptotic behavior of the equation of state will be $p = \epsilon \sim n^2$.

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