ON THE ANALYTIC PROPERTIES OF THE ONE-FERMION GREEN'S FUNCTION IN THE QUANTUM THEORY OF MANY BODIES

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The analytic properties of the one-fermion Green's function and of the mass operator in the quantum theory of many bodies are investigated. In particular, the problem of the poles of the mass operator is examined.

WE shall investigate the analytic properties of the one-fermion Green's function of a system of interacting Fermi particles in the ground state. This problem has been treated recently in a paper by Luttinger.[1] Luttinger confines himself, however, to the consideration of "normal" Fermi systems, i.e., systems that do not have an energy gap in the spectrum of one-fermion excitations. The present paper deals with the analytic properties of the Green's function for systems of both types (both with and without a gap). In particular, a study is made of the problem of the connection of a gap in the spectrum of the one-fermion excitations with a pole of the mass operator of the fermion. Also it is shown that one subtraction may be necessary for the presentation of the mass operator in the form of a dispersion integral.

For simplicity we shall confine ourselves to spatially homogeneous systems, although the results we obtain can be extended to the case of inhomogeneous systems. Also we shall assume that the number of particles N in the system is very large, and accordingly shall neglect all correction terms of order N^{-1} in the quantities considered.

Let us define the one-fermion Green's function by the equation*

$$G(x, x') = i\langle T(\psi(x)\psi^+(x'))\rangle. \tag{1}$$

Using the fact that $G(x-x')$ depends only on the difference $x-x'$, we represent this function as a Fourier integral:

$$G(x) = \frac{1}{(2\pi)^4}\int d\mathbf{p}\, d\omega\, e^{i\mathbf{px}-i\omega t}\, G(p, \omega). \tag{2}$$

As Galitskii and Migdal[2] have shown, $G(p, \omega)$ has the integral representation

$$G(p, \omega) = \int_{-\infty}^{0} d\omega' \frac{A(p, \omega')}{\omega' - \omega + i\delta} + \int_{0}^{\infty} d\omega' \frac{A(p, \omega')}{\omega' - \omega - i\delta}, \tag{3}$$

where ω is the energy measured from the chemical potential μ of the system, and the function $A(p, \omega)$ is nonnegative, i.e., $A(p, \omega) \geq 0$, and satisfies the condition

$$\int_{-\infty}^{\infty} d\omega A(p, \omega) = 1, \tag{4}$$

which is a consequence of the canonical commutation relation

$$\{\psi_\alpha(\mathbf{x}, t), \psi_\beta^+(\mathbf{x}', t)\} = \delta_{\alpha\beta}\,\delta(\mathbf{x} - \mathbf{x}'). \tag{5}$$

We write down Dyson's equation for $G(p, \omega)$:

$$-G^{-1}(p, \omega) = -G_0^{-1}(p, \omega) + M(p, \omega). \tag{6}$$

Here $G_0(p, \omega)$ is the Green's function of the system of noninteracting particles, and $M(p, \omega)$ is what is usually called the mass operator for the fermion.

Equation (6) is essentially a definition of $M(p, \omega)$. As is shown in the paper of Galitskii and Migdal,[2]

$$G_0(p, \omega) = -[\omega + \delta\mu - \zeta_p + i\delta\varepsilon(\zeta_p)]^{-1}$$
$$= -[\omega_0 - \zeta_p + i\delta\varepsilon(\zeta_p)]^{-1},$$
$$\delta\mu = \mu - E_F, \quad \zeta_p = p^2/2m - E_F, \tag{7}$$

where E_F is the Fermi energy, ω_0 is the energy measured from the Fermi energy, and $\epsilon(x) = x|x|^{-1}$.

It follows from Eqs. (3) and (4) that

$$\lim_{|\omega|\to\infty} \omega G(p, \omega) = -1, \tag{8}$$

from which, when we use Eq. (6) and (7), we get

$$\lim_{|\omega|\to\infty} \omega^{-1} M(p, \omega) = 0, \tag{9}$$

i.e., when the absolute value of ω is large $M(p, \omega)$ can increase, but more slowly than ω. It must be noted that Eq. (9) restricts the possible character

*In the literature one often encounters a definition of the Green's function which differs from ours in sign.

of the increase of $M(p, \omega)$ at infinity. Indeed, in general the function $M(p, \omega)$ does not necessarily have to increase for large ω. It can approach a constant or even decrease.

From Eq. (3) there follows an important property of $G(p, \omega)$, regarded as a function of the complex variable ω. First, $G^*(p, \omega) = G(p, \omega^*)$, from which it follows that also $M^*(p, \omega) = M(p, \omega^*)$. Furthermore, as is easily verified, the sign of the imaginary part of $G(p, \omega)$ agrees with that of the imaginary part of the argument. Following Castillejo, Dalitz, and Dyson,[3] we shall call functions that have this property R-functions (see also a paper by Ansel'm and others[4]). It is easy to see that R-functions have the following property: if $H(\omega)$ is an R-function, then $-H^{-1}(\omega)$ is also an R-function. Thus $-G^{-1}(p, \omega)$ and $-G_0^{-1}(p, \omega)$ are also R-functions. When we now use Eqs. (6), (7), and (9), we can verify that $M(p, \omega)$ is also an R-function.

This fact enables us to write a general expression for $M(p, \omega)$ analogous to the representation (3) for the Green's function. To do so we first note that to the zeroes of the Green's function (if there are any) there correspond poles of the function $M(p, \omega)$. On the other hand, an R-function can have poles only of the first order, and these poles must lie on the real axis and have real and negative residues. We readily convince ourselves of this if we note that near a pole any function may be replaced by its principal part (pole term) at that pole. If our function is an R-function, then this principal part is obviously also an R-function. It is not hard to verify that an expression of the type $a(\omega - \Omega)^{-n}$ can be an R-function only for $n = 1$, $a > 0$, and real Ω, and from this there follows the assertion made above about the poles of $M(p, \omega)$.

As has already been noted, the poles of $M(p, \omega)$ correspond to the zeroes of the Green's function. It follows from Eq. (3) that $G(p, \omega)$ can be zero only at points at which $A(p, \omega) = 0$, since $\operatorname{Im} G(p, \omega) \sim A(p, \omega)$ for real ω. On the other hand, as Galitskii and Migdal[2] have shown, $A(p, \omega)$ can be defined in the following way:

$$A(p, \omega) = \begin{cases} \sum_n |\langle n|\psi^+(0)|0\rangle|^2, & \omega > 0 \\ \sum_n |\langle n|\psi(0)|0\rangle|^2 & \omega < 0, \end{cases} \tag{10}$$

where the sum is taken over all states of the system which have the momentum p and the energy ω.

From physical considerations it is improbable that this expression would be zero at any isolated value $\omega = \omega(p)$; it is still more improbable that the function $\operatorname{Re} G(p, \omega)$ would simultaneously vanish at $\omega = \omega(p)$. In any case, we cannot indicate any reasonable physical ideas corresponding to such a situation. Therefore in what follows we shall assume that the zeroes of the Green's function can lie only in regions of values of ω where $A(p, \omega) \equiv 0$.

It also follows from Eq. (3) that in these regions $G(p, \omega)$ is continuous and has a continuous positive derivative. Therefore in each region of values of ω where $A(p, \omega) \equiv 0$ there can be not more than one zero of the function $G(p, \omega)$. We shall assume that $A(p, \omega)$ can be zero only in a region near zero, i.e., for values of ω in the range

$$-\Delta_-(p) < \omega < \Delta_+(p); \quad \Delta_\pm(p) > 0, \tag{11}$$

and it follows from Eq. (10) that to this there corresponds a gap in the spectrum of the one-fermion excitations [the connection between the pole of $M(p, \omega)$ and the gap in the one-fermion spectrum has also been investigated by Migdal[5]].

Thus we must consider two cases: 1) there is no gap in the spectrum of the one-fermion excitations, and 2) there is a gap in this spectrum.

To begin with, let us consider the first case (the only one treated by Luttinger[1]). Assuming that $M(p, \omega)$ increases with increasing ω, we have

$$M(p, \omega) = M(p) + \omega \int_{-\infty}^{0} d\omega' \frac{\alpha(p, \omega')}{\omega'(\omega' - \omega + i\delta)} + \omega \int_{0}^{\infty} d\omega' \frac{\alpha(p, \omega')}{\omega'(\omega' - \omega - i\delta)}, \tag{12}$$

where $M(p) = M(p, 0)$ is a real quantity, and

$$\alpha(p, \omega) = \varepsilon(\omega) \operatorname{Im} M(p, \omega) > 0. \tag{13}$$

It is easy to obtain this expression if we make use of the following fact. The Green's function $G(p, \omega)$ is the limiting value of the function defined by Eq. (3) (if we regard it as a function of the complex variable ω) as ω approaches the real axis from above, if $\omega > 0$, and from below, if $\omega < 0$. It is obvious that $M(p, \omega)$ must have this same property. Applying Cauchy's theorem to $\omega^{-1} M(p, \omega)$ and taking C_1 as the path of integration, we get a function whose limiting value as ω approaches the real axis from above coincides with $M(p, \omega)$ for

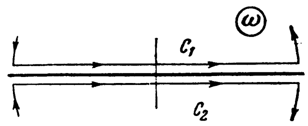

FIG. 1

$\omega > 0$ (Fig. 1). Then taking as the path of integration the contour C_2 of Fig. 1, we get in a similar way an expression for $M(p, \omega)$ for $\omega < 0$. These two expressions are combined in Eq. (12), if we use the fact that $M^*(p, \omega) = M(p, \omega^*)$; it is easy to see that $\alpha(p, \omega)$ must be positive in order for $M(p, \omega)$ to be an R-function.[3,4] If, on the other hand, $M(p, \omega)$ remains bounded (or decreases) as ω increases, then in just the same way, applying Cauchy's theorem to the difference $M(p, \omega) - M(p, \infty)$, we get

$$M(p, \omega) = M(p) + \int_{-\infty}^{0} d\omega' \frac{\alpha(p, \omega')}{\omega' - \omega + i\delta} + \int_{0}^{\infty} d\omega' \frac{\alpha(p, \omega')}{\omega' - \omega - i\delta}, \tag{14}$$

where $M(p) = M(p, \infty)$ is real and $\alpha(p, \omega) > 0$.

Let us go on to the second case. The expression (3), regarded as a function of the complex variable ω, now defines a single analytic function in the plane with cuts from $-\infty$ to $-\Delta_-(p)$ and from $\Delta_+(p)$ to ∞. Therefore $M(p, \infty)$ must also be an analytic function of ∞ in the plane with the same cuts. Applying Cauchy's theorem to the function $M(p, \omega)(\omega - \omega_1)^{-1}$ and integrating along the path shown in Fig. 2, we get as an expression for $M(p, \omega)$

$$M(p, \omega) = M(p, \omega_1) + \delta_p^2 [(\omega_p - \omega)^{-1} - (\omega_p - \omega_1)^{-1}] + \int_{-\infty}^{-\Delta_-(p)} d\omega' \alpha(p, \omega') \left(\frac{1}{\omega' - \omega + i\delta} - \frac{1}{\omega' - \omega_1}\right) + \int_{\Delta_+(p)}^{\infty} d\omega' \alpha(p, \omega') \left(\frac{1}{\omega' - \omega - i\delta} - \frac{1}{\omega' - \omega_1}\right), \tag{15}$$

where $\delta_p^2 \geq 0$, $-\Delta_-(p) < \omega_p < \Delta_+(p)$, $\alpha(p, \omega) > 0$, and $\delta_p^2 \to 0$ if $\Delta_\pm(p) \to 0$; the function $M(p, \omega_1)$ is real, and ω_1 is an arbitrary quantity lying in the interval from $-\Delta_-(p)$ to $\Delta_+(p)$. In particular, we may set $\omega_1 = 0$.

In this expression we have taken account of the fact that, as was shown earlier, $M(p, \omega)$ can have a pole in the region where $A(p, \omega) \equiv 0$, and that the residue at this pole must be negative. If $M(p, \omega)$ remains bounded for large ω (or decreases), then Eq. (15) can be replaced by the expression

$$M(p, \omega) = M(p) + \frac{\delta_p^2}{\omega_p - \omega} + \int_{-\infty}^{-\Delta_-(p)} \frac{d\omega' \alpha(p, \omega')}{\omega' - \omega + i\delta} + \int_{\Delta_+(p)}^{\infty} \frac{d\omega' \alpha(p, \omega')}{\omega' - \omega - i\delta}, \tag{16}$$

where $M(p)$ is real, and the conditions satisfied by δ_p^2, $\alpha(p, \omega)$, and ω_p are the same as before. We note further that a necessary and sufficient

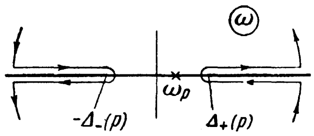

FIG. 2

condition for the existence of a pole of $M(p, \omega)$ are the inequalities

$$G(p, -\Delta_-(p)) < 0, \quad G(p, \Delta_+(p)) > 0. \tag{17}$$

It is interesting to consider as an example the problem of the form of the Green's function and the energies of the elementary excitations in the case in which we can confine ourselves to just the pole term and a constant term in the expression for $M(p, \omega)$. As is well known, the spectrum of the elementary excitations is determined by the zeroes of the expression (6). In our approximation we get from this condition

$$\omega^2 - \omega(\omega_p + \Sigma_p) + \omega_p \Sigma_p - \delta_p^2 = 0, \quad \Sigma_p = \zeta_p - \delta\mu - M(p). \tag{18}$$

Solving this equation, we have

$$\varepsilon_p^{\pm} = \tfrac{1}{2}[\omega_p + \Sigma_p \pm \sqrt{(\omega_p - \Sigma_p)^2 + \delta_p^2}], \tag{19}$$

where ϵ_p^+ is the energy for particles, and ϵ_p^- is the energy for holes. It can be seen from Eq. (19) that $\epsilon_p^+ > \epsilon_p^-$ and equality, $\epsilon_p^+ = \epsilon_p^-$, is impossible, provided only δ_p^2 is not zero. The minimum of the difference $\epsilon_p^+ - \epsilon_p^-$ gives the width of the energy gap in the spectrum of the one-fermion excitations.

The Green's function is obviously of the form

$$G(p, \omega) = -\frac{u_p^+}{\omega - \varepsilon_p^+ + i\delta} - \frac{u_p}{\omega - \varepsilon_p^- - i\delta}, \tag{20}$$

$$u_p^{\pm} = \left(1 + \frac{\partial M}{\partial \omega}\right)^{-1}\Bigg|_{\omega = \varepsilon_p^{\pm}} = \frac{(\omega_p - \varepsilon_p^{\pm})^2}{(\omega_p - \varepsilon_p^{\pm})^2 + \delta_p^2}. \tag{21}$$

The number density of the particles in the system is given by the equation[2]

$$n = \frac{2}{(2\pi)^3} \int d\mathbf{p} \int_{-\infty}^{\infty} d\omega e^{i\omega\delta} G(p, \omega) = \frac{2}{(2\pi)^3} \int d\mathbf{p}\, u^-(p), \tag{22}$$

which can be regarded as an equation for $\delta\mu$.

If the spectrum of the system is symmetrical, i.e., if the energy of a particle, taken as a function of the momentum, differs only in sign from the energy of a hole, then obviously $\omega_p = -\Sigma_p$, and all of the expressions have particularly simple forms:

$$\varepsilon_p^+ = -\varepsilon_p^- = \sqrt{\Sigma_p^2 + \delta_p^2} = \varepsilon_p, \quad u_p^{\pm} = (\varepsilon_p \pm \Sigma_p)(2\varepsilon_p)^{-1}; \quad u_p^+ + u_p^- = 1. \tag{23}$$

For $\delta\mu = 0$ and $M(p) = 0$ these expressions take the form that they usually have in the theory of superconductivity (cf. the paper by Gor'kov[6]).

In the general case the Green's function has poles in the complex plane, whose positions determine the energies and dampings of the elementary excitations. These poles lie on a sheet of the analytic function $G(p, \omega)$ different from that defined by the expression (3). For particles we have at the pole $\mathrm{Re}\,\omega > 0$ and $\mathrm{Im}\,\omega < 0$, and for holes the opposite signs. Using this fact and separating off the pole terms, we have

$$G(p, \omega) = -\frac{u_+(p)}{\omega - \varepsilon_p^+ + i\Gamma_p^+} - \frac{u_-(p)}{\omega + \varepsilon_p^- - i\Gamma_p^-} - \varphi(p, \omega), \quad (24)$$

where $\varphi(p, \omega)$ has no singularities of the pole type, ϵ_p^+ and Γ_p^+ are the energy and damping for particles, and ϵ_p^- and Γ_p^- are those for holes;

$$u_\pm(p) = \left(1 + \frac{\partial M}{\partial \omega}\right)^{-1}\Bigg|_{\omega = \pm(\varepsilon_p^\pm - i\Gamma_p^\pm)}. \quad (25)$$

Obviously as the momentum increases the energy of an elementary excitation must approach the energy of a free particle. In this case the imaginary part of the energy of an elementary excitation, i.e., the damping, is of the form

$$\Gamma_p^+ \approx \pi\alpha(p, \zeta_p). \quad (26)$$

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