ON THE INTERACTION HAMILTONIAN IN QUANTUM FIELD THEORY

A. D. SUKHANOV

Mathematical Institute, Academy of Sciences, U.S.S.R.

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The problem of obtaining the interaction Hamiltonian in quantum field theory is considered. An analysis is made of the expression for the Hamiltonian that follows from the Bogolyubov method, in particular for theories with derivative couplings. It is shown that this expression satisfies the condition for integrability of the Tomonaga-Schwinger equation for any renormalized theory. It is also shown how in this method one can accomplish the removal from the S matrix of the nonphysical dependence on the shape of the intermediate surfaces and achieve gauge invariance of the S matrix for scalar electrodynamics in the Klein-Gordon formalism. All of the results are obtained without taking "surface divergences" into account, and the problem of these divergences remains an open one.

1. INTRODUCTION

A step of great importance in the development of quantum field theory was the introduction of the interaction representation and the transition to the covariant Tomonaga-Schwinger equation [1,2]

$$i\delta\Phi(\sigma)/\delta\sigma(x) = H_{int}(x;\sigma)\,\Phi(\sigma), \qquad (1)$$

where

$$H_{int}(x;\sigma) = i\frac{\delta S(\sigma)}{\delta\sigma(x)}S^+(\sigma) \qquad (2)$$

is the interaction Hamiltonian density in the interaction representation, expressed in terms of the matrix $S(\sigma)$, which is the solution of (1) with the initial condition $S(-\infty) = 1$.

Since it is usually the problem of the theory to obtain $S(\sigma)$ for a known $H_{int}(x;\sigma)$, and not conversely, Eq. (2) does not give us a concrete expression for $H_{int}(x;\sigma)$, and we have to determine it from other considerations. Historically, the first way of obtaining $H_{int}(x;\sigma)$ was through various attempts [3,4] based on the use of considerations of correspondence with the Hamiltonian of the ordinary Schrödinger equation. This approach, however, encountered a number of difficulties in theories in which the interaction Lagrangian includes couplings with derivatives (or vector fields), since the $H_{int}(x;\sigma)$ so obtained did not satisfy the condition of integrability of the Tomonaga-Schwinger equation in the form

$$i\frac{\delta H_{int}(x;\sigma)}{\delta\sigma(y)} - i\frac{\delta H_{int}(y;\sigma)}{\delta\sigma(x)} + [H_{int}(x;\sigma), H_{int}(y;\sigma)] = 0$$

$$\text{for } x \sim y \text{ or } x = y. \qquad (3)$$

The appearance of these difficulties when correspondence arguments are applied directly is due to the fact that in the ordinary Schrödinger equation the quantity that has physical meaning is only the Hamiltonian

$$H(t) = \int_{-\infty}^{\infty} H(x)\Big|_{x^0=t} d\mathbf{x},$$

and not the Hamiltonian density, which, according to the usual canonical formalism, is written in the form

$$H(x) = \sum_k \frac{\partial L}{\partial(\partial u_k/\partial x^0)}\frac{\partial u_k}{\partial x^0} - L(x). \qquad (4)$$

At the same time, what appears in the Tomonaga-Schwinger equation is a Hamiltonian density $H_{int}(x;\sigma)$ which itself has physical meaning, and accordingly must be a covariant function of the field operators and satisfy the condition (3). Naturally when one uses for the $H_{int}(x;\sigma)$ of Eq. (1) the quantity without physical meaning given by Eq. (4) the difficulties we have mentioned arise in cases in which $H_{int}(x;\sigma) \neq -L(x)$.*

Ways of solving the problem were found by Matthews [5] and by Kanesawa and Koba, [6] who proposed two different covariant ways of obtaining $H_{int}(x;\sigma)$ when one has a known Lagrangian.

The idea of the Matthews method is that in going from the ordinary Schrödinger equation to the Tomonaga-Schwinger equation it is not enough merely to go over from state amplitudes $\Phi(t)$ to

*In cases in which the Lagrangian contains a vector field an analogous situation arises in taking the supplementary condition into account.

state amplitudes $\Phi(\sigma)$ and formally go over from the Hamiltonian to the Hamiltonian density; it is necessary to generalize the canonical formalism in a suitable way to the case of arbitrary spacelike surfaces. Then instead of the expression (4) there appears [7] the physically meaningful covariant expression

$$H(x;\sigma)=\sum_k\left(n_\alpha\frac{\partial L}{\partial(\partial u_k/\partial x^\alpha)}\right)\left(n_\beta\frac{\partial u_k}{\partial x^\beta}\right)-L(x), \quad (5)$$

where n_α is the timelike unit vector normal to the arbitrary spacelike surface passing through x, and $(n_\alpha)^2 = 1$, $n_0 > 0$.

It must be emphasized that Eq. (5) establishes a connection between the total Lagrangian and the total Hamiltonian, and in general cannot be applied as a connection between the interaction Lagrangian and the interaction Hamiltonian.

Therefore Matthews [5] has proposed the following way of obtaining $H_{int}(x;\sigma)$, starting from Eq. (5). One must first obtain $\mathbf{H}_{tot}(x;\sigma)$ in the Heisenberg representation by Eq. (5), then separate off from it $H_{int}(x;\sigma)$, and after this go over to the interaction representation. In this way one gets in any theory an $H_{int}(x;\sigma)$ which satisfies the condition (3). For example, for scalar electrodynamics, for which

$$L_{int}(x)=ie:\left(\varphi^*(x)\frac{\partial\varphi}{\partial x^\alpha}-\frac{\partial\varphi^*}{\partial x^\alpha}\varphi(x)\right)A_\alpha(x):$$
$$+e^2:\varphi^*(x)\varphi(x)A_\alpha^2(x):, \quad (6)$$

one finds that

$$H_{int}(x;\sigma)=-L_{int}(x)+e^2:\varphi^*(x)\varphi(x)[n_\alpha A_\alpha(x)]^2:. \quad (7)$$

As can be seen from Eq. (7), in cases in which the interaction Lagrangian contains a coupling with derivatives (or vector fields) $H_{int}(x;\sigma)$ contains terms which have quadratic dependence on the normal to the spacelike surface σ passing through x, and thus is not only a function of x, but also a functional of the surface σ. The method of Yang and Feldman [8] is also one of the variations on the Matthews method.

A different method for obtaining $H_{int}(x;\sigma)$ directly in the interaction representation has been proposed by Kanesawa and Koba, [6] who took as the starting point $L_{int}(x)$ and proposed to look for $H_{int}(x;\sigma)$ in the form

$$H_{int}(x;\sigma)=-L_{int}(x)+A(x;\sigma), \quad (8)$$

where $A(x;\sigma)$ is a function of the field operators and a functional of the surface σ, and is chosen so that the condition (3) is satisfied for $H_{int}(x;\sigma)$.

If one attentively analyzes the expressions for $H_{int}(x;\sigma)$ obtained by the two methods described above, one can perceive the following connection which holds [as Eq. (5) does not] between the interaction Lagrangian and the interaction Hamiltonian in the interaction representation:

$$H_{int}(x;\sigma)=\frac{1}{2}\sum_k\left(n_\alpha\frac{\partial L_{int}}{\partial(\partial\varphi_k/\partial x^\alpha)}\right)\left(n_\beta\frac{\partial L_{int}}{\partial(\partial\varphi_k/\partial x^\beta)}\right)$$
$$-L_{int}(x). \quad (9)$$

Neither in the Matthews method nor in the Kanesawa-Koba method, however, does this connection appear in a natural way in the course of the calculations. Therefore one would like to have a method for obtaining $H_{int}(x;\sigma)$ [which we shall hereafter call simply $H(x,\sigma)$] in which a connection of the form (9) arises legitimately.

A general feature of the methods expounded here for obtaining $H(x,\sigma)$ is that one starts originally from the bare classical interaction Lagrangian. Therefore in the derivation of the S matrix [9] by the solution of Eq. (1) two problems arise. First, both the expression

$$S(\sigma)=T\exp\left\{-i\int_{-\infty}^{\sigma}H(x;\sigma)\,dx\right\}, \quad (10)$$

and the expression for $S(\infty)$ obtained from it in the limit $\sigma\to\infty$ are divergent expressions and require the use of a regularization procedure. Second, for the case of a coupling with derivatives $H(x,\sigma)$ contains terms quadratically dependent on the normals, which in $S(\sigma)$ are [10] the normals to a family of intermediate surfaces used in the solution of Eq. (1) by the method of successive approximations. Thus $S(\sigma)$ turns out to depend on the particular choice of such a family of surfaces, and this is of course physically meaningless.

As is well known, it has not yet been possible to get an expression for $S(\sigma)$ which is free from divergences. If we reason formally and treat $S(\sigma)$ as an intermediate quantity, this indeed is not a serious trouble, since after going from $S(\sigma)$ to $S(\infty)$ we can remove the divergences by performing a suitable regularization. If, however, we do not remove from $S(\sigma)$ the nonphysical dependence on the shape of the intermediate surfaces, it is still present after the passage to the limit $S(\infty)$. A procedure for the removal of this dependence was also first proposed by Matthews, [5] who showed by extremely cumbersome calculations and far from obvious operations with singular functions that in the calculation of $S(\sigma)$ to second order one can bring to light a term which cancels the terms in the Hamiltonian that contains the normals. Since, however, this calculation is not mathematically clear, one would like to know a deeper reason for the results Matthews obtained.

In this paper we shall follow the method for obtaining the scattering matrix and the interaction Hamiltonian proposed by Bogolyubov and Shirkov[11]; in this method one starts at the beginning with an effective interaction Lagrangian in the interaction representation

$$\mathbf{L}(x; g) = L(x) + \sum_{n=2} \frac{1}{n!} \times \int_{-\infty}^{\infty} \Lambda_n (x, x_1 \ldots x_{n-1}) g(x_1) \ldots g(x_{n-1}) dx_1 \ldots dx_{n-1}, \tag{11}$$

which assures the finiteness of the S matrix for the proper choice of the T-product.

We shall consider the possibility of obtaining in the Bogolyubov method an interaction Hamiltonian which will satisfy the condition of integrability of the Tomonaga-Schwinger equation for any renormalizable theory and will have a natural connection of the form (9) with the effective interaction Lagrangian. We shall first examine how to solve in this method the problem of eliminating from the S matrix the nonphysical dependence on the shape of the intermediate surfaces in theories with derivative couplings. At the end we shall consider some features of the structure of the S matrix in scalar electrodynamics.

2. THE INTERACTION HAMILTONIAN IN THE BOGOLYUBOV METHOD

The fundamental quantity in the Bogolyubov method[11] is the matrix S(g), which is a functional of sufficiently smooth functions g(x). The apparatus of the matrix S(g) is, however, insufficient for the solution of all the problems confronting the theory, and it is necessary to introduce the apparatus of the Schrödinger equation. The proposed variational analog of this equation, for sufficiently smooth functions g(x), is

$$i\delta\Phi(g)/\delta g(x) = H(x; g)\Phi(g), \tag{12}$$

where

$$H(x; g) = i\frac{\delta S(g)}{\delta g(x)} S^+(g) \tag{13}$$

is the generalized interaction Hamiltonian density.

The conditions of relativistic covariance, unitarity, and causality for the matrix S(g) must completely determine an expression for H(x; g) which satisfies the conditions of relativistic covariance, Hermiticity, locality, and integrability. That the first two conditions for H(x; g) are satisfied is obvious. The locality condition of the form

$$\delta H(x; g)/\delta g(y) = 0 \quad \text{for } y \lesssim x \tag{14}$$

also follows directly from the condition of causality for S(g). As for the integrability condition, it is also satisfied, as we shall discuss in more detail in Sec. 4. Thus no new problems arise in connection with the generalized Hamiltonian H(x; g).

The problem arises when we want to go from Eq. (12) to Eq. (1) by going to the limit $g \to \theta_\sigma \to \theta(\tau_\sigma - x^0)$, i.e., when we want to obtain in this way an expression for the Hamiltonian H(x; σ), which, unlike the generalized Hamiltonian H(x; g), has physical meaning and must have local character in the ordinary sense—that is, must depend on the state of the fields only in an infinitely small neighborhood of the point x. As has been shown (cf. [11]), the condition (14) is sufficient to assure the local character of H(x; σ) when we go to the limit $g \to \theta_\sigma$. In practice, however, a number of difficulties arose in carrying out this passage to the limit, in particular the problem of "surface divergences."

First of all it must be emphasized that the expression for H(x; σ) is not equal to the limit of H(x; g), as one might have thought, but must be written in the form

$$H(x; \sigma) = \lim_{g\to\theta_\sigma} \int_{-\infty}^{\infty} H(x; g) g'(T_\sigma - x^0) dx^0, \tag{15}$$

where the passage to the limit is made after the integration.

Furthermore, in[11] use was made of the formula

$$\int_{-\infty}^{\infty} H_n(x, x_1 \ldots x_n) g(x_1) \ldots g(x_n) dx_1 \ldots dx_n = -\int_{-\infty}^{\infty} \Lambda_{n+1}(x, x_1 \ldots x_n) g(x_1) \ldots g(x_n) dx_1 \ldots dx_n, \tag{16}$$

where Λ_{n+1} are the same quasi-local operators as in Eq. (11). Equation (16) is obtained on the assumption that at first the regularization masses occurring in Λ_{n+1} remain fixed while $g \to \theta_\sigma$. At the same time it must be admitted that a different order in the passages to limits seems more reasonable, namely that of first letting all $M_i^2 \to \infty$, and then $g \to \theta_\sigma$ (private communication from D. A. Slavnov). But even for the terms in H(x; g) that are linear in the contractions, for which the question of the order of the limits does not arise at all, the analysis made in Sec. 4 shows that when there are derivatives in the Lagrangian there must be some additional terms in the right member of Eq. (16).

The final expression for $H(x;\sigma)$ obtained in [11] by the use of (16) is of the form

$$H(x;\sigma)=\lim_{g\to\theta_\sigma}\Big\{-L(x)-\sum_{n=1}\frac{1}{n!}\int_{-\infty}^{\infty}\Lambda_{n+1}(x,x_1\ldots x_n)\times g(x_1)\ldots g(x_n)\,dx_1\ldots dx_n\Big\}. \quad (17)$$

Since in the general case the quasi-local operators Λ_{n+1} contain derivatives of δ functions and the differentiations can get transferred to the functions $g(x)$, the limit process $g\to\theta_\sigma$ can lead, as is shown in [11], to the appearance of additional nonintegrable expressions of the type of products of δ functions of equal arguments. A more careful analysis, based on passage to the limit by the formula (15) and proper attention to the symmetry properties of the quasi-local operators, shows that, first, the main term in the expression obtained for $H(x;\sigma)$ is $-L(x;1)$, and second, the class of diagrams leading to "surface divergences" is much narrower than it seemed at first glance. In particular, there is no such problem in the vacuum diagrams of any theory, and in the linearly divergent diagrams. In this case also, however, there is still a problem of "surface divergences" for the quadratically divergent diagrams,* which is essentially a reflection of the additional difficulties which arise in the construction of a finite $S(\sigma)$ from a finite $S(g)$, and which are due to the fact that in going over from smooth functions $g(x)$ to θ functions it is in general necessary to redefine the coefficient functions of $S(\sigma)$ as integrable generalized functions. It must be noted that the problem of "surface divergences" in $S(\sigma)$ is not a peculiarity of the Bogolyubov method, but is present implicitly also in the Dyson method, as has also been remarked by Stueckelberg.[12]

Thus although the way of constructing $H(x;\sigma)$ in the Bogolyubov method seems to be the most natural one, it cannot be carried through at present because of the problem of "surface divergences." In this connection it must be emphasized that the operator structure of the "surface diverging" terms that appear in the Bogolyubov method has nothing in common with the operator structure of the terms depending quadratically on the normals which appear in the Hamiltonians of theories that contain derivative couplings (cf. e. g., [7]). These terms are in principle of a different nature. Whereas the "surface diverging" terms depend on the form of the passage to the limit $g\to\theta_\sigma$ for arbitrary σ, the ordinary surface terms depend on the shape of the surface σ itself after the passage to the limit.

Accordingly, while leaving to one side the problem of "surface divergences," we wish to examine the question as to whether both $H(x;g)$ and the $H(x;\sigma)$ obtained from it in the limit satisfy the conditions of integrability of the corresponding equations for theories with derivatives in their bare Lagrangians, since it would seem that the formula (17) does not contain the terms necessary for this. Actually the situation is even more serious, since even when there are no derivatives in the bare Lagrangian the counterterms of all effective Lagrangians, except that of Hurst and Thirring, contain terms with derivatives, which according to Eq. (9) must lead to additional terms in the Hamiltonian. For example, for the second-order boson self-energy diagram we in fact have $L'(x)=ae^2\colon(\partial\varphi/\partial x_\alpha)^2\colon$. Therefore the corresponding $H'(x;\sigma)$ must be of the form

$$H'(x;\sigma)=-L'(x)+\tfrac{1}{2}a^2e^4\colon\Big(n_\alpha\frac{\partial\varphi}{\partial x^\alpha}\Big)^2\colon.$$

Thus the problem of satisfying the integrability conditions of the Tomonaga-Schwinger equation in the Bogolyubov method is exceptionally significant, since it requires the appearance in $H(x;\sigma)$ of a large number of terms with quadratic dependence on the normals. We shall return to this problem in Sec. 4.

3. THE S MATRIX IN THEORIES WITH DERIVATIVE COUPLINGS

Since in the Bogolyubov method the Hamiltonian is found from a known S matrix, we must first elucidate some features of the derivation of the S matrix in theories with derivative couplings. We may ask the question: how is it possible in the methods of Dyson and Bogolyubov to get identical expressions for the S matrix in these theories, although the two methods formally apply the same operation (the T-product) to different original expressions?

As is well known, in deriving the S matrix in such theories by the Dyson method we are forced at first to include in the Hamiltonian terms quadratically dependent on the normals, starting from the condition (3), and then to eliminate these terms from the S matrix by a rather complicated procedure. Naturally this inclusion and subsequent elimination of certain terms is not due to the physics of the problem, but to peculiarities of the mathematical apparatus employed. In fact, in the

*A more detailed communication on this problem will be published later.

Bogolyubov method one is able to avoid these steps and construct the S matrix directly from the Lagrangian, on the basis of a number of general arguments.

A point of primary importance here is that the causality condition in second order (in both its differential [11] and its integral [13] forms) leads to the compatibility condition in the form

$$[L(x),\ L(y)] = 0 \quad \text{for} \quad x \sim y\,(x \neq y), \tag{18}$$

which is valid for any renormalizable theory and in the Bogolyubov method replaces the stronger condition (3).

Next the difference between the actual methods of constructing the S matrix comes into play. In the Bogolyubov method we arrive at once at T-products of Lagrangians. Here, if the Lagrangian contains derivatives, there arises in general the problem of a finite arbitrariness in the definition of the second derivative of $D^c(x-y)$, if the definition of $D^c(x-y)$ for coincident arguments is fixed. In fact, knowing that [11]

$$D^c(x-y) = \theta(x^0-y^0)\,D^-(x-y) - \theta(-x^0+y^0)\,D^+(x-y) = \frac{1}{(2\pi)^4}\int \frac{e^{ik(x-y)}dk}{m^2-k^2-i\varepsilon}, \tag{19}$$

we can define $\partial^2 D^c(x-y)/\partial x^\alpha \partial y^\beta = \tilde{D}^c_{\alpha\beta}(x-y)$ either in the form

$$\tilde{D}^c_{\alpha\beta}(x-y) = \theta(x^0-y^0)\,\tilde{D}^-_{\alpha\beta}(x-y) - \theta(-x^0+y^0)\,\tilde{D}^+_{\alpha\beta}(x-y) - n_\alpha n_\beta \delta(x-y) = \frac{1}{(2\pi)^4}\int \frac{(-k_\alpha k_\beta)\,e^{ik(x-y)}dk}{m^2-k^2-i\varepsilon}, \tag{20}$$

or in the form

$$\tilde{D}^c_{\alpha\beta}(x-y) = \theta(x^0-y^0)\,\tilde{D}^-_{\alpha\beta}(x-y) - \theta(-x^0+y^0)\,\tilde{D}^+_{\alpha\beta}(x-y), \tag{21}$$

or in any other form which differs from the expressions (20) and (21) by a quasi-local covariant operator.

Since, however, the S-matrix apparatus is adapted to the momentum representation, from the point of view of this apparatus the simplest and most physically reasonable definition is Eq. (20), and this is the one adopted in [11]. With this definition of $D^c_{\alpha\beta}(x-y)$ one gets for the S matrix in a theory with derivative coupling the expression

$$S(\infty) = T\exp\left\{i\int_{-\infty}^{\infty} L(x)\,dx\right\}, \tag{22}$$

which is free from any nonphysical dependence on the shapes of the intermediate surfaces.

In Dyson's method, on the other hand, in solving the Tomonaga-Schwinger equation we first arrive at advanced products of repeated commutators of Hamiltonians, and only subsequently, on regrouping them, do we come to the corresponding T-products. In doing so one uses, for example, the formula

$$D^{adv}(x-y) = D^c(x-y) - D^-(x-y), \tag{23}$$

where $D^c(x-y)$ is of the form (19) and

$$D^{adv}(x-y) = -\theta(-x^0+y^0)\,D(x-y). \tag{24}$$

If we now formally differentiate both sides of Eq. (23) twice, we still have equality of the two sides if on both the right side and the left side we simultaneously either do or do not differentiate the θ functions. The actual situation is different, however. In solving the Tomonaga-Schwinger equation we arrive, for example, at an expression $\theta(-x^0+y^0)\,[H(x;\sigma),\ H(y;\sigma)]$, and when we calculate this out in a theory with derivative coupling we get among other quantities a term

$$-\theta(-x^0+y^0)\,\tilde{D}_{\alpha\beta}(x-y), \tag{25}$$

which from the point of view of the apparatus of the Tomonaga-Schwinger equation it is most natural to take as the definition of $\tilde{D}^{adv}_{\alpha\beta}(x-y)$, because in solving this equation the situation in which one would differentiate the θ function in the expression (24) never arises.

Thus for the most reasonable definitions of $\tilde{D}^c_{\alpha\beta}(x-y)$ and $\tilde{D}^{adv}_{\alpha\beta}(x-y)$ the contraction of the type (23) does not occur and is replaced by the formula

$$\tilde{D}^{adv}_{\alpha\beta}(x-y) = \tilde{D}^c_{\alpha\beta}(x-y) - \tilde{D}^-_{\alpha\beta}(x-y) + n_\alpha n_\beta \delta(x-y). \tag{26}$$

This makes understandable a second point of difference of the two methods for constructing the S matrix in the case of a derivative coupling. Solving the Tomonaga-Schwinger equation, we at first arrive at the expression $[\tilde{D}^{adv}_{\alpha\beta}(x-y) + \tilde{D}^-_{\alpha\beta}(x-y)]$; we can call this quantity $\tilde{D}^c_{\alpha\beta}(x-y)$ [in accordance with Eq. (21)], and use for the S matrix the formula

$$S(\infty) = T\exp\left\{-i\int_{-\infty}^{\infty} H(x;\sigma)\,dx\right\}. \tag{27}$$

The expression (21), however, is not the most convenient definition of $\tilde{D}^c_{\alpha\beta}(x-y)$ from the point of view of the S-matrix apparatus. On the other hand, if we wish to use the definition (20), we have to go over from $\tilde{D}^{adv}_{\alpha\beta}(x-y)$ to $\tilde{D}^c_{\alpha\beta}(x-y)$ by Eq. (26), which automatically brings us from Eq. (27) to Eq. (22).

Thus the possibility in principle which Matthews has indicated for eliminating the nonphysical dependence on the shape of the intermediate surfaces for theories with derivative couplings can be connected with the possibility of giving different definitions* of $\widetilde{D}^c_{\alpha\beta}(x-y)$ and with the fact that the apparatus of the Tomonaga-Schwinger equation leads in the most natural way to the definition (21). Furthermore one can use for the S matrix either a formula of the type (27) or one of the type (22); one need only remember that actually different [in the part with $\widetilde{D}^c_{\alpha\beta}(x-y)$] definitions of the T-product are used in these formulas. We emphasize that with this approach to the problem there is a natural unity between the cases of a Lagrangian with derivatives and a Lagrangian containing a vector field. In both cases one can get an S matrix which has physical meaning by defining the second derivative of $D^c(x-y)$ in a suitable way.

4. THE INTEGRABILITY CONDITION IN THE BOGOLYUBOV METHOD

As is clear from Sec. 3, for the construction of the S matrix by the Bogolyubov method it is not necessary that the condition (3) be satisfied; it suffices for the weaker condition (18) to hold. In this method, however, there exists a Schrödinger equation as well as an S matrix. Here we shall examine in detail what the situation is as to satisfying the integrability condition for this equation and for the Tomonaga-Schwinger equation obtained from it by passage to the limit.

The integrability condition for the variational Schrödinger equation is of the form

$$i\,\delta H(x;g)/\delta g(y) - i\,\delta H(y;g)/\delta g(x) + [H(x;g),\ H(y;g)] = 0 \quad \text{for } x\sim y \text{ or } x=y, \tag{28}$$

and in its form is reminiscent of the condition (3), which, as is well known, is violated at the point $x=y$ for $H(x;\sigma) = -L(x)$ if the Lagrangian contains a coupling involving derivatives.

In our present case, when we use the definition of $H(x;g)$ by Eq. (13) and the fact that $S(g)$ is unitary, the condition (28) takes the form

$$\frac{\delta^2 S(g)}{\delta g(y)\,\delta g(x)} S^+(g) + \frac{\delta^2 S(g)}{\delta g(\ )\,\delta g(y)} S^+(g) - [H(x;g),\ H(y;g)] + [H(x;g),\ H(y;g)] = 0 \quad \text{for } x\sim y \text{ or } x=y. \tag{29}$$

In the expression (29) the commutators cancel and the equality of the second variational derivatives of $S(g)$ at the point $x=y$ does not arouse any doubts.

Thus the integrability condition of the variational Schrödinger equation is satisfied automatically for any theory and does not depend at all on the value of the commutator $[H(x;g),\ H(y;g)]$ (which in second order is $[L(x),\ L(y)]$ at the point $x=y$. There is nothing surprising in this result, since we did not obtain the expression for the generalized Hamiltonian from any sort of collateral arguments, but essentially from the physically meaningful solution of the equation (12).

The question of what happens to the integrability condition for $g\to\theta_\sigma$ calls for closer examination in view of the singular character of the approach to the limit. This is true especially because if we approach Eq. (16) uncritically we would have to admit that for all theories except the Hurst-Thirring field the integrability condition is violated in the limit.

Nevertheless, we are enabled to clear up this question by the analysis of the structure of the S matrix for theories with derivative couplings which we made in Sec. 3. We merely note that every ordinary product or T-product can be represented by an expression of the form

$$T(L(x)L(y)) = :L(x)L(y): + :\frac{\partial L(x)}{\partial\varphi(x)}\frac{\partial L(y)}{\partial\varphi(y)}: \frac{1}{i} D^c(x-y) + \ldots$$

Therefore if the Lagrangian involves a derivative coupling, then according to Eqs. (13) and (15) the Hamiltonian will contain, along with other terms which are linear in the contractions, a term of the type:

$$\Delta H(x;\sigma) = -\lim_{g\to\theta_\sigma}\int_{-\infty}^{\infty} dx^0\,dy g'(T_\sigma - x^0) \times g(T_\sigma - y^0) : \frac{\partial L(x)}{\partial(\partial\varphi/\partial x^\alpha)}\frac{\partial L(y)}{\partial(\partial\varphi/\partial y^\beta)} : \{\widetilde{D}^c_{\alpha\beta}(x-y) - \widetilde{D}^-_{\alpha\beta}(x-y)\}. \tag{30}$$

By the arguments which lead to Eq. (16) this term is to be regarded as equal to zero, since it does not depend at all on the order in which the limits are taken. If, however, we recall that in the

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*The arbitrariness in the definition of the T-product which we have indicated leads to an additional arbitrariness [14] in the definition of the interpolating field [15] (for a given S matrix); this arbitrariness is of the following character: in the definition of a vector interpolating field there is an additional arbitrariness as compared with the case of a scalar field, owing to the fact that we can have different definitions of $D^c_{\alpha\beta}(x-y)$ with the same S matrix.

Bogolyubov method S(g) depends on the Lagrangian and consequently $\tilde{D}^{c}_{\alpha\beta}(x-y)$ is defined by Eq. (20), then it is clear that to obtain $\Delta H(x;\sigma)$ we must go over to $\tilde{D}^{adv}_{\alpha\beta}(x-y)$ by Eq. (26). We then have

$$\Delta H(x;\sigma) = -\lim_{g\to\theta_\sigma}\int_{-\infty}^{\infty} dx^0\,dyg'(T_\sigma - x^0)$$
$$\times g(T_\sigma - y^0) : \frac{\partial L(x)}{\partial(\partial\varphi/\partial x^\alpha)}\frac{\partial L(y)}{\partial(\partial\varphi/\partial y^\beta)} : \{\tilde{D}^{adv}_{\alpha\beta}(x-y)$$
$$- n_\alpha n_\beta \delta(x-y)\}. \quad (31)$$

Here the first term in the curly brackets goes to zero in the limit, and the second term gives the nonvanishing contribution

$$\Delta H(x;\sigma) = \frac{1}{2} : \left[n_\alpha \frac{\partial L(x)}{\partial(\partial\varphi/\partial x^\alpha)}\right]\left[n_\beta \frac{\partial L(x)}{\partial(\partial\varphi/\partial x^\beta)}\right] \quad (32)$$

which assures that the integrability condition of the Tomonaga-Schwinger equation is satisfied both for the bare Lagrangian and for the counter-terms of arbitrary order, since in the Bogolyubov method what one has as L(x) is the effective Lagrangian L(x; 1). In second order in scalar electrodynamics

$$\Delta H(x;\sigma) = e^2 : \varphi^*(x)\varphi(x)[n_\alpha A_\alpha(x)]^2 :,$$

which corresponds to the well known usual expression.

Combining the results of Secs. 2 and 4, we can write (if we drop "surface-diverging" counter-terms) the expression for $H(x;\sigma)$ which follows from the Bogolyubov method:

$$H(x;\sigma) = -L(x;1)$$
$$+\frac{1}{2} : \left[n_\alpha \frac{\partial L(x;1)}{\partial(\partial\varphi/\partial x^\alpha)}\right]\left[n_\beta \frac{\partial L(x;1)}{\partial(\partial\varphi/\partial x^\beta)}\right] :. \quad (33)$$

Thus a formula of the type of Eq. (9) arises legitimately in this method.

It is also not hard to understand that with this way of obtaining the Hamiltonian no terms of fourth or higher order in the number of normals can arise, since when there is one derivative in the Lagrangian such terms will correspond to unconnected diagrams, which do not contribute to the Hamiltonian,[11] and when the Lagrangian contains products of derivatives some of the normals that appear will have to be identical, and therefore because $(n_\alpha)^2 = 1$ these terms also will not have more than two normals. Thus in any renormalizable theory a formula of the type of Eq. (33) gives an expression which is exhaustive from this point of view; this is in agreement with the mathematical treatment carried out by Nishijima.[16]

Thus in this method (if we disregard the problem of "surface divergences") both the generalized Hamiltonian H(x; g) and the physical Hamiltonian $H(x;\sigma)$ obtained from it in the limit $g\to\theta_\sigma$ satisfy the corresponding integrability conditions for any renormalizable theory. This result is all the more attractive because the integrability condition is important not only mathematically but also physically. In particular, for theories with derivative couplings it is only when this condition[16] is satisfied that the energy-momentum conservation law holds.

5. SOME FEATURES OF THE CONSTRUCTION OF THE S MATRIX FOR SCALAR ELECTRODYNAMICS

As is well known,[17,18] the Klein-Gordon and Duffin-Kemmer formalisms can be used with equal success in the construction of the S matrix for scalar electrodynamics by the Dyson method. In particular, the proof of renormalizability has been given in both formalisms. In doing this in the Dyson method one starts from the interaction Hamiltonian, which in one formalism has the form (7) and in the other the form

$$H^{D-K}(x;\sigma) = -ie : \bar{\psi}(x)\Gamma_\alpha\psi(x)A_\alpha(x) :$$
$$- e^2 m^{-1} : \bar{\psi}(x)\Gamma_\alpha\Gamma_\beta\psi(x)A_\alpha(x)A_\beta(x) :$$
$$+ e^2 m^{-1} : \bar{\psi}(x)\Gamma_\alpha\Gamma_\beta[n_\delta\Gamma_\delta]^2\psi(x)A_\alpha(x)A_\beta(x): \quad (34)$$

where Γ_α are the Duffin-Kemmer matrices, and then in obtaining the S matrix one goes over to the Lagrangian, of the form (6) or the form

$$L^{D-K}(x) = ie : \bar{\psi}(x)\Gamma_\alpha\psi(x)A_\alpha(x) :. \quad (35)$$

As can be seen from a comparison of the formulas (7)—(34) and (6)—(35), in going from the Hamiltonian to the Lagrangian in the Dyson method one eliminates not only the terms quadratic in the normals, but also, in the Duffin-Kemmer formalism, all the terms quadratic in the charge, while in the Klein-Gordon formalism the term e^2: $\varphi^*(x)\varphi(x)A^2_\alpha(x)$: remains. As was shown above, the possibility of eliminating from the S matrix the terms that depend quadratically on the normals is due to the fact that there is an extra finite arbitrariness in the definition of the T-product. In all probability the possibility of eliminating the other term of order e^2 in the Duffin-Kemmer formalism is due to this same circumstance.

It is clear from the foregoing that one of the advantages of the construction of the S matrix by the Bogolyubov method is that in this method the situation with the terms quadratically dependent on the normals is much simpler. Up to this time, however, the S matrix for scalar electrodynamics has been obtained by the Bogolyubov method only in the Duffin-Kemmer formalism[19]; one can ascribe this in particular to the fact that $L^{D-K}(x)$, unlike $L^{K-G}(x)$, contains only a term linear in the charge. In the Bogolyubov method this is of importance in the use of the correspondence arguments, which allow us to have only a first-order term in the bare Lagrangian.

Nevertheless, a careful analysis shows that the S matrix for scalar electrodynamics can also be constructed in the Klein-Gordon formalism by the Bogolyubov method. In fact, following the course of the arguments given in[11], we find from correspondence considerations that

$$S_1(x) = -e : \left(\varphi^*(x)\frac{\partial\varphi}{\partial x^\alpha} - \frac{\partial\varphi^*}{\partial x^\alpha}\varphi(x)\right) A_\alpha(x) :, \qquad (36)$$

because we can include in $S_1(x)$ only terms linear in $g(x)$. If, however, we continue with the construction of the S matrix, we can arrive at the usual formula. Namely, in second order we have

$$S_2(x, y) = e^2 T\left[: \left(\varphi^*(x)\frac{\partial\varphi}{\partial x^\alpha} - \frac{\partial\varphi^*}{\partial x^\alpha}\varphi(x)\right) A_\alpha(x) : \times : \left(\varphi^*(y)\frac{\partial\varphi}{\partial y^\beta} - \frac{\partial\varphi^*}{\partial y^\beta}\varphi(y)\right) A_\beta(y) : \right], \qquad (37)$$

and $S_2(x, y)$ is defined everywhere except at the point $x = y$. This indefiniteness in the T-product enables us not only to make $S_2(x, y)$ an integrable generalized function, but also to secure the gauge invariance of the matrix S(1).

In Sec. 3 we pointed out the existence of a finite arbitrariness in the definition of $\widetilde{D}^c_{\alpha\beta}(x - y)$ with a fixed definition of $D^c(x - y)$. In particular, we can add to the definition (20), which was adopted in [11], a term $g^{\alpha\beta}\delta(x - y)$. This definition of $\widetilde{D}^c_{\alpha\beta}(x - y)$ has the result of bringing out from the diagram that describes the meson Compton effect in second order a term

$$e^2 \int_{-\infty}^{\infty} dxdyg(x)g(y) : \varphi^*(x)\varphi(y) A_\alpha(x) A_\beta(y) : g^{\alpha\beta}\delta(x-y) = e^2 \int_{-\infty}^{\infty} dxg^2(x) : \varphi^*(x)\varphi(x) A^2_\alpha(x) :. \qquad (38)$$

We can return to the previous definition of $\widetilde{D}^c_{\alpha\beta}(x - y)$, which is more convenient for the further calculations, and include the term (38) in the effective interaction Lagrangian, which to second order takes the form

$$L(x; g) = ie : \left(\varphi^*(x)\frac{\partial\varphi}{\partial x^\alpha} - \frac{\partial\varphi^*}{\partial x^\alpha}\varphi(x)\right) A_\alpha(x) : + e^2 : \varphi^*(x)\varphi(x) A^2_\alpha(x) : g(x) + \frac{1}{2!}\int_{-\infty}^{\infty} \Lambda_2(x, y) g(y)\, dy. \qquad (39)$$

A peculiarity of the counterterm $e^2 : \varphi^*(x)\varphi(x) \times A^2_\alpha(x) : g(x)$ is that its appearance is associated not with the problem of securing the finiteness of the S matrix, but with that of securing its gauge invariance. Thus the requirement of gauge invariance [at least for S(1)] must be included from the very beginning among the fundamental requirements imposed on the scattering matrix.

At first glance it may seem that there is a great difference between the processes of constructing the S matrix by the Bogolyubov method in the different formalisms of scalar electrodynamics, since in the Klein-Gordon formalism we have to redefine the T-product in order to secure the gauge invariance of the S matrix, whereas in the Duffin-Kemmer formalism it is secured automatically, as it were. The difference, however, is an apparent one. Actually the appearance of the term $e^2 : \varphi^*(x)\varphi(x) A^2_\alpha(x) :$ when we go from the Duffin-Kemmer formalism to the Klein-Gordon formalism is due to the fact that in the definition of the chronological contraction of $\psi(x)$ operators in the momentum representation there is a term $m^{-1}Y$, where Y is a combination of Γ_α matrices. This term is a typically quasi-local one and can be eliminated from the definition of the contraction if we ignore the requirements of gauge invariance.

Thus in the Klein-Gordon formalism we can use for the S matrix, besides formulas of the forms of Eqs. (27) and (22), the expression

$$S = T\exp\left\{ i \int_{-\infty}^{\infty} L'(x)\, dx \right\}, \qquad (40)$$

where

$$L'(x) = ie : \left(\varphi^*(x)\frac{\partial\varphi}{\partial x^\alpha} - \frac{\partial\varphi^*}{\partial x^\alpha}\varphi(x)\right) A_\alpha(x) :. \qquad (41)$$

Thus there are no essential differences between the Klein-Gordon and Duffin-Kemmer formalisms in the construction of the S matrix either by the Dyson method or by the Bogolyubov method.

As for the Schrödinger equation, independently of whether we assign the additional term indicated here to the Lagrangian or to the T-product, in the limit $g \to \theta_\sigma$ we get a Hamiltonian which agrees with the usual Hamiltonian of scalar electrodynamics. In the former case this term appears by the same arguments as the usual counterterms. In the latter case it arises in the Hamiltonian in the same way as the terms that depend quadratically on the normals.

Finally, it must be pointed out that the treatment given in Secs. 4 and 5 requires that we make one addition to the apparatus of local dynamical variables introduced by Bogolyubov and Shirkov.[11] One of the important requirements of this apparatus is that in the limit $g \to 0$ the quantities $B(x; g)$ must become identical with the corresponding expressions $B(x)$ of the free-field theory. If we apply this requirement to $H(x; g)$ for the scalar electrodynamics, then in the limit $g \to 0$ this expression goes over into $L'(x)$ [cf. Eq. (41)], which is neither the Hamiltonian nor even the Lagrangian of the free fields. The same difficulty is found with the corresponding current. A way out of this situation can be found by requiring that in the limit $g \to 0$ a quantity $B(x; g)$ is to agree with only the part of the corresponding free-field quantity that is linear in the charge.

6. CONCLUSION

Thus we have shown that if we leave to one side the problem of "surface divergences" the Bogolyubov method applied to any renormalizable theory gives in a legitimate way an expression for $H(x; \sigma)$ of the form (33) which satisfies the integrability condition for the Tomonaga-Schwinger equation. Since this Hamiltonian includes the usual counterterms, the solution of this equation will give an expression for the S matrix which is free from "ultraviolet" divergences. Thus the derivation of $H(x; \sigma)$ by the Bogolyubov method can be regarded as a third method for obtaining the interaction Hamiltonian in quantum field theory, and the most natural of the existing methods. It is also quite clear how in this method one can eliminate the terms that depend quadratically on the normals from the S matrix obtained by the solution of the Tomonaga-Schwinger equation.

As for the problem of "surface divergences," a consistent treatment of this will in all probability require the use of a more rigorous mathematical apparatus of the type of the new R operation.[11] Only such an approach to the problem will make it possible to decide finally whether it is of mathematical or physical origin, since there also exists the opinion[20] that there is no S matrix which has physical meaning. It is a matter of very great interest to settle this problem, since it arises both in the Bogolyubov method and in the Dyson method.[12] Furthermore, if the problem of "surface divergences" can be positively solved, we shall for the first time have to do with a Hamiltonian which will lead through the solution of the Tomonaga-Schwinger equation to an expression for the matrix $S(\sigma)$ which is free from both ordinary divergences and "surface divergences."

It must be emphasized, however, that even in this case new difficulties of a "surface" character can arise in the construction of the apparatus of local dynamical variables by means of the matrix $S(\sigma)$ (for example, in the construction of the expression for the interpolating field[15]).

Finally, if the Hamiltonian so obtained is used not for the derivation of the S matrix, but in some other apparatus, then the regularizing masses which it contains can lead to expressions which are infinite in the usual sense of the word. Therefore along with the solution to the problem of "surface divergences" one must look for a new and mathematically more rigorous approach to the entire set of questions associated with the Schrödinger equation and the apparatus of local dynamical variables.

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Note added in proof (November 19, 1961). We must emphasize that our use of the notation $\tilde{D}^c_{\alpha\beta}(x-y)$ in Eq. (21) is of a somewhat conditional character, because if taken too literally it could even lead to such an absurd result as $(\Box_x - m^2) \times D^c(x) = 0$. Indeed, in Eqs. (20) and (21) we have written different definitions of the contraction $T\left(\frac{\partial\varphi}{\partial x^\alpha}\frac{\partial\varphi}{\partial y^\beta}\right)$, which are used in the Bogolyubov and Dyson methods, respectively. Furthermore, although the definition (21) follows directly from the intuitive meaning of the T-product for $x \neq y$, the definition (20), for which the equation

$$T\left(\frac{\partial\varphi}{\partial x^\alpha}\frac{\partial\varphi}{\partial y^\beta}\right) = \frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y^\beta} T(\varphi(x)\varphi(y))$$

holds right down to $x = y$, is more convenient. The same can be said about the notation $\tilde{D}^{adv}_{\alpha\beta}(x-y)$ in Eq. (25).

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