INVARIANT PARAMETRIZATION OF THE RELATIVISTIC SCATTERING MATRIX

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Invariant parametrization of the relativistic amplitude for scattering through given angles is carried out for reactions involving particles of arbitrary spin. The analysis is applicable to both particles with vanishing and nonvanishing rest mass.

1. As is well known, the scattering matrix S is invariant and may therefore be a function of only invariant parameters, characterizing the initial and final states of the system. However the state vector of an arbitrary quantum-mechanical system depends also on noninvariant kinematic variables. Thus, for example, the state of a free particle may be described by the vector

$$|\mathbf{p}, \varkappa, s, m, \alpha\rangle$$

which contains invariant variables—the mass κ, the spin s, the "charge" α —as well as noninvariant variables—the momentum **p** and the projection m of the spin onto the z axis. Consequently matrix elements of the scattering matrix taken between the initial and final states in an arbitrary coordinate system will also depend on noninvariant quantities, such as the momenta of the particles and the projections of their spins. It is in this connection that we are faced with the problem of parametrizing the S matrix, i.e., of finding a transformation connecting the elements of the S matrix in its noninvariant form with elements that depend only on invariant parameters.

The solution of this problem is needed for scattering theory, for the problem of a complete experiment,[1] for dispersion relations.

Depending on the choice of invariant parameters it is possible to obtain various parametrizations of the scattering matrix. Thus, if one chooses as invariant parameters the energy, spins, and the orbital and total angular momenta of the initial and final states one obtains a "phase shift" parametrization, useful in a phase shift analysis of the experimental data. Such a parametrization of the S matrix has been fully carried out for both the non-relativistic (see e.g., [2-4]) and the relativistic case (see, e.g., [5]).

Such a parametrization, however, is not useful in the cases where the phase shift analysis does not yield satisfactory results, as for example in experiments on multiple scattering of high energy particles.

For the special case of scattering of spin 0 and $1/2$ particles Ashkin and Wolfenstein[6] gave in the nonrelativistic approximation an "angular" parametrization of the scattering matrix, i.e., using the scattering angle (beside the energy, spins, etc.) as an invariant parameter. The nonrelativistic parametrization for the case of arbitrary spins was obtained by Ritus,[7] Bilen'kii et al,[8] and Fischer and Ciulli.[9]

In the present work the relativistically invariant angular parametrization of the S matrix is obtained.

In order to parametrize the S matrix in an arbitrary (laboratory) coordinate system it is convenient to parametrize it first in the center-of-mass system (c.m.s.), and then transform to the laboratory system by means of a unitary Lorentz rotation U:

$$S_{\text{lab}} = U^{+} S_{\text{c.m.s.}} U. \tag{1}$$

In Sec. 2 we give various versions of parametrizing the S matrix in the c.m.s. The investigation of $S_{\text{c.m.s.}}$ is by its nature nonrelativistic and therefore our parametrization of $S_{\text{c.m.s.}}$ does not differ in principle from the parametrizations in[7-9], however it preserves the convenient matrix form in which the invariantly parametrized elements of the S matrix are written, it does not introduce any nonphysical parameters of the type of the total spin of the initial and final channels, and is directly applicable when the initial and final states of the particles are specified in terms of their helicities as is necessary in reactions involving particles with zero mass.

2. The scattering of two particles in the c.m.s. is described by the scattering matrix

$$\langle \mathbf{n}, \varkappa_1, \varkappa_2, s_1, s_2, m_1, m_2, \alpha | S(\varkappa) | \mathbf{n}', \varkappa_1', \varkappa_2', s_1', s_2', m_1', m_2', \alpha' \rangle \quad (2)$$

where κ_1, κ_2, κ_1', κ_2' are the particles' masses, s_1, s_2, s_1', s_2' —their spins, m_1, m_2, m_1', m_2' —projections of their spins on the axis z_N of the arbitrarily chosen coordinate system (system N); **n**, **n′** are unit vectors in respectively the directions of the final and initial momenta (in the c.m.s.); α, α' are other parameters invariant under proper Lorentz transformations (charge, parity, etc.); κ is the total mass of the system (initial, as well as final); the S matrix is diagonal in κ. In the following the parameters κ, κ_1, κ_2, κ_1', κ_2', α, α' will be omitted.

In order to obtain an invariant parametrization of $S_{c.m.s.}$ it is necessary to choose a coordinate system in which the spins of the particles are specified invariantly, i.e., are rigidly connected with the invariant directions given by the vectors **n**, **n′** and $\mathbf{k} = \mathbf{n} \times \mathbf{n}'/|\mathbf{n} \times \mathbf{n}'|$. Depending on the character of the problem one or another parametrization of the S matrix turns out to be more convenient. Thus, for example, for nucleon-nucleon scattering it is desirable to determine the scattering amplitude in triplet and singlet states separately. In the general case this corresponds to the introduction of the spin of the channel as an invariant parameter of the S matrix. Such a parametrization corresponds to the LS coupling case in the phase shift parametrization.

a) To obtain such a parametrization we choose invariant coordinate systems A, A′ in which the axes z_A and $z_{A'}$ are parallel to **n** and **n′** respectively, and the axes y_A, $y_{A'}$ are taken perpendicular to the scattering plane (i.e., parallel to the vector **k**). At that the spins of the initial particles are determined in the system A′, and the spins of the final particles in the system A. We then rotate the original coordinate system N to make it coincide with the system A; this rotation is specified by the Euler angles α, β, and γ. The same coordinate system can be made to coincide with the sytem A′ by a rotation specified by the Euler angles α', β', and γ'. The Euler angles in these rotations are given by the relations

$$\alpha = \varphi, \quad \alpha' = \varphi', \quad \beta = \theta, \quad \beta' = \theta',$$
$$\gamma = \arccos\left[\frac{\cos\theta\cos\vartheta - \cos\theta'}{\sin\theta\sin\vartheta}\right],$$
$$\gamma' = \arccos\left[\frac{\cos\theta - \cos\theta'\cos\vartheta}{\sin\theta'\sin\vartheta}\right]. \quad (3)$$

Here φ, θ, φ', θ' are the spherical coordinates of the vectors **n**, **n′** in the system N; ϑ is the scattering angle given by the well known formula

$$\cos\vartheta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi' - \varphi). \quad (4)$$

$S_{c.m.s.}$ can now be written in the form

$$\langle \mathbf{n}, s_1, s_2, m_1, m_2, | S | \mathbf{n}', s_1', s_2', m_1', m_2' \rangle \quad (5)$$
$$= \sum_{jj'\lambda\lambda'} \langle s_1 s_2 m_1 m_2 | jm \rangle D^j_{m\lambda}(\alpha\beta\gamma)$$
$$\times \langle s_1, s_2, j, \lambda, \mathbf{n} | S | s_1', s_2', j', \lambda', \mathbf{n}' \rangle D^{*j'}_{m'\lambda'}(\alpha'\beta'\gamma')$$
$$\times \langle s_1' s_2' m_1' m_2' | j' m' \rangle,$$
$$m = m_1 = m_2; \qquad m' = m_1' + m_2'.$$

Here j′, j are the spins of the initial and final channels respectively, $\langle s_1, s_2, m_1, m_2 | jm \rangle$ are Clebsch-Gordan coefficients, $D^j_{m\lambda}(\alpha\beta\gamma)$ are the finite rotation matrices as defined by Edmonds.[10] The star denotes complex conjugation. The sole noninvariant parameters left now are **n** and **n′**, since λ, λ' are invariant being the projections of the channel spin onto the invariant directions **n** and **n′**.

Since

$$\langle s_1, s_2, j, \lambda, \mathbf{n} | S | s_1', s_2', j', \lambda', \mathbf{n}' \rangle$$

is invariant, it can depend only on invariant combinations of **n** and **n′** and not on **n** or **n′** separately, i.e., only on $\mathbf{n}\cdot\mathbf{n}' = \cos\vartheta$ (let us recall **n** and **n′** are unit vectors, i.e., $n^2 = n'^2 = 1$). Indeed,

$$\langle s_1, s_2, j, \lambda, \mathbf{n} | S | s_1', s_2', j', \lambda', \mathbf{n}' \rangle$$
$$= \sum_{l,l',m,m'} \langle \mathbf{n} | lm \rangle \langle l'm' | \mathbf{n}' \rangle$$
$$\langle s_1, s_2, j, \lambda, l, m | S | s_1', s_2', j', \lambda', l', m' \rangle = \sum_{l,m} \langle \mathbf{n} | lm \rangle$$
$$\langle lm | \mathbf{n}' \rangle \langle s_1, s_2, j, \lambda, l \| S \| s_1', s_2', j', \lambda', l \rangle$$
$$= \frac{1}{4\pi} \sum_l (2l+1) P_l(\cos\vartheta)$$
$$\langle s_1, s_2, j, \lambda, l \| S \| s_1', s_2', j', \lambda', l \rangle$$
$$= \langle s_1, s_2, j, \lambda \| S(\cos\vartheta) \| s_1', s_2', j', \lambda' \rangle. \quad (6)$$

We finally have

$$\langle s_1, s_2, m_1, m_2, \mathbf{n} | S | s_1', s_2', m_1', m_2', \mathbf{n}' \rangle$$
$$= \sum_{j,j',\lambda,\lambda'} \langle s_1 s_2 m_1 m_2 | jm \rangle D^j_{m\lambda}(\alpha\beta\gamma)$$
$$\times \langle s_1, s_2, j, \lambda \| S(x) \| s_1', s_2', j', \lambda' \rangle$$
$$\times D^{*j'}_{m'\lambda'}(\alpha'\beta'\gamma') \langle s_1' s_2' m_1' m_2' | j'm' \rangle, \qquad x = \cos\vartheta. \quad (7)$$

b) If one chooses for the invariant parameters the helicities λ_1, λ_2 (λ_1', λ_2') of the particles rather than j, λ (j′, λ'), one obtains a somewhat different parametrization:

$$\langle s_1, s_2, m_1, m_2, \mathbf{n} | S | s_1', s_2', m_1', m_2', \mathbf{n}' \rangle$$
$$= \sum_{\lambda_1, \lambda_2, \lambda_1', \lambda_2'} D^{s_1}_{m_1\lambda_1}(\alpha\beta\gamma) D^{s_2}_{m_2\lambda_2}(\alpha+\pi; \pi-\beta; \pi-\gamma)$$
$$\times \langle s_1, s_2, \lambda_1, \lambda_2 \| S(x) \| s_1', s_2', \lambda_1', \lambda_2' \rangle D^{*s_1'}_{m_1'\lambda_1'}(\alpha'\beta'\gamma') D^{*s_2'}_{m_2'\lambda_2'}$$
$$\times (\alpha'+\pi; \pi-\beta'; \pi-\gamma') \quad (8)$$

The arguments of the functions D:. are so chosen that the spins of the particles 2 and 2′ are determined in coordinate systems $\tilde{A}$ and $\tilde{A}'$, for which the axes $z_{\tilde{A}}$ and $z_{\tilde{A}'}$ are along the directions $-\mathbf{n}$ and $-\mathbf{n}'$ respectively, and the axes $y_{\tilde{A}}$ and $y_{\tilde{A}'}$ coincide and are parallel to **k**. Making use of the relation

$$D^j_{mm'}(\alpha+\pi; \pi-\beta; \pi-\gamma) = (-1)^{j-m'} D^j_{m-m'}(\alpha\beta\gamma), \quad (9)$$

we can write Eq. (8) in the form

$$\sum_{\lambda_1, \lambda_2, \lambda_1', \lambda_2'} (-1)^{s_2+s_3'-\lambda_2-\lambda_2'} D^{s_1}_{m_1\lambda_1}(\alpha\beta\gamma) D^{s_2}_{m_2-\lambda_2}(\alpha\beta\gamma)$$
$$\times \langle s_1, s_2, \lambda_1, \lambda_2 \| S(x) \| s_1', s_2', \lambda_1', \lambda_2' \rangle D^{*s_1'}_{m_1'\lambda_1'}(\alpha'\beta'\gamma')$$
$$\times D^{*s_2'}_{m_2'-\lambda_2'}(\alpha'\beta'\gamma'). \quad (10)$$

This parametrization clearly corresponds to the jj coupling case in the phase shift parametrization.

c) Jacob and Wick[4] showed that the calculation of polarization tensors is significantly simplified, if the particles in the initial and final states are specified by their helicities λ_i (i = 1, 1′, 2, 2′). The helicity of a particle is defined by the relation

$$|\mathbf{n}, s, \lambda\rangle = \sum_m D^s_{m\lambda}(\varphi; \theta; \psi) |\mathbf{n}, s, m\rangle, \quad (11)$$

where φ and θ are the angular coordinates of the vector **n** in the system N; the choice of ψ is arbitrary and depends on the representation chosen. Thus, in the representation used in [4,11] $\psi = -\varphi$, whereas in the representation used in [12] $\psi = 0$. Let us emphasize that the definition of helicity is not only nonunique (owing to the arbitrariness of ψ) but also noninvariant, since the angle ψ in any representation is determined in the laboratory coordinate system and therefore, generally speaking, changes under three-dimensional rotations with the result that, although the value of λ remains unchanged, $|\mathbf{n}, s, \lambda\rangle$ becomes multiplied by a noninvariant phase factor. For this reason the transition to an invariant parametrization of the S matrix reduces to an invariant fixing of the reference line from which the angle ψ is to be measured.

To accomplish this we go by means of three-dimensional rotations from the coordinate system N to the invariant systems A, $\tilde{A}$, A′, $\tilde{A}'$:

$$|\mathbf{n}, s, \lambda\rangle = \sum_{\lambda'} D^{*s}_{\lambda\lambda'}(0,0,\delta) |n, s, \lambda'\rangle_A = e^{i\lambda\delta} |\mathbf{n}, s, \lambda\rangle_A, \quad (12)$$

where $|\mathbf{n}, s, \lambda\rangle_A$ is the state vector of a particle with helicity λ, defined in the system A. The angle δ depends on the definition of ψ. For $\psi = 0$

$$\delta_1 = \gamma; \quad \delta_2 = \pi-\gamma; \quad \delta_1' = \gamma'; \quad \delta_2' = \pi-\gamma'. \quad (13)$$

We may now write

$$\langle s_1, s_2, \lambda_1, \lambda_2, \mathbf{n} | S | s_1', s_2', \lambda_1', \lambda_2', \mathbf{n}' \rangle$$
$$= (-1)^{\lambda_2'-\lambda_2} \exp\{i(\lambda_2-\lambda_1)\gamma\}$$
$$\times \langle s_1, s_2, \lambda_1, \lambda_2 \| S(x) \| s_1', s_2', \lambda_1', \lambda_2' \rangle$$
$$\times \exp\{-i(\lambda_2'-\lambda_1')\gamma'\}. \quad (14)$$

Here the invariant matrix elements $\langle \ldots \| S(x) \| \ldots \rangle$ are the same as in Eq. (10).

It should be noted that for forward scattering ($\mathbf{n} = \mathbf{n}'$) the vector **k** becomes arbitrary (while remaining perpendicular to **n**), i.e., the angles $\gamma = \gamma'$ are arbitrary. However in that case [e.g., for parametrization a)] the projection of the total angular momentum onto **n** coincides with λ' in the initial state and with λ in the final state. As a consequence of the conservation of the projection of the total angular momentum onto any direction in space we have $\lambda = \lambda'$, and therefore

$$\langle s_1, s_2, m_1, m_2, \mathbf{n} | S | s_1', s_2', m_1', m_2', \mathbf{n} \rangle = \sum_{j, j', \lambda} \langle s_1 s_2 m_1 m_2 | jm \rangle$$
$$\times D^j_{m\lambda}(\alpha, \beta, 0) e^{-i\lambda\gamma} \langle s_1, s_2, j, \lambda \| S(x) \| s_1', s_2', j', \lambda \rangle$$
$$\times D^{*j'}_{m'\lambda}(\alpha, \beta, 0) e^{i\lambda\gamma} \langle s_1' s_2' m_1' m_2' | j'm' \rangle, \qquad x = 1. \quad (15)$$

It is clear that Eq. (15) does not depend on γ. The situation is analogous for scattering in the backward direction.

d) In scattering experiments it is often the transverse, and not the longitudinal, polarization that is measured. Thus, in the scattering of an unpolarized beam by an unpolarized target, the scattered beam can have only transverse polarization, as is well known. We therefore give also the parametrization of the S matrix, for example for the case of "LS coupling," when the transverse polarizations μ and μ' are chosen as invariant parameters. They are invariant because they are defined in the invariant coordinate system B, in which the axis z_B is perpendicular to the scattering plane and the axis y_B is parallel to the vector $\mathbf{q} = \mathbf{n} + \mathbf{n}'$. At that

$$\langle s_1, s_2, m_1, m_2, \mathbf{n} | S | s_1', s_2', m_1', m_2', \mathbf{n}' \rangle$$
$$= \sum_{j, j', \mu, \mu'} \langle s_1 s_2 m_1 m_2 | jm \rangle D^j_{m\mu}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$$
$$\times \langle s_1, s_2, j, \mu \| S(x) \| s_1', s_2', j', \mu' \rangle D^{*j'}_{m'\mu'}$$
$$\times (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) \langle s_1' s_2' m_1' m_2' | j'm' \rangle, \quad (16)$$

where $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ are the Euler angles specifying the rotation of system N into system B:

$$\operatorname{tg}\tilde{\alpha} = -(\operatorname{tg}\theta\cos\varphi - \operatorname{tg}\theta'\cos\varphi')/(\operatorname{tg}\theta\sin\varphi - \operatorname{tg}\theta'\sin\varphi'),$$
$$\cos\tilde{\beta} = \sin\theta\,\sin\theta'\,\sin(\varphi'-\varphi)/\sin\vartheta,$$
$$\cos\tilde{\gamma} = (\cos\theta' - \cos\theta)/2\sin\tilde{\beta}\,\sin(\vartheta/2). \quad (17)^*$$

For forward (or backward) scattering the choice of the z_B axis becomes arbitrary, however once a choice of z_B is made the determination of the invariantly parametrized elements of the scattering matrix is unique.

3. Invariance under time inversion imposes on the scattering matrix the well known condition

$$T^{-1}ST = S^T, \quad (18)$$

where the superscript T denotes transposition, and the operator T is defined by the relation† (see, e.g., [13])

$$T\,|\,\mathbf{p}, \varkappa, s, m, \alpha\rangle = (-1)^{s-m}\,|-\mathbf{p}, \varkappa, s, -m, \alpha\rangle. \quad (19)$$

Let us see what restrictions are imposed by Eq. (18) on the invariant matrix elements. Using Eqs. (7) and (18) we get

$$(-1)^{s_1+s_2+s_1'+s_2'-m_1-m_2-m_1'-m_2'}\langle s_1, s_2, -m_1, -m_2, -\mathbf{n}\,|\,S\,|\,s_1', s_2', -m_1', -m_2', -\mathbf{n}'\rangle = \langle s_1', s_2', m_1', m_2', \mathbf{n}'\,|\,S\,|\,s_1, s_2, m_1, m_2, \mathbf{n}\rangle, \quad (20)$$

whence, using Eqs. (7), (10), (14), and (16) and taking into account the orthogonality of the Clebsch-Gordan coefficients and the unitarity of the matrices $D^j_{mm'}(\alpha, \beta, \gamma)$, we get for the parametrization of item a)

$$\langle s_1 s_2 j\lambda\,\|\,S(x)\,\|\,s_1' s_2' j'\lambda'\rangle = (-1)^{\lambda'-\lambda}\langle s_1' s_2' j'\lambda'\,\|\,S(x)\,\|\,s_1 s_2 j\lambda\rangle, \quad (21)$$

for the parametrizations of items b) and c)

$$\langle s_1 s_2 \lambda_1 \lambda_2\,\|\,S(x)\,\|\,s_1'\,s_2'\,\lambda_1'\,\lambda_2'\rangle = (-1)^{\lambda_1'-\lambda_2'-\lambda_1+\lambda_2}\langle s_1'\,s_2'\,\lambda_1'\,\lambda_2'\,\|\,S(x)\,\|\,s_1 s_2 \lambda_1 \lambda_2\rangle \quad (22)$$

and for the parametrization of item d)

$$\langle s_1 s_2 j\mu\,\|\,S(x)\,\|\,s_1' s_2' j'\mu'\rangle = (-1)^{\mu'-\mu}\langle s_1' s_2' j'\mu'\,\|\,S(x)\,\|\,s_1 s_2 j\mu\rangle. \quad (23)$$

Let us pass now to conditions imposed on the S matrix by space inversions. The operator P for space inversions is defined by the relation (see [13])

$$P\,|\,\mathbf{p}, \varkappa, s, m, \alpha\rangle = (-1)^s\,|-\mathbf{p}, \varkappa, s, m, \alpha\rangle. \quad (24)$$

For simplicity we assume here that space inversion does not affect the charge variable α (see the footnote). The condition that the S matrix be invariant under space inversions is written in the form

$$P^{-1}SP = S, \quad (25)$$

which gives for the parametrization of item a)

$$\langle s_1, s_2, j, \lambda\,\|\,S(x)\,\|\,s_1', s_2', j', \lambda'\rangle = (-1)^{s_1+s_2-s_1'-s_2'+j+j'-\lambda-\lambda'} \times \langle s_1, s_2, j, -\lambda\,\|\,S(x)\,\|\,s_1', s_2', j', -\lambda'\rangle, \quad (26)$$

for the parametrizations of items b) and c)

$$\langle s_1, s_2, \lambda_1, \lambda_2\,\|\,S(x)\,\|\,s_1', s_2', \lambda_1', \lambda_2'\rangle = (-1)^{\lambda_1'-\lambda_2'-\lambda_1+\lambda_2} \times \langle s_1, s_2, -\lambda_1, -\lambda_2\,\|\,S(x)\,\|\,s_1', s_2', -\lambda_1', -\lambda_2'\rangle \quad (27)$$

and for the parametrization of item d)

$$\langle s_1, s_2, j, \mu\,\|\,S(x)\,\|\,s_1', s_2', j', \mu'\rangle = (-1)^{s_1+s_2-\mu-s_1'-s_2'+\mu'}\langle s_1, s_2, j, \mu\,\|\,S(x)\,\|\,s_1', s_2', j', \mu'\rangle. \quad (28)$$

4. In order to construct the S matrix in an arbitrary (laboratory) system of coordinates it is necessary to know how state vectors transform under a Lorentz transformation. As is known,[5] the transformation of the vector $|\,\mathbf{p}, \kappa, s, m\rangle$ under a Lorentz transformation with the four-velocity $u_\mu = (\mathbf{u}; iu_0)$

$$\mathbf{p} = \mathbf{p}' + \mathbf{u}\,(\mathbf{u}\mathbf{p}')/(u_0+1) + \mathbf{u}p_0',$$
$$p_0 = (\mathbf{u}\mathbf{p}') + u_0 p_0' \quad (29)^*$$

(where $p_0 = \sqrt{\mathbf{p}^2 + \kappa^2}$ is the particle energy) is given by

$$|\,\mathbf{p}, \varkappa, s, m\rangle = \sqrt{p_0'/p_0}\sum_{m'} U^s_{mm'}(\mathbf{p}, \mathbf{u})\,|\,\mathbf{p}'(\mathbf{p}), \varkappa, s, m'\rangle, \quad (30)$$

$$\mathbf{p}'(\mathbf{p}) = \mathbf{p} + \mathbf{u}\,(\mathbf{u}\mathbf{p})/(u_0+1) - \mathbf{u}p_0,$$
$$p_0'(p) = -(\mathbf{u}\mathbf{p}) + p_0 u_0. \quad (31)$$

The operator $U(\mathbf{p}, \mathbf{u})$ is for spin $1/2$ of the form (see [15])

$$U^{1/2}(\mathbf{p}, \mathbf{u}) = \frac{(p_0+\varkappa)(u_0+1) - (\mathbf{u}\sigma)(\mathbf{p}\sigma)}{[2\,(p_0+\varkappa)(u_0+1)(u_0 p_0 - \mathbf{u}\mathbf{p} + \varkappa)]^{1/2}}, \quad (32)$$

where σ is the vector Pauli matrix. The operator $U(\mathbf{p}, \mathbf{u})$ describes a three-dimensional relativistic spin rotation, i.e., the matrix $U^s_{mm'}(\mathbf{p}, \mathbf{u})$ is the finite rotations matrix $D^{*s}_{mm'}(a, b, c)$ with appropriate values of the Euler angles:

$$U^s_{mm'}(\mathbf{p}, \mathbf{u}) = D^{*s}_{mm'}(a\,(\mathbf{p}\mathbf{u}), b\,(\mathbf{p}, \mathbf{u}), c\,(\mathbf{p}, \mathbf{u})). \quad (33)$$

The angles a, b, c turn out to be equal to

$$a = \operatorname{arc\,tg}\frac{[(p_0+\varkappa)(u_0+1) - \mathbf{u}\mathbf{p}]\,r_x + r_y r_z}{[(p_0+\varkappa)(u_0+1) - \mathbf{u}\mathbf{p}]\,r_y - r_x r_z},$$

$$b = \operatorname{arc\,cos}\left\{1 - \frac{\mathbf{r}^2 - r_z^2}{(u_0+1)(p_0+\varkappa)(u_0 p_0 - \mathbf{u}\mathbf{p} + \varkappa)}\right\},$$

$$c = \operatorname{arctg}\frac{r_y r_z - r_x\,[(p_0+\varkappa)(u_0+1) - \mathbf{u}\mathbf{p}]}{r_x r_z + r_y\,[(p_0+\varkappa)(u_0+1) - \mathbf{u}\mathbf{p}]}, \qquad \mathbf{r} = [\mathbf{p}\mathbf{u}]. \quad (34)^\dagger$$

*tg = tan.

†It is assumed here, for simplicity, that the operators T and P (see below) do not act on the charge variables α. The transformation of α under time and space inversions is discussed in detail in [14].

*$\mathbf{u}\mathbf{p} = \mathbf{u}\cdot\mathbf{p}$.

†arctg = $\tan^{-1}$, $[\mathbf{p}\mathbf{u}] = [\mathbf{p}\times\mathbf{u}]$.

The transformation of the vector $|\mathbf{p}, s, \lambda\rangle$ under Lorentz rotations can be gotten from $U^{s}_{mm'}$ if use is made of the definition of helicity, Eq. (11):

$$|\mathbf{p}, s, \lambda\rangle = \sqrt{p_0'/p_0} \sum_{\lambda'} W^{s}_{\lambda\lambda'}(\mathbf{p}, \mathbf{u})\, |\mathbf{p}'(\mathbf{p}), s, \lambda'\rangle, \qquad (35)$$

where

$$W^{s}_{\lambda\lambda'} = \sum_{mm'} D^{s}_{m\lambda}(\varphi, \theta, \psi)\, U^{s}_{mm'}(\mathbf{p}, \mathbf{u})\, D^{*s}_{m'\lambda'}(\varphi, \theta', \psi'). \qquad (36)$$

Here φ, θ and φ', θ' are the spherical coordinates of the vectors $\mathbf{p}/|\mathbf{p}|$ and $\mathbf{p}'/|\mathbf{p}'|$. Making use of the formula

$$\sum_{m'} D^{j}_{mm'}(\alpha, \beta\, \gamma)\, D^{j}_{m'n}(\alpha', \beta', \gamma') = D^{j}_{mn}(\alpha_0, \beta_0, \gamma_0), \qquad (37)$$

$$\alpha_0 = \alpha + \operatorname{arctg} \frac{\sin\beta' \sin(\gamma + \alpha')}{\cos\beta' \sin\beta + \cos\beta \sin\beta' \cos(\gamma + \alpha')},$$

$$\beta_0 = \arccos[\cos\beta \cos\beta' - \sin\beta \sin\beta' \cos(\gamma + \alpha')],$$

$$\gamma_0 = \gamma' + \operatorname{arc\,tg} \frac{\sin\beta \sin(\gamma + \alpha')}{\cos\beta \sin\beta' + \cos\beta' \sin\beta \cos(\gamma + \alpha')}, \qquad (38)$$

it is easy to obtain the matrix $W^{s}_{mm'}(\mathbf{p}, \mathbf{u})$ explicitly for arbitrary $\mathbf{p}$ and $\mathbf{u}$, however the formulas are unwieldy and will not be given here.

In a different form the matrices $U^{s}_{mm'}$ and $W^{s}_{mm'}$ [for the case $\psi = -\varphi$ in Eq. (11)] were obtained by Ritus.[11]

We can now express the S matrix in the laboratory frame in terms of $S_{\text{c.m.s.}}$:

$$\langle \mathbf{p}_1, \mathbf{p}_2, \varkappa_1, \varkappa_2, s_1, s_2, m_1, m_2, \alpha \,|\, S_{\text{lab}} \,|\, \mathbf{p}_1', \mathbf{p}_2', \varkappa_1', \varkappa_2', s_1', s_2',$$
$$\times\, m_1', m_2', \alpha'\rangle = \delta(\mathbf{K} - \mathbf{K}')\, \delta(\varkappa - \varkappa')$$
$$\times\, [\tilde{p}_{01}\tilde{p}_{02}\tilde{p}_{01}'\tilde{p}_{02}'/p_{01}p_{02}p_{01}'p_{02}']^{1/2} \sum U^{*s_1}_{m_1\tilde{m}_1}(\mathbf{p}_1, \mathbf{u})\, U^{*s_2}_{m_2\tilde{m}_2}(\mathbf{p}_2, \mathbf{u})$$
$$\times \langle \varkappa_1, \varkappa_2, s_1, s_2, \tilde{m}_1, \tilde{m}_2, \mathbf{n}, \alpha \,|\, S_{\text{c.m.s.}}(\varkappa) \,|\, \varkappa_1', \varkappa_2', s_1', s_2',$$
$$\times\, \tilde{m}_1', \tilde{m}_2', \mathbf{n}', \alpha'\rangle\, U^{s_1'}_{m_1'\tilde{m}_1'}(\mathbf{p}_1', \mathbf{u})\, U^{s_2'}_{m_2'\tilde{m}_2'}(\mathbf{p}_2', \mathbf{u}),$$
$$\mathbf{K} = \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2', \qquad \mathbf{u} = \mathbf{K}/\varkappa,$$
$$\mathbf{n} = \tilde{\mathbf{p}}_1/|\tilde{\mathbf{p}}_1|, \qquad \mathbf{n}' = \tilde{\mathbf{p}}_1'/|\tilde{\mathbf{p}}_1'|, \qquad (39)$$

where the summation is over $\tilde{m}_1$, $\tilde{m}_2$, $\tilde{m}_1'$, $\tilde{m}_2'$.

An analogous formula may be written down in the case when the particles' helicities λ_i are specified.

It is clear that the above discussion applies equally well to reactions involving particles with vanishing rest mass. In that case the matrix for the Lorentz transformation of the state vectors is diagonal in λ, since for such particles the helicity is an invariant under the Lorentz group (see [11]).

The parametrization of the scattering matrix for reactions with an arbitrary number of particles in the final state proceeds by the method discussed automatically:

$$\langle s_1, \ldots, s_i, m_1, \ldots, m_i, \mathbf{n}_1, \ldots, \mathbf{n}_{i-1} \,|\, S_{\text{c.m.s.}} \,|\, s_1', s_2',$$
$$m_1', m_2', \mathbf{n}') = \sum D^{s_1}_{m_1\lambda_1}(\alpha_1\beta_1\gamma_1) \ldots D^{s_i}_{m_i\lambda_i}(\alpha_i\beta_i\gamma_i)$$
$$\times \langle s_1, \ldots, s_i, \lambda_1, \ldots, \lambda_i \,\|\, S(x_1, \ldots, x_q) \,\|\, s_1', s_2', \lambda_1', \lambda_2'\rangle$$
$$\times\, D^{*s_1'}_{m_1'\lambda_1'}(\alpha_1'\beta_1'\gamma_1')\, D^{*s_2'}_{m_2'\lambda_2'}(\alpha_2'\beta_2'\gamma_2'), \qquad q = 3(i-2) \qquad (40)$$

(summation over $\lambda_1, \ldots \lambda_i, \lambda_1', \lambda_2'$).

As the "basic" invariant coordinate system, to which are "attached" the invariant coordinate systems of each particle, one may choose any pair of relative momenta of the particles $\mathbf{n}_k = \mathbf{p}_k/|\mathbf{p}_k|$ and $\mathbf{n}_l = \mathbf{p}_l/|\mathbf{p}_l|$ and the vector $\mathbf{k}_{kl} = \mathbf{n}_k \times \mathbf{n}_l$. The method for transforming expression (40) into a relativistically invariant form is obvious.

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