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DEPOLARIZATION OF μ^- MESONS IN THE FORMATION OF μ-MESIC ATOMS WITH SPIN-½ NUCLEI

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Besides a fine structure, the levels of μ-mesic atoms formed with nuclei of nonzero spin also have a hyperfine structure. The muon depolarization will be affected if the hyperfine splitting of the weakly excited levels is large in comparison with their width. The magnitude of the effect is estimated for the case in which the nuclear spin is ½. If the hyperfine splitting of the lower excited levels is large in comparison with their width, the polarization of a muon in the K shell should be approximately ⅓ that of the case with zero nuclear spin. If, on the other hand, the level width considerably exceeds the hyperfine splitting, the hyperfine structure will involve only the K shell and the polarization will decrease to ½ that of the case with zero nuclear spin.

1. INTRODUCTION

THE question of the depolarization of μ^- mesons in the formation of μ-mesic atoms has been considered for spin-zero nuclei by several authors.[1-4] A nonzero spin of the nucleus leads to a hyperfine structure in the mesic-atom levels and to an additional depolarization of the muons. Überall[5] and Lubkin[6] estimated this additional depolarization by taking into account only the hyperfine splitting of the ground-state level 1s. Actually, when the nuclear spin is different from zero, depolarization also occurs in the excited levels, owing to the interaction between the muons and the magnetic moment of the nucleus. This is due to the fact that the hyperfine splitting of weakly excited levels in the μ-mesic atoms is comparable in most cases with the width of the levels and sometimes even considerably exceeds it. We shall therefore consider here the depolarization of the muons with allowance for this effect.

The chief role in the depolarization of the μ^- meson in the formation of μ-mesic atoms is played by the spin-orbit interaction. Of importance here is the ratio between the level width and the fine-structure splitting. The radiative width is always small in comparison with the fine-structure splitting, but in the strongly excited upper levels, the probability of Auger transitions is very large and the total width is greater than the fine-structure splitting. It can be assumed that the depolarization does not occur until the muon reaches the level whose fine-structure splitting is comparable to or larger than the Auger width. For the lower levels we can assume the level width to be small in comparison with the fine structure. This means that the time spent by the muon in the levels is sufficient for the reversal of the spin under the action of the spin-orbit interaction, as a result of which the muon is depolarized.

If the nucleus has a nonzero spin, then below some level the hyperfine-structure interaction begins to have an influence on the muon polarization. This effect becomes appreciable when the level width is comparable to the hyperfine splitting. For nuclei of not very large Z (for example, P^{31}, which was investigated in [7]) the hyperfine splitting is, on the average, of the same order as the radiative width. Consequently, the influence of the hyperfine structure becomes evident in the lower levels, where the probability of Auger transitions is small. Hence there exists an intermediate group of levels for which the total width is much less than the fine-structure, but larger than the hyperfine-structure splitting.

Hence the entire process of a cascade transition to the K shell from highly excited states can be split into three stages: in the first stage the width of the level occupied by the muon, owing to the Auger effect, is larger than the fine-structure splitting; in the second stage the width is smaller than the fine-structure splitting, but greater than the hyperfine splitting; in the third stage it is comparable to the hyperfine splitting. In the first stage practically no depolarization occurs; in the second stage, depolarization takes place as a result of the spin-orbit interaction; in the third stage it also occurs as a result of the hyperfine interaction between the magnetic moments of the muon and the nucleus. The separation into the second and third stages was introduced here to facilitate a qualitative explanation of the process. Actually, to estimate the depolarization we simultaneously consider the fine and hyperfine structure splitting.

In order to illustrate the relation between the fine-structure splitting and the level width, we have shown in the table the values of the quantities $\omega_f\tau$ and $\omega_h\tau$ (ω_f and ω_h are the values of the fine and hyperfine splitting, and τ is the lifetime of the given level) for different states of the μ-mesic atom P^{31} with a principal quantum number $n = 5$. Also shown are the probability for Auger transitions Γ_a and the total probability for transitions from the given level $\Gamma = 1/\tau$ in units of 10^{15} sec^{-1}. In order to find the probability of radiative transitions we used Table 15 of [8] and calculated the probability for Auger transitions by multiplying the probability for radiative transitions by the conversion coefficient for the corresponding transition in a nucleus of charge $Z - 1 = 14$. It is seen from the table that the conditions $\omega_f\tau \gg 1$ and $\omega_h\tau < 1$, which should hold in the third stage, are already fulfilled for the levels with $n = 5$, except for the s and p levels. We note that the width of the p level has a basically radiative character and the main contribution to the width comes from the probability of a transition to the K shell. But the ratio of the hyperfine splitting to the radiative width depends weakly on the principal quantum number. Hence, for all p levels of a given mesic atom (in any case, for levels that are not very high), $\omega_h\tau$ is approximately constant.

Thus we shall assume that in the initial state the muon is in a level whose fine splitting is small in comparison to the width. The polarization which occurs in this state will be preserved throughout the entire first stage, where the width is larger than the fine-structure splitting.

Initial state	Width, 10^{15} sec^{-1}		$\omega_f\tau$	$\omega_h\tau$	β	$\bar{\beta}$
	Γ_r	$\Gamma = 1/\tau$				
$5s_{1/2}$	0.028	0.16	—	26	0.26	0.26
$5p_{1/2}$	0.43	0.64	73	2.17	−0.20	0.31
$5p_{3/2}$				0.86	0.56	
$5d_{3/2}$	0.15	0.46	34	0.72	−0.23	0.27
$5d_{5/2}$				0.46	0.60	
$5f_{5/2}$	0.076	0.60	13	0.25	−0.31	0.23
$5f_{7/2}$				0.19	0.64	
$5g_{7/2}$	0.045	0.90	5	0.10	−0.33	0.22
$5g_{9/2}$				0.08	0.66	

At the end of the first stage the muon drops to a level whose fine-structure splitting is large in comparison to the width, and this condition is maintained for all subsequent levels occupied by the muon. This second stage has already been considered in detail.[4] For large orbital angular momenta l, which apparently occur most frequently, the probabilities of states with total angular momenta $j = l + 1/2$ and $j = l - 1/2$ are approximately the same, and the muon polarization in each of them is about $1/3$. We assume that l remains large for the entire second stage. Then the muons will undergo a transition, with a probability close to unity, from states with $j = l + 1/2$ to states with $j_1 = l_1 + 1/2$ and from states with $j = l - 1/2$ to states with $j_1 = l_1 - 1/2$, where $l_1 = l - 1$. In transitions of the first type the muon polarization does not change and in transitions of the second type it acquires a factor $(j+1)j_1/j(j_1+1)$; if l is still large at the end of the second stage, we can neglect the difference between this factor and unity. It thus follows that at the end of the second stage the muon polarization in each of the fine-structure states can also be taken equal to $\sim 1/3$.

In the third stage the hyperfine structure will have an appreciable influence on the muon polarization. The following sections of this article will be devoted to an analysis of this influence for the special case of a nuclear spin $I = 1/2$.

A study of the process of muon depolarization in capture by nuclei with spins $I > 1/2$ requires calculations of much greater complexity.

The basic results obtained in the present work reduce to the following.

For nuclei with $I = 1/2$ the 1s level to which the muon drops as a result of all the transitions consists, as is known, of two sublevels of hyperfine structure with a total angular momentum F equal to zero and unity; the separation between the sublevels is much greater than $\hbar/\tau_\mu$, where τ_μ is the muon lifetime. It is natural to expect, and this will also be shown below, that the probability of dropping to one of these sublevels is proportional to the statistical weight. In the state with $F = 0$ the muon polarization is zero, and therefore the average spin of the muon in the K shell is $3/4$ of the average value of the spin in the state with $F = 1$.

We denote by β the ratio of the muon polarization in the K shell to the polarization at the end of the second stage. If the muon is in a state with $j = l + 1/2$ at the end of the second stage, then we have $\beta = 1/2$ in the absence of hyperfine splitting of the excited levels and $\beta \approx 2/3$ for very strong splitting. If the muon was in a state with $j = l - 1/2$ at the beginning of the third stage, then the influence of the hyperfine splitting of the excited levels will be more important. If there is no hyperfine splitting of the excited levels $\beta \approx 0$, and if the splitting is large $\beta \approx -1/3$, i.e., in this case there is a spin flip. If we take the average over the states $j = l \pm 1/2$ at the beginning of the third stage, we obtain a value of $\sim 1/12$ for the ratio of the final polarization to the initial polarization in the absence of hyperfine splitting in the upper levels and $\sim 1/18$ for strong hyperfine splitting. These results will be discussed in more detail in the following sections. In order to illustrate the results of the calculations, the values of β are given in the table for transitions from different states with a principal quantum number $n = 5$ for the μ-mesic atom P^{31}. The last column of the table gives the quantity

$$\bar{\beta} = \frac{l}{2l+1}\beta_{l-1/2} + \frac{l+1}{2l+1}\beta_{l+1/2}.$$

2. DERIVATION OF BASIC FORMULAS

Since the hyperfine splitting of the μ-mesic atom levels is small, we include these levels in one group. We denote such groups by the letters A, B, C, . . . , M, N and the sublevels of each of these groups by the indices $\alpha, \beta, \gamma, \ldots, \mu, \nu$. Then we readily obtain in the usual way[9] the formula relating the matrix densities $\rho^{(f)}$ and ρ in the final and initial states:

$$\rho^{(f)}_{\nu\nu'} = \exp\{-i\omega_{\nu\nu'}t\} \underset{n}{\mathrm{S}} w_n \rho^{(f_n)}_{\nu\nu'}, \tag{1}$$

where

$$\rho^{(f_n)} = N_n \sum_{\mu\mu'} \frac{H_{\nu\mu}H^+_{\mu'\nu'}}{1 + i(\omega_{\mu\mu'} - \omega_{\nu\nu'})\tau_M} \sum_{\varepsilon\varepsilon'} \frac{H_{\mu\varepsilon}H^+_{\varepsilon'\mu'}}{1 + i(\omega_{\varepsilon\varepsilon'} - \omega_{\nu\nu'})\tau_L} \cdots$$
$$\cdots \sum_{\alpha\alpha'} \frac{H_{\beta\alpha}H^+_{\alpha'\beta'}}{1 + i(\omega_{\alpha\alpha'} - \omega_{\nu\nu'})\tau_A}\rho_{\alpha\alpha'}. \tag{2}$$

Here $H_{\alpha\beta}$ is the matrix element for a transition of the system from a state α to a state β with the emission of a quantum or Auger electron; τ_A, τ_B, . . . are the lifetimes of the levels A, B, . . . ; $\omega_{\beta\beta'} = (E_\beta - E_{\beta'})/\hbar$, and E_β and $E_{\beta'}$ are the energies of the sublevels β and β' referring to one group B; the symbol S denotes the summation over different cascades; w_n is the probability for the n-th cascade and N_n is a normalizing factor determined from the condition $\mathrm{Sp}\,\rho^{(f_n)} = 1$. In (2) the summation includes all possible states of the emitted quanta and Auger electrons.

The μ-meson lifetime τ_μ is much greater than $\hbar/\Delta E$, where ΔE is the hyperfine splitting of the ground state level. Hence, in averaging the matrix $\rho^{(f)}$ over the time, the elements for which $\omega_{\nu\nu'} \neq 0$ vanish, owing to the factor $\exp(-i\omega_{\nu\nu'}t)$. Consequently, for the matrix $\rho^{(F)}$ averaged over the time we can write

$$\rho^{(F)} = \underset{n}{\mathrm{S}} w_n \rho^{(F_n)}; \tag{3}$$

$$\rho^{(F_n)} = N_n \mathrm{D}\Big(\sum_{\mu\mu'} \frac{H_{\nu\mu}H^+_{\mu'\nu'}}{1 + i\omega_{\mu\mu'}\tau_M} \sum_{\varepsilon\varepsilon'} \frac{H_{\mu\varepsilon}H^+_{\varepsilon'\mu'}}{1 + i\omega_{\varepsilon\varepsilon'}\tau_L}$$
$$\cdots \sum_{\alpha\alpha'} \frac{H_{\beta\alpha}H^+_{\alpha'\beta'}}{1 + i\omega_{\alpha\alpha'}\tau_A}\rho_{\alpha\alpha'}\Big), \tag{4}$$

where the symbol D means that in the matrix in the parenthesis the elements connecting states with different energies are assumed to vanish (i.e., those for which $\omega_{\nu\nu'} \neq 0$).

The levels A, B, C, . . . occupied by the μ meson are characterized by a principal quantum number n, an orbital angular momentum l, and a quantum number j ($\mathbf{j} = \mathbf{l} + \mathbf{s}$, where $\mathbf{s}$ is the muon spin), and the sublevels $\alpha, \beta, \gamma, \ldots$ differ in the values of the total angular momentum F and its projection M ($\mathbf{F} = \mathbf{j} + \mathbf{I}$, where $\mathbf{I}$ is the spin of the nucleus). The symbol D thus signifies the separation of the matrix elements which are diagonal in the F representation.

We limit ourselves to the consideration of the case $I = 1/2$ and write the general expression for the density matrix ρ corresponding to a group of states with given values of n, l, and j. This expression should be linear in $\mathbf{n}$, where $\mathbf{n}$ is a unit vector in the direction of the initial polarization of the muon. Using also the condition that ρ is Hermitian and invariant, we obtain

$$\rho = N\Big(1 + a_1\mathbf{jI} + \mathbf{n}\Big\{a_2\mathbf{j} + a_3\mathbf{I} + a_4[\mathbf{jI}]$$
$$+ a_5\Big(\mathbf{j}(\mathbf{jI}) + (\mathbf{jI})\mathbf{j} - \tfrac{2}{3}j(j+1)\mathbf{I}\Big)\Big\}\Big), \tag{5}$$

where the coefficients a_i are real. The normalization condition $\mathrm{Sp}\,\rho = 1$ gives $N = (2j+1)^{-1} \times (2I+1)^{-1}$. We shall consider the density matrix ρ given in the form (5) and calculate the matrix $\rho^{(F_n)}$ from formula (4). To do this, we first consider in (4) one of the terms in the sum over n corresponding to a given cascade and we carry out the summation over $\alpha\alpha'$:

$$\rho^{(1)}_{\beta\beta'} = \sum_{\alpha\alpha'} H_{\beta\alpha}H^+_{\alpha'\beta'}\rho_{\alpha\alpha'}/(1 + i\omega_{\alpha\alpha'}\tau_A). \tag{6}$$

It will be convenient to represent the matrix

$$\rho'_{\alpha\alpha'} = \frac{\rho_{\alpha\alpha'}}{1 + i\omega_{\alpha\alpha'}\tau_A} = \rho_{\alpha\alpha'} + \eta_{\alpha\alpha'},$$

$$\eta_{\alpha\alpha'} = -\frac{i\omega_{\alpha\alpha'}\tau_A}{1 + i\omega_{\alpha\alpha'}\tau_A}\rho_{\alpha\alpha'} \tag{7}$$

in a form similar to (5). For this purpose we provisionally denote states with $F = j + 1/2$ and $F = j - 1/2$ by the subscripts 1 and 2, respectively; then $\omega_{12} = -\omega_{21} = \omega$; omitting the subscript in τ_A, we have

$$\eta_{11} = \eta_{22} = 0;$$

$$\eta_{12} = -\frac{i\omega\tau}{1+i\omega\tau}\rho_{12} = -\frac{\omega^2\tau^2}{1+\omega^2\tau^2}\rho_{12} - \frac{i\omega\tau}{1+\omega^2\tau^2}\rho_{12},$$

$$\eta_{21} = \frac{i\omega\tau}{1-i\omega\tau}\rho_{21} = -\frac{\omega^2\tau^2}{1+\omega^2\tau^2}\rho_{21} + \frac{i\omega\tau}{1+\omega^2\tau^2}\rho_{21},$$

i.e.,

$$\eta = -\frac{\omega^2\tau^2}{1+\omega^2\tau^2}\eta' + \frac{i\omega\tau}{1+\omega^2\tau^2}\eta'';$$

$$\eta' = \rho - \mathrm{D}(\rho), \qquad \eta'' = \eta'\mathrm{P}. \tag{8}$$

Here D denotes the separation of the diagonal parts in the F representation and P is the operator with matrix elements $P_{11} = -P_{22} = 1$, $P_{12} = P_{21} = 0$; it is readily seen that

$$\mathrm{P} = (4\mathbf{jI} + 1)/(2j+1) \tag{9}$$

is such an operator.

In order to separate the diagonal part from (5) we note that the operators $\mathbf{jI}$ and $\mathbf{j} + \mathbf{I} = \mathbf{F}$ are diagonal in the F representation, while $[\mathbf{jI}] = i(\mathbf{j}(\mathbf{jI}) - (\mathbf{jI})\mathbf{j})$ contains no diagonal elements in the F representation. Moreover, we have

$$\mathrm{D}(\mathbf{j}) = \frac{j(j+1) + F(F+1) - 3/4}{2F(F+1)}\mathbf{F}$$

$$= \left[\frac{2j}{2j+1}\frac{j+1+2\mathbf{jI}}{2j+1} + \frac{2(j+1)}{2j+1}\frac{j-2\mathbf{jI}}{2j+1}\right]\mathbf{F}$$

$$= \frac{4}{(2j+1)^2}[j(j+1) - \mathbf{jI}](\mathbf{j}+\mathbf{I})$$

$$= \mathbf{j} - \frac{2}{(2j+1)^2}[\mathbf{j}(\mathbf{jI}) + (\mathbf{jI})\mathbf{j} + \mathbf{j} - 2j(j+1)\mathbf{I}],$$

$$\mathrm{D}[\mathbf{j}(\mathbf{jI})] = \mathrm{D}[(\mathbf{jI})\mathbf{j}]$$

$$= \frac{F(F+1) - j(j+1) - 3/4}{2}\,\frac{F(F+1) + j(j+1) - 3/4}{2F(F+1)}\mathbf{F}$$

$$= \frac{1}{2}[\mathbf{j}(\mathbf{jI}) + (\mathbf{jI})\mathbf{j}] + \frac{1}{2(2j+1)^2}[\mathbf{j}(\mathbf{jI})$$

$$+ (\mathbf{jI})\mathbf{j} + \mathbf{j} - 2j(j+1)\mathbf{I}]. \tag{10}$$

In these formulas we have used the relations

$$2I_iI_k = \tfrac{1}{2}\delta_{ik} + ie_{ikl}I_l; \qquad j_ij_k - j_kj_i = ie_{ikl}j_l. \tag{11}$$

With the aid of (5), (8), (9), and (10) we obtain

$$\eta' = a_4\mathbf{n}[\mathbf{jI}] + \frac{2a_0}{(2j+1)^2}\mathbf{n}[\mathbf{j}(\mathbf{jI}) + (\mathbf{jI})\mathbf{j} + \mathbf{j} - 2j(j+1)\mathbf{I}],$$

$$\eta'' = \frac{ia_4}{2j+1}\mathbf{n}[\mathbf{j}(\mathbf{jI}) + (\mathbf{jI})\mathbf{j} + \mathbf{j} - 2j(j+1)\mathbf{I}] - \frac{2ia_0}{2j+1}\mathbf{n}[\mathbf{jI}], \tag{12}$$

where $a_0 = a_2 - a_3 + (1/6)a_5(2j+3)(2j-1)$, and finally, owing to (8) and (7), we have

$$N^{-1}\rho' = 1 + b_1\mathbf{jI} + \mathbf{n}\Big\{b_2\mathbf{j} + b_3\mathbf{I} + b_4[\mathbf{jI}]$$

$$+ b_5\Big[\mathbf{j}(\mathbf{jI}) + (\mathbf{jI})\mathbf{j} - \frac{2}{3}j(j+1)\mathbf{I}\Big]\Big\}; \tag{13}$$

$$b_1 = a_1, \quad b_2 = a_2 - \frac{1}{2j+1}\frac{\omega\tau}{1+\omega^2\tau^2}\left(a_4 + \frac{2a_0\omega\tau}{2j+1}\right),$$

$$b_4 = \frac{1}{1+\omega^2\tau^2}\left(a_4 + \frac{2a_0\omega\tau}{2j+1}\right),$$

$$b_3 = a_3 + \frac{4}{3}\frac{j(j+1)}{2j+1}\frac{\omega\tau}{1+\omega^2\tau^2}\left(a_4 + \frac{2a_0\omega\tau}{2j+1}\right),$$

$$b_5 = a_5 - \frac{1}{2j+1}\frac{\omega\tau}{1+\omega^2\tau^2}\left(a_4 + \frac{2a_0\omega\tau}{2j+1}\right). \tag{14}$$

For convenience we introduce the notation:

$$A = (j+1)\left(\frac{2a_0}{2j+1} - ia_4\right), \qquad x = \frac{1}{1+i\omega\tau}; \tag{15}$$

then formulas (12) can be rewritten in the form

$$b_1 = a_1, \quad b_2 = a_2 - \frac{\mathrm{Re}\,A(1-x)}{(j+1)(2j+1)}, \quad b_4 = -\frac{\mathrm{Im}\,Ax}{j+1},$$

$$b_3 = a_3 + \frac{4}{3}\frac{j}{2j+1}\mathrm{Re}\,A(1-x),$$

$$b_5 = a_5 - \frac{\mathrm{Re}\,A(1-x)}{(j+1)(2j+1)}. \tag{16}$$

We now carry out the summation in (6). To do this we note that the matrix elements of the different operators occurring in formula (13) between states with different μ and λ (μ and λ are the projections of $\mathbf{j}$ and $\mathbf{I}$) have the following form (the subscripts of the vectors denote the cyclical coordinates -1, 0, 1:

$$\langle\mu\lambda|1|\mu'\lambda'\rangle = \delta_{\mu\mu'}\delta_{\lambda\lambda'}, \quad \langle\mu\lambda|j_i|\mu'\lambda'\rangle = [j(j+1)]^{1/2}\delta_{\lambda\lambda'}C^{j\mu}_{j\mu'1i},$$
$$\langle\mu\lambda|I_i|\mu'\lambda'\rangle = [I(I+1)]^{1/2}\delta_{\mu\mu'}C^{I\lambda}_{I\lambda'1i},$$

$$\langle\mu\lambda|j_ij_k + j_kj_i - \tfrac{2}{3}j(j+1)\delta_{ik}|\mu'\lambda'\rangle$$
$$= \tfrac{1}{3}[10j(j+1)(2j-1)(2j+3)]^{1/2}C^{2i+k}_{1i1k}C^{j\mu}_{j\mu'2i+k}\delta_{\lambda\lambda'}, \tag{17}$$

where C are the Clebsch-Gordan coefficients. Moreover, the matrix elements of the operators H_{LN} and H^{+}_{LN} (L-th and N-th multipole moment of the transition and its projection) depend on μ and λ in the following way:

$$\langle j_1\mu_1\lambda_1|H_{LN}|j\mu\lambda\rangle = Q\delta_{\lambda\lambda_1}C^{j_1\mu_1}_{j\mu LN},$$

$$\langle j\mu'\lambda'|H^{+}_{LN}|j_1\mu_1'\lambda_1'\rangle = Q^*\delta_{\lambda'\lambda_1'}C^{j_1\mu_1'}_{j\mu' LN}$$

(j, μ, μ', λ, λ' refer to the level A; j_1, μ_1, μ_1', λ_1, λ_1' refer to the level B), where the coefficient Q does not depend on the projection of the angular momenta and can therefore appear only in the form of an unessential multiplicative factor in front of the entire matrix, owing to the need for normalization. Since the probability of dipole transitions is very large, we can set L equal to unity (apart from the transition 2s – 1s, for which the polarization does not change). We carry out the summation over N in formula (6).

In the calculation of $\rho^{(1)}$ we can carry out the summation over F, replace M by the summation over $\mu\lambda$ and use Racah's method.[10] The matrix $\rho^{(1)}$ will then be represented in the form (5) with the coefficients $a_i^{(1)}$, which are related in the following way to the coefficients b_i appearing in the matrix ρ':

$$\frac{a_1^{(1)}}{b_1}=\frac{a_2^{(1)}}{b_2}=\frac{a_4^{(1)}}{b_4}=\left[\frac{j(j+1)(2j+1)(2j_1+1)}{j_1(j_1+1)}\right]^{1/2}W(jjj_1j_1;\,1L)(-1)^{-j-j_1+L+1}=\frac{j_1(j_1+1)+j(j+1)-L(L+1)}{2j_1(j_1+1)},$$

$$a_3^{(1)}=b_3,$$

$$a_5^{(1)}=b_5\left[\frac{j(j+1)(2j+1)(2j+3)(2j-1)(2j_1+1)}{j_1(j_1+1)(2j_1+3)(2j_1-1)}\right]^{1/2}W(jjj_1j_1;2L)(-1)^{-j-j_1+L}$$
$$=b_5\left\{\frac{2[j_1(j_1+1)+j(j+1)-L(L+1)]^2+(j_1-j)^2(j_1+j+1)^2-L^2(L+1)^2}{2j_1(j_1+1)(2j_1+3)(2j_1-1)}\right.$$
$$\left.-\frac{(2L^2+2L+3)[j_1(j_1+1)+j(j+1)-L(L+1)]}{2j_1(j_1+1)(2j_1+3)(2j_1-1)}\right\}, \tag{18}$$

where W are the Racah coefficients.

According to (4), we should find the sum

$$\rho^{(2)}=\sum_{\beta\beta'}H_{\gamma\beta}H^{+}_{\beta'\gamma'}\rho^{(1)}_{\beta\beta'}/(1+i\omega_{\beta\beta'}\tau_B),$$

which will be characterized by the coefficients $a_i^{(2)}$ expressed through $a_i^{(1)}$ in a similar way, etc. After carrying out the last summation, we should separate from the obtained matrix the part diagonal in the F representation, as a result of which we obtain the final density matrix $\rho^{(Fn)}=D(\rho^{(fn)})$. This last operation is not difficult to perform if we note that, according to (7), $\rho'=D(\rho)$ for $\omega\tau=\infty$. Thus representing $\rho^{(fn)}$ in the form (5) with the coefficients $a_i^{(fn)}$, we find from formulas (14) the coefficients $b_i^{(fn)}$; in them we set $\omega\tau=\infty$ and obtain the corresponding coefficients a_i^{Fn} of the matrix $\rho^{(Fn)}$.

We note, moreover, that $j=1/2$ in the lowest level (of the K shell) and therefore some of the five coefficients of the final density matrix prove to be unimportant. As a matter of fact, it is not difficult to show that for $j=1/2$ we have $\mathbf{j}(\mathbf{jI})+(\mathbf{jI})\mathbf{j}+(2/3)j(j+1)\mathbf{I}=0$. Since from formula (14) it follows for $\omega\tau=0$ and $j=1/2$ that $b_4=0$, $b_2=b_3=1/2(a_2+a_3)$, then the final density matrix $\rho^{(Fn)}$ will be characterized by two coefficients a_1^{Fn} and $a_2^{Fn}=a_3^{Fn}$. Of course, the same will hold for the matrix $\rho^{(F)}$, for which, according to (3), $a_i^F=\sum_n w_n a_i^{Fn}$. Therefore

$$\rho^{(F)}=\tfrac{1}{4}(1+a_1^F\mathbf{jI}+a_3^F\mathbf{nF})=\tfrac{1}{4}\left[\left(1+\tfrac{1}{4}a_1^F+a_3^F\mathbf{nF}\right)P_+\right.$$
$$\left.+\left(1-\tfrac{3}{4}a_1^F+a_3^F\mathbf{nF}\right)P_-\right]=\tfrac{1}{3}p_+\left(1+\tfrac{3}{2}\lambda^F\mathbf{nF}\right)P_++p_-P_-. \tag{19}$$

Here $P_+=1/4(3+4\mathbf{jI})$ and $P_-=1/4(1-4\mathbf{jI})$ are the operators of the projection on states with $F=1$ and $F=0$, while P_+ and P_- are the probabilities of the corresponding states:

$$p_+=\tfrac{3}{4}\left(1+\tfrac{1}{4}a_1^F\right),\qquad p_-=\tfrac{1}{4}\left(1-\tfrac{3}{4}a_1^F\right). \tag{20}$$

The parameter $\lambda^F=(2/3)a_3^F/(1+1/4a_1^F)$ is equal to the polarization of the muon in the triplet state of the K shell.

In a state corresponding to the beginning of the second stage, when the interaction leading to the hyperfine splitting is still very small, the density matrix is equal to the direct product of the muon density matrix in a level with given values of n, l, j[11] and the unpolarized nucleus:

$$\rho=\frac{1}{2(2j+1)}\left(1+\frac{3\lambda}{j+1}\mathbf{jn}\right). \tag{21}$$

The following values of the coefficients a_i correspond to this state, which we shall call the initial state:

$$a_1=a_3=a_4=a_5=0,\qquad a_2=a_0=3\lambda/(j+1); \tag{22}$$

the parameter λ is related to the mean value of the spin $\langle\mathbf{s}\rangle$ in this state by the expression

$$\langle s\rangle=\lambda\mathbf{n}\,[j(j+1)-l(l+1)+3/4]/2(j+1) \tag{23}$$

From formulas (16) and (18) it is seen that the coefficient $a_1=b_1$ remains equal to zero in all the subsequent states, too. Consequently, $a_1^F=0$, $p_+=3/4$, and $p_-=1/4$. In this way we arrive at the natural result that the probabilities of the muon dropping to the K-shell levels with $F=1$ and $F=0$ are proportional to their statistical weights. As a result, for the characteristics of the polarized state of the μ-mesic atom in the K shell it is sufficient to know one coefficient a_3^F, which is connected with the mean value of the muon spin in the K shell $\langle \mathbf{s}_K \rangle$ by the relation

$$\langle s_K \rangle = \mathrm{Sp}\, \rho^{(F)} \mathbf{s} = \tfrac{1}{4} a_3^F \mathbf{n}. \tag{24}$$

Taking into account (23), we obtain for the depolarization coefficient $\beta_K = \langle \mathbf{s}_K \mathbf{n} \rangle / \langle \mathbf{sn} \rangle$:

$$\beta_K = \frac{a_3^F}{2\lambda} \frac{j+1}{j(j+1) - l(l+1) + 3/4} = \frac{a_3^F}{2\lambda} \times \begin{cases} 1, & j = l + 1/2 \\ -(j+1)/j, & j = l - 1/2 \end{cases}. \tag{25}$$

It is obvious that

$$\beta_K = \sum_n w_n \beta_n, \tag{26}$$

where β_n is the depolarization coefficient corresponding to a given cascade and the summation is carried out over the entire cascade beginning with the initial level.

3. ESTIMATE OF THE DEPOLARIZATION AND DISCUSSION OF RESULTS

We now determine the change in the polarization in any cascade as the muon goes from the initial state to the K shell. Let the muon be initially in a level with the angular momenta l and $j = l + 1/2$. From such a state a transition is possible to a state with $l_1 = l - 1$ and $j_1 = l - 1/2$ or to a state with $l_1 = l + 1$ and $j_1 = l + 1/2$ or $l + 3/2$. Transitions with an increase in $l\,(l_1 = l + 1)$ are unlikely, as is seen, e.g., from Table 15 in [8], and we shall therefore neglect them (except for transitions from s states).

If we insert $j_1 = j - 1$ into formulas (18), they then take the form

$$a_2^{(1)} = b_2 \frac{j+1}{j}, \quad a_3^{(1)} = b_3, \quad a_4^{(1)} = b_4 \frac{j+1}{j},$$
$$a_5^{(1)} = b_5 \frac{(j+1)(2j+3)}{j(2j+1)}. \tag{27}$$

Here the quantity

$$A^{(1)} = j\left(\frac{2a_0^{(1)}}{2j-1} - ia_4^{(1)}\right),$$
$$a_0^{(1)} = a_2^{(1)} - a_3^{(1)} + \tfrac{1}{6} a_5^{(1)} (2j-3)(2j+1)$$

can, after some calculations in which we take (15) and (16) into account, be expressed in terms of A as

$$A^{(1)} = Ax + 2\gamma/(4j^2 - 1),$$
$$\gamma = 2a_2(j+1) - a_3 - \tfrac{2}{3} a_5 (j+1)(2j+3). \tag{28}$$

Constructing the expression $\gamma^{(1)} = 2a_2^{(1)} j - a_3^{(1)} - 2/3\, a_5^{(1)} j(2j+1)$, we can prove with the aid of formulas (16), (15) and (27) that $\gamma^{(1)} = \gamma$. Hence if the transitions occur only with a decrease in j, then it is sufficient to calculate γ once in the initial state, after which we can find the quantity $A^{(i)}$ by applying formula (28) the required number of times.

It is seen from formulas (16) and (18) that in each transition a_3 changes by the quantity $(4/3)\, j(2j+1)^{-1}\, \mathrm{Re}\, A(1-x)$. Hence in the final state

$$a_3^F = a_3 + \frac{4}{3} \sum_{i=0}^{f} \frac{j_i}{2j_i + 1} \mathrm{Re}\, A^{(i)} (1 - x_i),$$
$$A^{(i+1)} = A^{(i)} x_i + \frac{2\gamma}{4j_i^2 - 1}, \tag{29}$$

where the sum runs over all intermediate levels in the given cascade transition and i = 0 corresponds to $j_i = j$. In these formulas referring to a given cascade, we omit for simplicity the subscript that labels the different cascades. In the final state $x_f = 0$, $j_f = 1/2$.

According to formulas (22), (15), and (28), we have in the initial state

$$A = 6\lambda/(2j+1), \qquad \gamma = 6\lambda. \tag{30}$$

Formulas (29) and (30) enable us to determine the coefficient a_3^F for a given cascade transition and, together with this, the depolarization from (25).

We shall consider by way of example the depolarization in two extreme cases: 1) all $x_i = 1$, except for x_f, i.e., the hyperfine splitting is small in comparison to the width of the levels everywhere except for the K shell; 2) $x_i = 0$ for all lower excited levels important for depolarization, i.e., the hyperfine splitting is larger than the level width.

In the first case, formula (28) gives

$$A^{(1)} = \frac{6\lambda}{2j+1} + \frac{12\lambda}{4j^2-1} = \frac{6\lambda}{2j-1}, \quad A^{(2)} = \frac{6\lambda}{2j-3} \text{ etc.}$$
$$A^{(i)} = \frac{6\lambda}{2j_i + 1}, \quad A^{(f)} = 3\lambda, \tag{31}$$

and formula (29) reduces to $a_3^F = (1/3)\, \mathrm{Re}\, A^{(f)} = \lambda$, since from (25) we have $\beta = 1/2$ (for $j = l + 1/2$). Since this occurs for all important cascades, then, according to (26), $\beta_K = 1/2$. This result should also have been expected, for under these conditions the depolarization occurs only as a result of the hyperfine splitting in the K shell. If x is close to unity, but not exactly equal to it, then, if we neglect products with $(1 - x_i)$, we obtain the following formula:

$$\beta = \frac{1}{2} + \mathrm{Re} \sum_{i=0}^{f-1} \frac{2j_i - 1}{(2j_i + 1)^2} (1 - x_i). \tag{32}$$

In the second case $(x_i = 0)$, we have

$$A^{(i)} = \frac{12\lambda}{4(j_i+1)^2 - 1},$$

$$a_3^F = 16\lambda \sum_{i=1}^{f} \frac{j_i}{(2j_i+1)^2(2j_i+3)} + \frac{8\lambda j}{(2j+1)^2}$$

$$\approx \lambda\left[3 - \frac{\pi^2}{6} - \frac{2(6j+1)}{3(2j+1)^3}\right]. \qquad (33)$$

Here j corresponds to the highest level occupied in the given cascade for which $x_i = 0$ [the last expression in (33) is valid apart from terms of the order $(2j+1)^{-4}$]. We thus see that $a_3^F > \lambda$. Assuming that j is primarily large, we arrive at the conclusion that $a_3^F \approx 4\lambda/3$. Consequently, according to (25), $\beta \approx 2/3$ and is approximately the same for all cascades; hence $\beta_K \approx 2/3$. Thus, in the second case (strong hyperfine splitting), the depolarization in the transition from states with $j = l + 1/2$ proves to be about $3/4$ that of the first case (absence of hyperfine structure everywhere but in the last level).

This result can be explained physically as follows: In the presence of hyperfine structure, the muon polarizes the nucleus during the transitions between excited levels. Hence, the muon is already considerably polarized just before the transition to the K shell, which contributes to the preservation of the muon polarization. When the hyperfine splitting at the upper levels is small the muon polarizes the nucleon only in the K shell, as a result of which it loses half its polarization.

If, assuming the x_i small, we neglect their products we obtain for β the formula

$$\beta = \frac{1}{2}\left[3 - \frac{\pi^2}{6} - \frac{2}{3}\,\frac{6j+1}{(2j+1)^3} - \mathrm{Re}\left(\frac{8x}{(2j-1)(2j+1)^2} + \sum_{i=1}^{f-1} \frac{16x_i}{(2j_i-1)(2j_i+1)^2(2j_i+3)}\right)\right]. \qquad (34)$$

We now consider transitions from states with $j = l - 1/2$. If we neglect transitions with an increase in l, then the possible transitions here are either $j_1 = j - 1$ or $j_1 = j$, where their relative probabilities are $(j+1)(2j-1)/j(2j+1)$ and $1/j(2j+1)$. For transitions of the first type, formulas (27) are valid, while for transitions of the second type we have the formulas

$$a_2^{(1)} = b_2\,\frac{j(j+1)-1}{j(j+1)}, \qquad a_3^{(1)} = b_3,$$

$$a_4^{(1)} = b_4\,\frac{j(j+1)-1}{j(j+1)}, \qquad a_5^{(1)} = b_5\,\frac{j(j+1)-3}{j(j+1)}. \qquad (35)$$

We now consider all cascades beginning with a given initial state (in which $j = l - 1/2$) and involving levels of given n and l. In each transition of these cascades, j decreases by unity until it attains the value j_r at which a transition would occur without a change in j. The relative probability of such a cascade is

$$w_r = \frac{j+1}{2j+1}\,\frac{1}{j_r(j_r+1)}. \qquad (36)$$

When all x_i are close to unity, we obtain for such a cascade

$$a_3^F = \lambda\,\mathrm{Re}\left\{\frac{j_r(j_r+1)-1}{j_r(j_r+1)}\left[1 - \sum_{i=0}^{r} \frac{2}{(2j_i+1)^2}(1 - x_i^{(-)}) + \sum_{i=r+1}^{f-1} \frac{2(2j_i-1)}{(2j_i+1)^2}(1 - x_i^{(+)})\right] + \sum_{i=0}^{r} \frac{4j_i}{(2j_i+1)^2}(1 - x_i^{(-)})\right\}, \qquad (37)$$

where we have neglected terms containing the factor $(1 - x_i)$. When $x_i \ll 1$ for all x_i we obtain neglecting products of x,

$$a_3^F = \lambda\,\mathrm{Re}\left\{\frac{2}{3}\,\frac{2j_r(j_r+1)-1}{j_r(j_r+1)} - \left[1 - \frac{4(4j_r+1)}{3(j_r+1)(2j_r+1)^2}\right] \times \sum_{i=0}^{r-1} \frac{16x_i^{(-)}}{(2j_i-1)(2j_i+1)^2(2j_i+3)} - \frac{8(4j_r+1)}{3(j_r+1)(2j_r+1)^4}x_r^{(-)} + \frac{16}{j_r(2j_r-1)(2j_r+1)^4}x_{r+1}^{(+)} - \frac{j_r(j_r+1)-1}{j_r(j_r+1)} \times \sum_{i=r+2}^{f-1} \frac{16x_i^{(+)}}{(2j_i-1)(2j_i+1)^2(2j_i+3)}\right\}. \qquad (38)$$

In formulas (37) and (38) we denote by $x_i^{(+)}$ and $x_i^{(-)}$ the quantities $1/(1 + i\omega\tau)$ for levels with $j_i = l_i + 1/2$ and $j_i = l_i - 1/2$, respectively.

Taking into account (25), (26), and (36), we obtain, after approximate summation over r,

$$\beta_K = -\frac{2(j+1)^2}{j(2j+1)}\,\mathrm{Re}\left[\frac{10-\pi^2}{4} - \frac{1}{4(j+1)} + \sum_{i=0}^{f-1} \frac{6j_i+1}{6(j_i+1)(2j_i+1)}(1 - x_i^{(-)}) + \sum_{i=1}^{f-1} \frac{2j_i-1}{2j_i(2j_i+1)^2}(1 - x_i^{(+)})\right], \qquad 1 - x_i \ll 1; \qquad (39)$$

$$\beta_K = -\frac{2(j+1)^2}{j(2j+1)}\,\mathrm{Re}\left[2 - \frac{\pi^2}{6} - \frac{1}{3(j+1)} - \frac{1}{9}x_{f-1}^{(-)} - \sum_{i=0}^{f-2} \frac{x_i^{(-)}}{j_i(j_i+1)(2j_i+1)(2j_i+3)} - \sum_{i=1}^{f-2} \frac{4x_i^{(+)}}{3(j_i+1)(2j_i-1)(2j_i+1)^2(2j_i+3)}\right], \qquad x_i \ll 1. \qquad (40)$$

From formula (39) it is seen that when $x_i \approx 1$ the depolarization coefficient is $\beta_K \approx 0$, i.e., in this case there is almost complete depolarization. If $x_i \ll 1$, then, as seen from (40), $\beta_K \approx 1/3$ for large initial j, which indicates a spin flip of the muon. Averaging with equal probabilities expression (39) and the corresponding expression (32) for the case with $j = l + 1/2$, we obtain for $j \gg 1$

$$\overline{\beta_K} = \frac{1}{4}\,\mathrm{Re}\left[1 - \sum_{i=0}^{f-1} \frac{6j_i + 1}{3(j_i+1)(2j_i+1)}\left(1 - x_i^{(-)}\right) + \sum_{i=0}^{f-1} \frac{(2j_i-1)^2}{j_i(2j_i+1)^2}\left(1 - x_i^{(+)}\right)\right], \qquad 1 - x_i \ll 1. \tag{41}$$

Similar averaging of expressions (40) and (34) for the case $x_i \ll 1$ leads to the following result for $j \gg 1$

$$\overline{\beta_K} = \frac{1}{6}\,\mathrm{Re}\left[1 + \frac{1}{3}x_{f-1}^{(-)} - \frac{1}{8}x_{f-1}^{(+)} + \sum_{i=0}^{f-2} \frac{3x_i^{(-)}}{j_i(j_i+1)(2j_i+1)(2j_i+3)} - \sum_{i=0}^{f-2} \frac{8(6j_i+5)\,x_i^{(+)}}{(j_i+1)(2j_i-1)(2j_i+1)^2(2j_i+3)}\right], \qquad x_i \ll 1. \tag{42}$$

In the summations in (42) the coefficients of x_i decrease rapidly with j_i, so that for small x_i we can limit ourselves to the terms with x_{f-1}:

$$\overline{\beta_K} \approx \frac{1}{6}\,\mathrm{Re}\left[1 + \frac{1}{3}x_{f-1}^{(-)} - \frac{1}{8}x_{f-1}^{(+)}\right] \qquad (x_i \ll 1). \tag{43}$$

As regards the case $x_i \sim 1$, we cannot carry out such an operation in formula (41), since the corresponding coefficients in the summations decrease more slowly. We can only note that the limiting value of $\bar{\beta}_K$ for $x_i \rightarrow 1$ is $1/4$. Denoting the polarization of the muon in the K shell by λ_K ($\lambda_K = \bar{\beta}_K \lambda$) and taking $\lambda \approx 1/3$, we have

$$\lambda_K = \frac{1}{12} \qquad \text{for } x_i = 1,\ \omega\tau = 0;$$

$$\lambda_K = \frac{1}{18}\left[1 + \frac{1}{3}\,\frac{1}{1+(\omega\tau)_{1/2}^2} - \frac{1}{8}\,\frac{1}{1+(\omega\tau)_{3/2}^2}\right] \qquad \text{for } x_i \ll 1,\ \omega\tau \gg 1. \tag{44}$$

The second formula of (44) contains the quantities $(\omega\tau)_{3/2}$ and $(\omega\tau)_{1/2}$ for the levels $p_{3/2}$ and $p_{1/2}$, for which, as we have already noted, $\omega\tau$ weakly depends on the principal quantum number. For $\omega\tau \gg 1$, we obtain $\lambda_K = 1/18$.

The experimental data for the depolarization of muons on nuclei with spin $I = 1/2$ change only for P^{31}, for which the mean value of the muon polarization in the K shell is $\lambda_K = 3(0.025 \pm 0.005)$.[7] For the levels $3p_{1/2}$, $3p_{3/2}$, $3d_{3/2}$, and $3d_{5/2}$ the values of $\omega_h\tau$ are equal to 3.15, 1.26, 2.27, and 1.45, respectively. We therefore have here the intermediate case $[(\omega_h\tau)_{\mathrm{eff}} \sim 1]$, and consequently we should expect a value between $1/12$ and $1/18$ for the muon polarization in the K shell λ_K. This is in agreement with the experimental data cited above.

In conclusion, we shall comment on the Z-dependence of the final polarization of the muon λ_K. For hyperfine splitting we have $\omega_h \sim Z^3$, while for the radiative width we have $\Gamma_r \sim Z^4$. Hence for the lower levels of the heavy elements the effective value of $\omega_h\tau$ is small. Consequently (if we disregard the possibility of a very large nuclear magnetic moment), in the formation of μ-mesic atoms with nuclei of spin $1/2$ and Z much larger than for phosphorus, λ_K should be close to $1/12 = 0.083$.

For the lowest levels of the μ-mesic atoms formed with light nuclei with $I = 1/2$, where the main contribution to the width also comes from the radiative transitions, we have $\omega_h\tau \gg 1$. But for such mesic atoms the Auger transitions are already important for weakly excited levels. Hence, although we can also expect λ_K to decrease with decreasing Z, it will apparently not attain the lower limit $1/18 = 0.056$ found here for the lightest nuclei.

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