DISPERSION REPRESENTATION OF THE DEUTERON FORM FACTOR

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The second anomalous singularity of the deuteron form factor situated at $s \sim 9\mu^2$ is considered. The dispersion representation is given for the simplest graph yielding this singularity. A method is developed which significantly simplifies the calculations of the discontinuities in the absorption parts of the diagrams with three free ends.

1. INTRODUCTION

THE deuteron form factor, the graph for which is shown in Fig. 1a, depends on one invariant—the square of the photon momentum $P^2 = s$. The squares of the momenta of the two deuterons are equal to the squares of their masses $P_1^2 = P_2^2 = D^2$. The deuteron form factor considered as a function of s has in addition to normal singularities also some anomalous ones. The simplest anomalous singularity of the deuteron is given by the graphs shown[1-2] in Fig. 1b. It is situated at $s = 16M\varepsilon$ (M is the nucleon mass, ε is the deuteron binding energy $D = 2M - \varepsilon$). Subsequent singularities of the form factor are situated at $s > 16M\varepsilon$. At $s = 4\mu^2$ (μ is the mass of the π meson) there exists a normal singularity which appears as the result of graphs of the type shown in Fig. 1c. Analogous graphs with three or four π-meson lines lead to normal singularities at $s = 9\mu^2$ and $s = 16\mu^2$. At

$$s = 4\mu^2 + 16\mu \sqrt{\varepsilon (M - \mu^2/4M)}$$

there exists a second anomalous singularity which results from graphs of a different type (cf., Fig. 1d.)

For sufficiently small values of s we can write the dispersion representation for the deuteron form factor by restricting ourselves to the contribution made to the dispersion part by graphs only of the type shown in Fig. 1b. If we want to write the dispersion representation more accurately, or if s is not sufficiently small, then we must include in the absorption part the contribution from the normal singularities and from graphs of Fig. 1d. The dispersion representation of the graphs of Fig. 1d is the most difficult one; it will be carried out in the present paper.

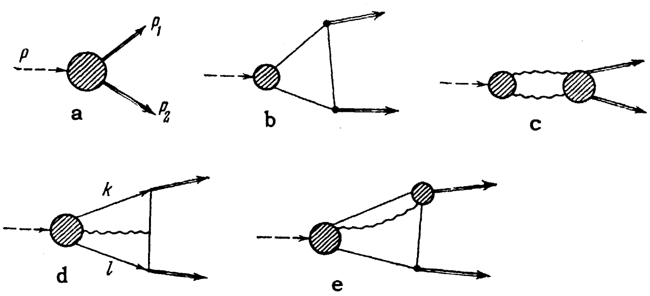

FIG. 1. Dotted line denotes a photon, wavy line denotes a meson, solid line denotes a nucleon, double line denotes a deuteron.

In the absence of anomalous singularities the dispersion representation of the vertex part has the form

$$\frac{1}{\pi} \int_{m^2}^{\infty} \frac{\varphi(s')}{s' - s} ds', \qquad (1)$$

where m is the smallest possible sum of the intermediate masses. If we begin to vary any of the masses in the graphs of the vertex part (1), then anomalous singularities can appear in this vertex part. In this case the contour of integration in (1) will not be a simple one: when the masses are varied the singularities in the absorption part of $\varphi(s)$ deform the contour. Such a process of the deformation of the contour of the dispersion integral (1) by graphs of the type shown in Fig. 1b has been discussed in detail by Mandelstam[3].

The graph shown in Fig. 1d will also deform the contour. In order to determine the manner in which the contour of integration is deformed in the latter case we consider the graph of Fig. 1d in which the shaded block is replaced by a dot (we denote this graph by the symbol 1d*). We shall observe with the aid of Landau graphs the behavior of the singularities of interest to us in the absorption part of such a graph as a function of the variation of the intermediate masses.

The Landau graph for 1d* is shown in Fig. 2a. The double line corresponds to the deuteron mass, the ordinary solid line corresponds to the nucleon mass, the wavy line corresponds to the π-meson mass. The dotted line gives the value of $\sqrt{s_0}$ where s_0 is the position of the singularity of the absorption part of graph 1d* and at the same time is also the position of the singularity of the whole graph 1d. Graphs 2b and 2c give the position of two other singularities of the absorption part. All these singularities of the absorption part are situated approximately at $s \sim \mu^2 - 9\mu^2$. The graph 1d* can in addition also have singularities of the same type as the graphs of Fig. 1e. Corresponding to this the absorption part of 1d* has singularities determined by graphs 2d and 2e. They are situated approximately at $s \sim 4M\mu + \mu^2$. The singularities of the absorption part of graph 1d* determined from Fig. 2 can be not the only possible ones. But these singularities play an essential role in the discussion of the deformation of the contour of integration.

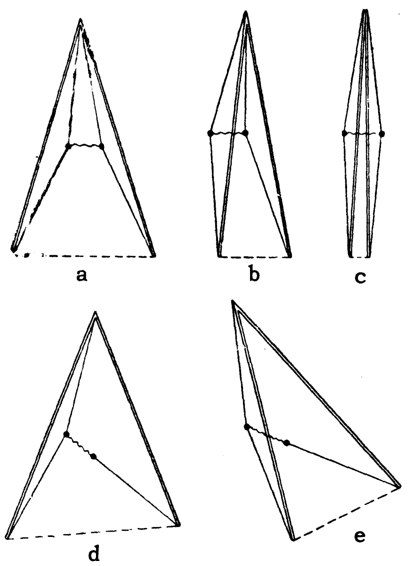

FIG. 2

If we consider the mass μ in graph 1d* to be large, say equal to 2M, then this graph does not have any anomalous singularities and its dispersion representation is given in the form (1), with $m = 2M + \mu$. By means of the graphs of Fig. 2 it can be easily shown that as μ is reduced (μ has a small negative imaginary part) the singularities of the absorption part of 2a and 2d deform the contour of integration, while the singularities of 2b, c and e move without deforming the contour in the direction of smaller values of s. When further decrease in μ causes the singularity of 2a to overtake the singularity of 2e, no further deformations of the contour occur, since μ has a negative imaginary part. The contour of integration in the dispersion representation of graph 1d* is in this case shown in Fig. 3. In the same figure is also shown the position of those singularities of the absorption part of 1d* which are determined by the graphs of Fig. 2. The dotted line shows the position of the cut in the absorption part arising as a result of the singularity a.

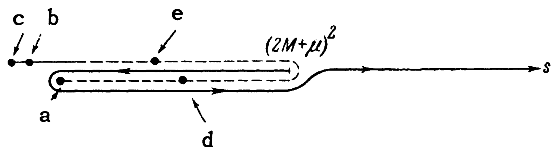

FIG. 3

The dispersion representation of graph 1d* has the form

$$\frac{1}{\pi}\int_{s_0}^{(2M+\mu)^2} ds' \frac{\Delta\varphi(s')}{s'-s} + \frac{1}{\pi}\int_{(2M+\mu)^2}^{\infty} ds' \frac{\varphi(s')}{s'-s},$$

$$s_0 = 4\mu^2 + 16M\mu\sqrt{\frac{\varepsilon}{M}\left(1-\frac{\mu^2}{4M^2}\right)}, \tag{2}$$

where $\Delta\varphi(s)$ is the discontinuity in the absorption part evaluated on the two edges of the cut coming from the singularity a. If we are interested in the dispersion representation of graph 1d* for small values of s we have to know the discontinuity in the absorption part near the singularity a.

We know the absorption part of graph 1d* for $s > 4D^2$. It is given by (for simplicity we assume the nucleon to be a particle without spin)

$$\varphi(s) = \int dk\, dl\, \delta(k^2 - M^2)\, \delta(l^2 - M^2)\, \delta((P-k-l)^2 - \mu^2)$$

$$\times [((P_1-k)^2 - M^2)((P_2-l)^2 - M^2)]^{-1}$$

$$= \frac{\pi^2}{16}\frac{1}{\sqrt{s\left(\frac{s}{4}-D^2\right)}}\int_{(M+\mu)^2}^{(\sqrt{s}-M)^2} d\sigma_1 \int_{\tau_{2-}}^{\tau_{2+}} \frac{d\tau_2}{\tau_2 - M^2}$$

$$\times \left(\sigma_1\left[\frac{(\sigma_1-\tau_2+D^2)^2}{4\sigma_1} - D^2\right]\right)^{-1/2} \ln\frac{\tau_{1+}-M^2}{\tau_{1-}-M^2};$$

$$\tau_{1\pm} = M^2 + D^2 - \frac{1}{2\sigma_1}(\sigma_1 + M^2 - \mu^2)(\sigma_1 - \tau_2 + D^2)$$

$$\pm 2\left(\left[\frac{(\sigma_1+M^2-\mu^2)^2}{4\sigma_1} - M^2\right]\left[\frac{(\sigma_1-\tau_2+D^2)^2}{4\sigma_1} - D^2\right]\right)^{1/2},$$

$$\tau_{2\pm} = M^2 + D^2 - \frac{1}{2}(s - \sigma_1 + M^2)$$

$$\pm 2((s/4 - D^2)[(s-\sigma_1+M^2)^2/4s - M^2])^{1/2}.$$

$$\tau_1 = (P_1-k)^2, \qquad \tau_2 = (P_2-l)^2, \qquad \sigma_1 = (P-l)^2. \tag{3}$$

We can make an analytic continuation of expression (3) towards $s \sim 9\mu^2$ and evaluate the discontinuity in the absorption part. However, such a method of procedure introduces a number of considerable difficulties. We shall proceed in a somewhat different manner. In graph 1d* we shall vary the masses μ and D, ascribing a negative imaginary part to μ, and a positive imaginary part to D. For μ, D > 2M the singularities a, b, c, d, e will go over to the region $s > 16M^2$. In this region we can easily evaluate the discontinuity in the absorption part in the neighborhood of the singularity a. We shall then continue this discontinuity analytically back towards the original values of the masses μ and E. In the next section we shall carry out this procedure.

2. EVALUATION OF THE DISCONTINUITY IN THE ABSORPTION PART OF GRAPH 1d*

The graph 1d* can be analytically continued with respect to the masses μ and D if we ascribe a negative imaginary part to the first mass, and a positive imaginary part to the second mass. As μ is increased the singularities a, b, c, d, e will begin to move towards larger values of s. When μ has almost reached 2M, but we still have $2M - \mu \gg 2M - D$, the position of the singularities will be the same as in Fig. 4a. As μ is increased further the singularities a and d will coincide with the point $(2M + \mu)^2$ (Fig. 4b). Further, the singularity a will move outside the contour of integration (Fig. 4c) and moving in the direction of large values of s will approach the point $s = 4D^2$, will go around it, and will start moving in the direction of smaller values of s (Fig. 4d and e). For $\mu > 2M$ the singularity a will go into the upper half-plane.

If we now increase $D + i\varepsilon$ in such a way that $D > 2M$, then the singularity a will return to the real axis. For $D > 2M$, $\mu > 2M$ and $\mu - 2M \gg D - 2M$ the position of the singularities is shown in Fig. 4f. This follows from the fact that for $|\mu - 2M| \gg |D - 2M|$, $\mu \sim 2M$, $D \sim 2M$ the position of the singularities is a function of the masses of the type

$$A(\mu - 2M) + B\sqrt{(\mu - 2M)(D - 2M)}$$

(this can be seen from the Landau graphs). In going over from D, $\mu < 2M$ to D, $\mu > 2M$ the square root does not change its sign.

Generally speaking, these may not be the only singularities possessed by the absorption part. But from the analytic continuation of (3), which will be carried out below, it can be seen that for $s > (2M + \mu)^2$ the absorption part of graph 1d* has no other singularities. We have to evaluate the discontinuity in the absorption part near the singularity a. This can be done simply if we know the absorption part for values of s situated to the right of the singularity a, and if we then continue it towards smaller values of s.

In the case shown in Fig. 4a, for sufficiently large values of s (for example, for $s \gg 4D^2$) the absorption part is given by expression (3). As μ and D are increased both the region of integration in (3) and the position of the singularities of the integrands will be altered. We begin to vary μ and D in the following order: we first increase $\mu - i\varepsilon$ in such a way that $\mu > 2M$; we then increase $D + i\varepsilon$ in such a way that $D > 2M$, but $\mu - 2M \gg D - 2M$. Then for s lying to the right of the singularity a (cf. Fig. 4b) the region of integration in expression (3) and the position of the singularities in the σ_1, τ_1 plane will be as shown in Fig. 5.

The region of integration is shaded. Singularities I and II come from the logarithm, singularity

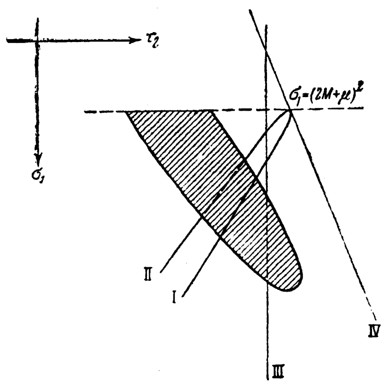

FIG. 5

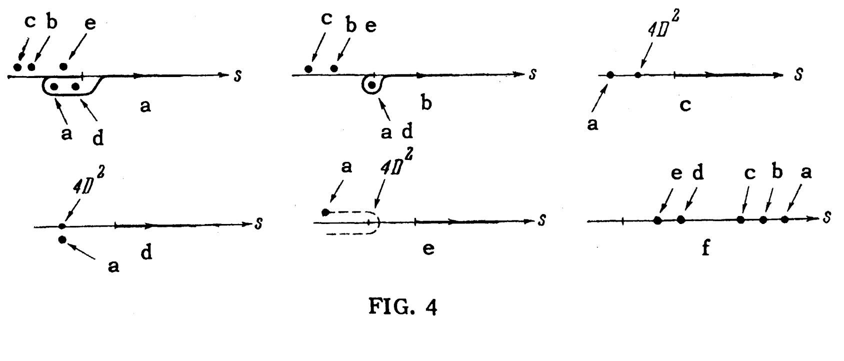

FIG. 4

III comes from the pole, and singularity IV comes from the square root. The fact that the line of singularity is shown by a dotted line signifies that in the complex domain of τ_2 the given singularity lies below the contour of integration.

If we evaluate the integral over τ_2, then the integrand which depends on σ_1 will have singularities which occur at such values of σ_1 at which the line limiting the contour of integration intersects with singularity III (we denote these points of intersection by σ_I, σ_{II}) or with singularity I (σ_{III}, σ_{IV}). Moreover, these singularities occur at σ_1, for which, for example, singularity III intersects with singularities I and II (σ_V and σ_{VI} respectively). The position of the contour of integration with respect to σ_1 and the positions of the singularities $\sigma_I - \sigma_{VI}$ in the case of large s and $\mu > 2M$, $D > 2M$ ($\mu - 2M \gg D - 2M$) are shown in Fig. 6a. If s has a positive imaginary part, then singularities σ_I, σ_{II}, σ_{III} and σ_{IV} are situated in the upper half-plane; therefore, singularities σ_{II} and σ_{IV} displace the contour of integration with respect to σ_1 slightly upwards. If s has a negative imaginary part, σ_I, σ_{II}, σ_{III} and σ_{IV} are displaced into the lower half-plane; singularities σ_I and σ_{III} deform the contour of integration in the direction of their motion.

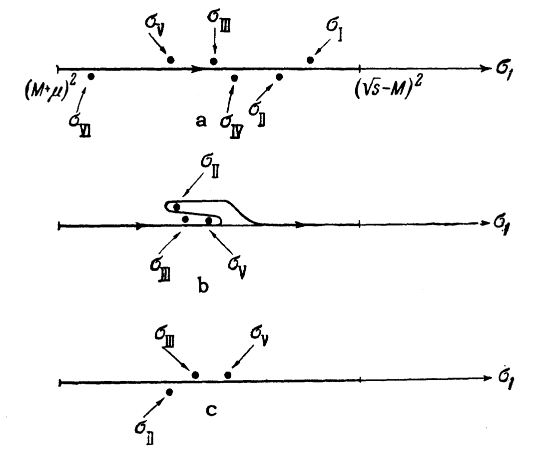

FIG. 6

As s decreases singularities $\sigma_I - \sigma_{IV}$ move in the direction of smaller values of σ_1. When s reaches the singularity a, the singularities σ_{II}, σ_{III} and σ_{IV} will coincide. If s will continue decreasing further retaining a positive imaginary part, then the contour of integration with respect to σ_1 will be deformed (Fig. 6b). If s decreases retaining a negative imaginary part, then there will be no deformation of the contour of integration (Fig. 6c). In this case (Im s < 0) the contour of integration with respect to τ_2 for $\sigma_{II} < \sigma_1 < \sigma_V$

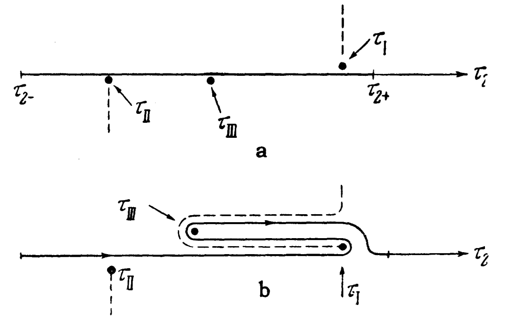

FIG. 7

will be as shown in Fig. 7a; τ_I, τ_{II} denote the positions of the logarithmic singularities, τ_{III} is the position of the pole singularity. Along the contour of integration the logarithm is real to the left of τ_{II} and to the right of τ_I, while in the interval between τ_I and τ_{II} the logarithm has an added imaginary term $i\pi$. For Im s > 0 and $\sigma_{II} < \sigma_1 < \sigma_V$ the contour of integration with respect to τ_2 will be deformed. The positions of the singularities and of the contour of integration are shown in Fig. 7b. For $\sigma_1 > \sigma_V$ and $\sigma_1 < \sigma_{II}$ the position of the contour of integration does not depend on the sign of the imaginary part of s.

It can now be easily shown that the discontinuity in the absorption part between the singularity a and b is given by

$$-\Delta\varphi'(s) = \varphi(s+i\varepsilon) - \varphi(s-i\varepsilon)$$

$$= \frac{\pi^2}{16}\frac{1}{\sqrt{s(s/4-D^2)}}\int_{\sigma_{II}}^{\sigma_V} d\sigma_1 \frac{(-2\pi i)^2}{\{\sigma_1[(\sigma_1-M^2+D^2)^2/4\sigma_1-D^2]\}^{1/2}};$$

$$\sigma_{II} = s/2 + M^2 - [s(s/4-D^2)(1-4M^2/D^2)]^{1/2},$$

$$\sigma_V = M^2 + D^2\mu^2/2M^2$$

$$+ [4D^2\mu^2(1-D^2/4M^2)(1-\mu^2/4M^2)]^{1/2}. \qquad (4)$$

This expression must be analytically continued back towards the physical values of D, μ and towards $s \geq 9\mu^2$, but in reverse order: we first decrease $D + i\varepsilon$ to the physical value of D. The singularity σ_V goes into the upper half plane, and the singularity σ_{II} goes into the lower half plane.

We now decrease $\mu - i\varepsilon$. For $\mu < 2M$ σ_V falls on the real axis, and the upper part of the cut between the singularities of the square root $\sigma_1 = (D-M)^2$, $\sigma_1 = (D+M)^2$. As a result of such continuation the sign in front of the square root in σ_V will not change. Moreover, we must diminish s in order to obtain the value of the discontinuity in the absorption part for $s \geq 9\mu^2$. The values of s must be varied along the path along which the singularity a moved when we were increasing the masses μ and D from their normal values towards μ, D > 2M. This means that we must first take s into the upper half plane, then bring it to the real axis at $s < 4D^2$, increase it to $S = 4D^2$, go around this point clockwise, and then decrease it along the real axis (cf. Fig. 4d, c, e). As a result of this, s reaches the lower edge of the cut from the root $(s/4 - D^2)^{1/2}$; the root in σ_{II} does not change sign.

We obtain the following formula for $\Delta\varphi$ for physical values of μ, D and $4\mu^2 + 16M\mu[\varepsilon(1 - \mu^2/4M^2)/M]^{1/2} < s < 16\mu^2$:

$$\Delta\varphi(s) = \frac{\pi^4}{\sqrt{s(4D^2 - s)}} \int_{\sigma_{II}}^{\sigma_V} d\sigma_1 \{[\sigma_1 - (D - M)^2] \times [-\sigma_1 + (D + M)^2]\}^{-1/2}; \tag{5}$$

σ_{II} and σ_V are given by formula (4). $\Delta\varphi$ vanishes, as it should, for $s = 4\mu^2 + 16M\mu[\varepsilon(1 - \mu^2/4M^2)/M]^{1/2}$.

We have represented the discontinuity in the absorption part in the form of an integral over $\sigma_1 = (P - l)^2$. In addition, we can introduce into formula (5) integration over $\sigma_2 = (P - k)^2$. As long as we are considering only the graph 1d*, generally speaking, this is not worth while doing. But in the more complicated graphs 1d we must carry out the integration both over σ_1 and over σ_2.

In graph 1d* integration over σ_2 can be introduced fairly simply. In the right hand side of (5) we must introduce into the integrand a factor equal to unity:

$$\frac{1}{\pi} \int_{z_-}^{z_+} dz\, [(z_1^2 - 1)(z_2^2 - 1) - (z - z_1 z_2)^2]^{-1/2};$$

$$z_\pm = z_1 z_2 \pm \sqrt{(z_1^2 - 1)(z_2^2 - 1)}, \tag{6a}$$

where z_1, z_2 and z are defined by the relations

$$\sigma_2 = M^2 + s - \frac{1}{2\sigma_1}(\sigma_1 + M^2 - \mu^2)(\sigma_1 + s - M^2) + 2z\left\{\left[\frac{(\sigma_1 + M^2 - \mu^2)^2}{4\sigma_1} - M^2\right]\left[\frac{(\sigma_1 + s - M^2)^2}{4\sigma_1} - s\right]\right\}^{1/2},$$

$$0 = D^2 - \frac{1}{2\sigma_1}(\sigma_1 + M^2 - \mu^2)(\sigma_1 - M^2 + D^2) + 2z_1\left\{\left[\frac{(\sigma_1 - M^2 + D^2)^2}{4\sigma_1} - D^2\right]\left[\frac{(\sigma_1 + M^2 - \mu^2)^2}{4\sigma_1} - M^2\right]\right\}^{1/2},$$

$$0 = s - \frac{1}{2\sigma_1}(\sigma_1 + s - M^2)(\sigma_1 - M^2 + D^2) + 2z_2\left\{\left[\frac{(\sigma_1 + s - M^2)^2}{4\sigma_1} - s\right]\left[\frac{(\sigma_1 - M^2 + D^2)^2}{4\sigma_1} - D^2\right]\right\}^{1/2}. \tag{6b}$$

Formulas (5) and (6) give the required value of the discontinuity in the absorption part near the singularity a. In the next section we shall formulate a rule which considerably simplifies the evaluation of the discontinuities in the absorption parts of diagrams with three free ends.

3. A SIMPLE RULE FOR THE EVALUATION OF THE DISCONTINUITIES IN THE ABSORPTION PARTS OF DIAGRAMS WITH THREE FREE ENDS

The evaluation of the discontinuity in the absorption part of graph 1d* consisted of three stages. We first increased the masses $\mu + i\varepsilon$, $D + i\varepsilon$ to μ, D > 2M, and in doing so we observed the behavior of the absorption part by means of Landau graphs. In the second stage we made an analytic continuation of the absorption part with respect to the masses from the region D, μ < 2M into the region D, μ > 2M, and we evaluated the discontinuity in this absorption part near the singularity of interest to us. In the third stage we made an analytic continuation in strictly reverse order of the discontinuity which we have obtained back towards the physical values of μ and D and towards $s \gtrsim 4\mu^2 + 16\mu[\varepsilon M(1 - \mu^2/4M^2)]^{1/2}$.

The second stage is considerably more difficult than the first and the third taken together. However, it is specifically for this stage that we can introduce a rule which in practice avoids all these difficult calculations. It has been noted already a long time ago that the discontinuities in the absorption parts are a result of the vanishing of the denominators of the functions Δ_F (private communication from V. N. Gribov and I. T. Dyatlov, cf. also[4]). In the region μ, D > 2M we can attempt to replace in (3) $[(P_1 - k)^2 - M^2]^{-1}[(P_2 - l)^2 - M^2]^{-1}$ by $\alpha\delta[(P_1 - k)^2 - M^2]\,\delta[(P_2 - l)^2 - M^2]$ and see whether we might not obtain the same result for the discontinuity in the absorption part as we obtain as a result of a direct calculation. It can be seen at once that this can be done if $\alpha = (-2\pi i)^2$. The discontinuity in the absorption part of (2) in the region μ, D > 2M can then be written in the form

$$(-2\pi i)^2 \int dk\,dl\,\delta(k^2 - M^2)\,\delta(l^2 - M^2)\,\delta((P - k - l)^2 - \mu^2) \times \delta((P_1 - k)^2 - M^2)\,\delta((P_2 - l)^2 - M^2). \tag{7}$$

In doing this we must remember that expression (7) differs from zero for s lying between the singularities a and c (cf. Fig. 4c). For s lying between the singularities a and b expression (7) gives one function, while for s lying between the singularities c and b it gives another function. We require the value of the discontinuity in the absorption part near the singularity a, and, therefore, in expression (7) we must keep s between the singularities a and b.

We can also carry out a similar procedure involving analytic continuation with respect to the masses in the case of a simple graph of type 1b. In order that in such a graph all the lines can be real, it must be reduced by varying the masses, for example, to the graph of Fig. 8. If we do this by varying $M_1 - i\varepsilon$, $D_1 + i\varepsilon$ and $D_2 + i\varepsilon$, then we shall see that the discontinuity in the absorption part of such a graph can be correctly evaluated by replacing $[(P_1 - k)^2 - M_1^2]^{-1}$ by $(-2\pi i)\,\delta \times [(P_1 - k)^2 - M_1^2]$.

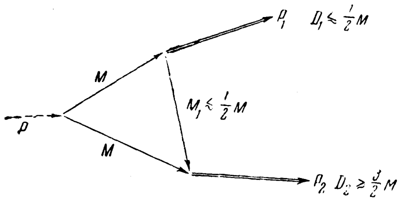

FIG. 8

From these two examples it can be seen that in order to evaluate the discontinuity in the absorption part of any graph corresponding to a diagram with three free ends we must go over to such masses that each vertex of this graph becomes fully decomposable. In the absorption part of this graph we must replace factors of the type $(k^2 - m^2)^{-1}$ by $(-2\pi i)\,\delta\,(k^2 - m^2)$. The discontinuity so obtained must then be continued analytically with respect to the internal masses having a negative imaginary part, and with respect to external masses having a positive imaginary part. The inverse continuation of the discontinuity towards the physical values of the masses is not a very difficult problem. Its solution is aided by an investigation of the behavior of the singularities by means of Landau graphs.

4. CONCLUSION

The introduction of graphs 1c and d into the dispersion representation of the deuteron form factor is of interest since these graphs determine the structure of the deuteron at distances of the order of μ^{-1}. We have presented the dispersion representation only for the graph 1d*. Apparently the representation of other graphs of type 1d is given by the same integrals (5) and (6), but with the integrand containing an additional function $A(s, \sigma_1, \sigma_2)$ corresponding to the shaded block in graph 1d. This is associated with the fact that this shaded block has no anomalous singularities for $s \sim 9\mu^2$ and, therefore, will not likely deform in any manner the contours of the dispersion integrals. It seems to us that by means of the method utilized in the present paper this can be shown more rigorously. Apparently a similar method of evaluating the discontinuities in the absorption parts can be successfully applied to diagrams with four free ends, and this must simplify the evaluation of $\rho(s, t)$ for complicated graphs.

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