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THE DEFORMATION FIELD IN AN ISOTROPIC ELASTIC MEDIUM CONTAINING MOVING DISLOCATIONS

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Differential equations are obtained for the deformation field created by dislocations in an elastic isotropic medium. It is shown that the field of the elastic deformation tensors ϵ_{ik} and the field of the displacement velocity vectors v_i of the elements of the medium can be determined in the general case if the density of the Burgers vector and the dislocation flux are known as functions of the coordinates and time. The deformation of an infinite medium is considered at large distances from the system of moving dislocations which creates the deformation. The intensity with which elastic waves are radiated by such a system is calculated.

DISLOCATIONS in a continuous medium must be regarded as defined singularities of the field of the elastic deformations which extend along lines that are either closed or terminated on the surface of the body; these singularities have the characteristic property that in their presence the elastic displacement vector **u** in the medium is not a single-valued function of the coordinates.[1-3] However, the distortion tensor $u_{ik} = \nabla_i u_k$ ($\nabla_i \equiv \partial/\partial x_i$) made up of the spatial derivatives of the vector **u**, remains a single-valued and differentiable function of the coordinates. The fact that the vector **u** is not single-valued is analogous to the similar property of the scalar potential of the magnetic field created by linear electric currents. And, just as the concept of the magnetic potential loses its meaning for a continuous distribution of currents, so does the concept of a vector elastic displacement **u** lose its meaning for a continuous distribution of dislocations. Therefore, the elastic deformation of a medium with a continuous distribution of dislocations is described directly by the distortion tensor u_{ik}, which satisfies the condition[1-4]

$$e_{ilm}\nabla_l u_{mk} = -D_{ik}, \qquad i, k = 1, 2, 3. \tag{1}$$

Here e_{ikl} is the unit antisymmetric third-rank tensor,* and D_{ik} is the dislocation density tensor (of the Burgers vector density),[1-4]

$$\nabla_i D_{ik} = 0, \tag{2}$$

which is the compatibility condition of the equations in (1).

For a linear dislocation

$$D_{ik} = \tau_i b_k \delta(\boldsymbol{\xi}), \tag{3}$$

where $\boldsymbol{\tau}$ is a unit vector tangent to the dislocation line, **b** is the Burgers vector, $\boldsymbol{\xi}$ is a two-dimensional radius vector starting from the axis of the dislocation and lying in the plane perpendicular to $\boldsymbol{\tau}$ at the given point, and $\delta(\boldsymbol{\xi})$ is a two-dimensional δ-function.

In a crystal the vectors $\boldsymbol{\tau}$ and **b** can have only a certain number of well-defined directions; therefore, the tensor D_{ik} in a crystal must have the form

$$D_{ik}(\mathbf{r}) = \sum_{\alpha,\beta}{}' \tau_i^\alpha b_k^\beta \rho^{\alpha\beta}(\mathbf{r}), \tag{4}$$

where α and β denote the possible directions of the vectors $\boldsymbol{\tau}$ and **b**, respectively, and $\rho^{\alpha\beta}$ is the density of distribution of the vectors $\boldsymbol{\tau}$ and **b** over the directions specified. It should be noted that directions of types α and β are in general different and nonequivalent directions in the crystal.

The dislocation singularity, by definition, is not associated with an additional volume force;[2] therefore the equilibrium conditions of an isotropic elastic medium containing dislocations have, in the absence of external volume forces, the usual form

$$\mu\nabla_k(u_{ik} + u_{ki}) + \lambda\nabla_i u_{kk} = 0, \tag{5}$$

where λ and μ are the Lamé coefficients.

For a given dislocation density Eqs. (1) and (5) comprise in the static case a complete system of

*In formula (1) and in those that follow, summation from 1 to 3 is carried out over repeated indices.

equations for determining the distortion tensor u_{ik}, i.e., they completely describe the deformed state of the medium.

For moving dislocations, corresponding differential equations have been proposed by Holländer.[5] However, their derivation is not completely satisfactory: first, they were derived under the assumption that the difference between the velocities of Rayleigh surface waves and shear waves in the solid body could be neglected; second, the quantities assumed given were not the density of the Burgers vector and its flux, but certain quantities not having a direct physical meaning.

In the present work it is shown that the system of equations determining the deformation field of moving dislocations can be formulated without using the stated assumptions. In this system the density of the Burgers vector and its flux are the sources of the deformation tensor field and the medium displacement velocity vector. One of the possible ways of solving this system of equations is equivalent to Nabarro's method,[6] which reduces to replacing the displacement of a linear dislocation by the formation of a dislocation loop. This method is convenient when only the time dependence of the deformation field is considered, but requires the additional solution of a static system of equations of type (1) and (5) to find u_{ik} uniquely.

We present here another method of solving the original system of equations, which is applicable in the general case and is based on the introduction of certain subsidiary quantities like field potentials. This method is also used to determine the field of a dislocation system at large distances from the system, a problem of independent interest.

1. THE DISLOCATION FLUX DENSITY TENSOR

We set up a complete system of differential equations describing the dynamics of an elastic body in which there are moving dislocations. We assume that motion of the dislocations is not accompanied by transport of mass. The equation of motion of an isotropic elastic medium, which replaces (5), must then be written as usual:

$$\rho \partial v_i/\partial t = \mu \nabla_k (u_{ik} + u_{ki}) + \lambda \nabla_i u_{kk}, \tag{6}$$

where ρ is the density of the medium, and $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$ is the displacement velocity of an element of the medium with coordinates $\mathbf{r}$ at the time t (there are no external volume forces).

Condition (1), which defines the dislocation singularity, does not depend on the state of motion of the medium, and therefore remains unchanged.

In order that the system (1) and (6) be complete, it is necessary to establish the relation between the spatial derivatives of the vector $\mathbf{v}$ and the time derivative of the tensor u_{ik}. If the dislocations remain immobile during the motion of the medium, then the obvious equality applies:

$$\nabla_i v_k = \partial u_{ik}/\partial t. \tag{7}$$

If the dislocations move, and the density of dislocations changes with time, equality (7) is incompatible with (1); we therefore, replace it by

$$\nabla_i v_k = \partial u_{ik}/\partial t - I_{ik}, \tag{8}$$

in which the tensor I_{ik} must be chosen so that (1) and (8) are compatible.

To simplify the tensor equations, we denote in what follows any tensor of second rank A_{ik} by the symbol $\hat{\mathbf{A}}$ and introduce the notation:[3] *

$$\operatorname{Rot} \hat{\mathbf{A}} \equiv (e_{ilm} \nabla_l A_{mk}), \qquad \operatorname{Div} \hat{\mathbf{A}} \equiv (\nabla_l A_{lk}), \quad \operatorname{Grad} \mathbf{a} \equiv (\nabla_i a_k).$$

Then the condition of compatibility (1) and (8) can be written as

$$\partial \hat{\mathbf{D}}/\partial t + \operatorname{Rot} \hat{\mathbf{I}} = 0. \tag{9}$$

Equation (9) is the differential form of the conservation law for the Burgers vector. Thus, it is natural to call the tensor $\hat{\mathbf{I}}$ the dislocation flux density and Eq. (9) the dislocation flux continuity equation. An equation formally identical with (9) was obtained by Holländer;[5] however, it involved different quantities (see below).

Starting from the physical concept of the dislocation flux density, we can easily establish the relation between the tensors $\hat{\mathbf{I}}$ and $\hat{\mathbf{D}}$ for a known velocity of the dislocations.

We consider first a linear dislocation with Burgers vector $\mathbf{b}$, for which each point of the loop moves with velocity $\mathbf{V} = \mathbf{V}(\mathbf{r}, t)$, and calculate the Burgers vector flux caused when such dislocations cross some line $\mathscr{L}$. If $\mathbf{dl}$ is an element of the line $\mathscr{L}$, and $\boldsymbol{\tau}$ is a unit vector tangent to the dislocation loop near the point of intersection with the line $\mathscr{L}$, crossing of the line $\mathscr{L}$ by the dislocation loop and transport of the Burgers vector occur only when there is a component of the velocity $\mathbf{V}$ perpendicular both to $\mathbf{dl}$ and to $\boldsymbol{\tau}$. It is apparent that the number of such crossings of the element $\mathbf{dl}$ by "parallel" loops of dislocations in a unit time is†

$$N\,[d\mathbf{l}, \boldsymbol{\tau}]\mathbf{V},$$

where N is the number of dislocations under consideration per unit area of the plane perpendicular to $\boldsymbol{\tau}$.

*Rot = Curl.
†$[\mathbf{dl}, \mathbf{t}] = \mathbf{dl} \times \mathbf{t}$.

Therefore, the flux of the component b_k of the Burgers vector through the line $\mathscr{L}$ is

$$\int_{\mathscr{L}} dl_i I_{ik} = \int_{\mathscr{L}} N\,[d\mathbf{l}, \boldsymbol{\tau}]\,\mathbf{V} b_k. \qquad (10)$$

It follows from (10) that in the case of linear dislocations with equal velocities at the point in space considered

$$I_{ik} = N e_{ilm} \tau_l b_k V_m. \qquad (11)$$

The generalization of (11) to the case of a continuous distribution of dislocations leads to a formula similar to (4):

$$I_{ik}(\mathbf{r}) = e_{ilm} \sum_{\alpha,\beta} \tau_l^{\alpha} b_k^{\beta} V_m^{\alpha\beta}(\mathbf{r})\, \rho^{\alpha\beta}(\mathbf{r}). \qquad (12)$$

Here $\mathbf{V}_{\alpha\beta}$ is the mean velocity of an element of the dislocation line with the corresponding directions (α and β) of the vectors $\boldsymbol{\tau}$ and $\mathbf{b}$, and $\rho(\mathbf{r})$ is the same distribution density as in (4).

Thus, the tensor $\hat{\mathbf{I}}$ has an independent meaning, and is a characteristic of the dislocation motion. To determine the field of the velocity displacements of the medium, the tensor $\hat{\mathbf{I}}$, as well as the tensor $\hat{\mathbf{D}}$, must be assumed given. For a given tensor $\hat{\mathbf{I}}$ the system of equations (1), (6), and (8) is complete.

Special interest attaches to the trace of the tensor $\hat{\mathbf{I}}$ ($\mathrm{Sp}\,\hat{\mathbf{I}} = I_{kk}$), which, in the case of a linear dislocation, is proportional to $[\mathbf{b} \times \boldsymbol{\tau}] \cdot \mathbf{V}$, on the direction perpendicular to the vectors $\boldsymbol{\tau}$ and $\mathbf{b}$; the direction perpendicular to the vectors $\boldsymbol{\tau}$ and $\mathbf{b}$ is, of course, perpendicular to the glide plane of the dislocation.

The quantity $\mathrm{Sp}\,\hat{\mathbf{I}}$ appears in the equation that is obtained by contracting (8):

$$\mathrm{div}\,\mathbf{v} - \partial u_{kk}/\partial t = -\,\mathrm{Sp}\,\hat{\mathbf{I}}. \qquad (13)$$

The physical content of (13) is easily clarified. In fact, the sum u_{kk} is the relative volume change of an element of the medium, and is obviously related to the corresponding relative change of density:

$$u_{kk} = -\,\delta\rho/\rho. \qquad (14)$$

Substituting (14) in (13) and using the linearity of the theory we arrive at the relation:

$$\rho^{-1}\{\mathrm{div}\,\rho\,\mathbf{v} + \partial\rho/\partial t\} = -\,\mathrm{Sp}\,\hat{\mathbf{I}}. \qquad (15)$$

If the elements of the medium move without destruction of continuity during the motion of the dislocation, then, by virtue of the equation of continuity, the left-hand side of (15) vanishes, and

$$\mathrm{Sp}\,\hat{\mathbf{I}} \equiv I_{kk} = 0. \qquad (16)$$

For linear dislocations, condition (16) has a very simple meaning: the dislocation velocity vector $\mathbf{V}$ always lies in the plane of the vectors $\boldsymbol{\tau}$ and $\mathbf{b}$, i.e., in the approximation considered mechanical motion of the dislocation can occur only in the corresponding glide plane.[1,2]

If the dislocation motion is accompanied by the formation of certain breaks in continuity, e.g., by a macroscopic accumulation of vacancies along a part of the dislocation line, then the left-hand side of (15) differs from zero and is equal to the rate of relative inelastic increase of mass of a certain elementary volume of the medium (or, correspondingly, decrease of its specific volume). We denote the relative inelastic increase of specific volume of the medium at the point $\mathbf{r}$ in a unit time by $q(\mathbf{r}, t)$; then

$$\mathrm{Sp}\,\hat{\mathbf{I}} = q(\mathbf{r}, t). \qquad (17)$$

It follows from (17) that motion of a dislocation in a direction perpendicular to its glide plane is possible through the formation (or annihilation) of some continuous chain of vacancies.

With regard to the tensor $\hat{\mathbf{I}}$, it should be specially noted that the dislocation flux density directly determines the rate of plastic deformation of the medium. In order to verify this, we form from (8) the symmetrical expression

$$\tfrac{1}{2}(\nabla_i v_k + \nabla_k v_i) = \partial\varepsilon_{ik}/\partial t - \mathbf{I}_{ik}^S, \qquad (18)$$

in which ϵ_{ik} is the elastic deformation tensor, equal to the symmetric part of u_{ik}, and I_{ik}^S is the symmetric part of I_{ik}.

Since the vector $\mathbf{v}$ is the rate of the total geometrical displacement of an element of the medium, the left-hand side of (18) is the tensor of the velocity of the total geometric deformation $\partial\epsilon_{ik}^g/\partial t$, and therefore

$$\partial(\hat{\varepsilon}^g - \hat{\varepsilon})/\partial t = -\,\hat{\mathbf{I}}^S, \qquad \partial\varepsilon_{ik}^g/\partial t = \tfrac{1}{2}(\nabla_i v_k + \nabla_k v_i).$$

The difference $\hat{\epsilon}^g - \hat{\epsilon}$ defines the part of the total deformation tensor which is not associated with elastic stresses, and is usually called the plastic deformation of the body. Denoting this quantity by $\hat{\epsilon}^p$, we obtain

$$\partial\hat{\varepsilon}^p/\partial t = -\,\hat{\mathbf{I}}^S. \qquad (19)$$

Thus, the change of the plastic deformation tensor at some point of the medium in a small time δt is

$$\delta\hat{\varepsilon}^p = -\,\hat{\mathbf{I}}^S \delta t. \qquad (20)$$

Comparing (20) with (16), we see that the statement that $\mathrm{Sp}\,\hat{\mathbf{I}}$ is zero is equivalent to saying that the corresponding plastic deformation does not involve a change in the volume of the body.

A relationship equivalent to (19) or (20), but expressed in a different form, was obtained by Kröner and Rieder.[7]

2. DEFORMATION FIELD OF MOVING DISLOCATIONS

Equations (1), (6), and (8) comprise the system

$$\rho \frac{\partial v_i}{\partial t} = \mu \nabla_k (u_{ik} + u_{ki}) + \lambda \nabla_i u_{kk},$$
$$e_{ilm} \nabla_l u_{mk} = -D_{ik}, \quad \nabla_i v_k = (\partial u_{ik}/\partial t) - I_{ik}, \qquad (21)$$

which, as already remarked in Sec. 1, is complete and allows the velocity and plastic deformation fields in the body to be found for given sources (i.e., dislocation densities and fluxes).

One of the possible methods of solving the system (21) is to differentiate the first equation of the system [Eq. (6)] with respect to time, and, using the third equation [Eq. (8)], obtain the dynamic Lamé equation in terms of the vector velocity **v**, with the vector $2\mu \operatorname{Div} \hat{\mathbf{I}}^S$ playing the part of the "force density." After **v** has been determined by integrating (8), the time-dependent part of the elastic distortion tensor u_{ik} can be found. The solution of the Lamé equation in terms of **v**, with subsequent use of (8) to determine the time-dependent part of u_{ik}, is equivalent to Nabarro's method.[6] To determine u_{ik} uniquely, such a method requires the additional solution of the static system, which consists of Eq. (1) with the right-hand side taken at some definite moment of time, and an equilibrium equation of type (5), but with the right-hand side equal to $\rho\, \partial \mathbf{v}/\partial t$ at the same instant of time.

We shall solve the system (21) using a different method, which is more convenient in some cases. We write the distortion tensor u_{ik} in the form

$$u_{ik} = \nabla_i U_k - g_{ik}, \qquad (22)$$

where **U** is a vector which is a single-valued function of the coordinates (conditionally—the displacement vector) and $\hat{\mathbf{g}}$ is a second-rank tensor so chosen that its trace is zero

$$\operatorname{Sp} \hat{\mathbf{g}} = 0, \qquad (23)$$

and that it satisfies (1), i.e.,

$$\operatorname{Rot} \hat{\mathbf{g}} = \hat{\mathbf{D}}. \qquad (24)$$

The vector velocity **v** is also resolved into

$$\mathbf{v} = \partial \mathbf{U}/\partial t + \mathbf{w}, \qquad (25)$$

where, if (22) and (25) are to be compatible, the following condition must be satisfied:

$$\operatorname{Grad} \mathbf{w} + \partial \hat{\mathbf{g}}/\partial t = -\hat{\mathbf{I}}. \qquad (26)$$

Equations (23), (24), and (26) do not uniquely determine the tensor $\hat{\mathbf{g}}$ and the vector **w**, and can therefore be related by some additional condition which, to a large extent, is arbitrary. It can be shown that the necessity for such an additional condition disappears if **U** is chosen to be the vector of the total geometrical displacement and $\hat{\mathbf{g}}$ to be the plastic distortion tensor. However, the total displacement vector and the plastic distortion (or deformation) tensor are not functions of the state of the body; they depend on the deformation process that has brought the body to the given state, and are therefore defined only accurate to the initial state of the body. The choice of the initial state becomes the additional condition referred to above.

We are not interested in plastic deformation, but consider those dynamic characteristics of the elastic body (**v** and u_{ik}) which can depend on the history of the deformation only by virtue of the delay of elastic waves. Bearing this in mind, we impose on $\hat{\mathbf{g}}$ and **w** a condition that allows the original system of equations to be separated simply into an equation for **U** and a system of equations for $\hat{\mathbf{g}}$ and **w**. As such a condition it is convenient to take the differential equation

$$\rho \frac{\partial \mathbf{w}}{\partial t} + \mu \operatorname{Div} \hat{\mathbf{g}} = \mu \mathbf{D}, \quad D_i = e_{ikl} D_{kl}. \qquad (27)$$

The equation for the vector displacement **U**, if (23) and (27) are taken into account, then reduces to the dynamic Lamé equation, in which the vector $-2\mu\mathbf{D}$ takes on the role of the force density:

$$\rho \partial^2 \mathbf{U}/\partial t^2 - \mu \Delta \mathbf{U} - (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{U} = -2\mu \mathbf{D}. \qquad (28)$$

The methods of solving dynamic Lamé equations have been well studied and are well known (see, for example, [8–10]), and we shall not, therefore, present a solution of (28) in general form. In Sec. 3 these methods will be illustrated with a specific example.

Equations (24), (26), and (27), which determine $\hat{\mathbf{g}}$ and **w**, form the system

$$\operatorname{Rot} \hat{\mathbf{g}} = \hat{\mathbf{D}}, \qquad \operatorname{Grad} \mathbf{w} + \partial \hat{\mathbf{g}}/\partial t = -\hat{\mathbf{I}},$$
$$\operatorname{Div} \hat{\mathbf{g}} + c^{-2}\, \partial \mathbf{w}/\partial t = \mathbf{D}, \qquad (29)$$

where c denotes the velocity of the elastic shear wave ($c^2 = \mu/\rho$). The conservation laws (2) and (9) are a consequence of the system (29).

We shall, by convention, call the deformation field described by the tensor $\hat{\mathbf{g}}$ and the vector **w** the dislocation deformation. In so far as $\operatorname{Sp} \hat{\mathbf{g}} = 0$

and div $\mathbf{w} = 0$, the dislocation deformation contributes only to the "transverse" (shear) part of the elastic deformation caused by the dislocations.

For screw dislocations $\mathbf{D} \equiv 0$; we can therefore take $\mathbf{U} = 0$, and, by introducing the corresponding notation, reduce the system (29) to Maxwell's system for the plane case (Eshelby[11]). This allows the electromagnetic analogy to be widely used when screw dislocations are considered.

Reverting to the solution of the system (29), we express $\hat{\mathbf{g}}$ and $\mathbf{w}$ in the form

$$\hat{\mathbf{g}} = \operatorname{Rot} \hat{\mathbf{A}} - \operatorname{grad} \chi - \rho \partial \hat{\mathbf{B}} / \partial t,$$

$$\mathbf{w} = \partial \chi / \partial t + \mu \operatorname{Div} \hat{\mathbf{B}}, \; \chi_l = e_{ikl} A_{kl}, \tag{30}$$

where $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are certain tensor potentials, with regard to which we can at once remark that they are not uniquely defined. We use this lack of uniqueness to impose on the potentials the conditions

$$\operatorname{Rot} \hat{\mathbf{B}} + \mu^{-1} \partial \hat{\mathbf{A}} / \partial t = 0, \quad \operatorname{Div} \hat{\mathbf{A}} = 0. \tag{31}$$

Then the equations for $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$, which are obtained by substituting (30) in (29), separate:

$$\Delta \hat{\mathbf{A}} - c^{-2} \partial^2 \hat{\mathbf{A}} / \partial t^2 = -\hat{\mathbf{D}}, \; \Delta \hat{\mathbf{B}} - c^{-2} \partial^2 \hat{\mathbf{B}} / \partial t^2 = -\hat{\mathbf{I}} / \mu. \tag{32}$$

As is seen from (32), the velocity of propagation of the small perturbations of the dislocation deformation is the velocity of transverse sound waves, which agrees with the remark made above concerning the shear character of the dislocation deformation.

For known coordinate and time variation of $\hat{\mathbf{D}}$ and $\hat{\mathbf{I}}$ the general solutions of (32) in an unbounded space are the corresponding retarded potentials, which can easily be written down.

For a known distribution of dislocations and their fluxes, the solution of (28) and (32) also allows us to determine with the aid of formulae (22), (25), and (30) the elastic distortion tensor u_{ik} and the vector displacement velocity of the medium $\mathbf{v}$.

To conclude this section, we shall demonstrate the relationship between the system (28) and (29) and equations[5] with which Holländer proposed to describe directly the elastic deformation tensor $\hat{\epsilon}$ and the velocity vector $\mathbf{v}$.

If the tensor $\hat{\mathbf{D}}$ is expressed as the sum of two tensors $\hat{\mathbf{D}} = -\hat{\alpha} - \hat{\beta}$, one of which ($\hat{\alpha}$) gives rise to Kröner's[3] symmetrical incompatibility tensor $\hat{\eta}$

$$\eta_{ik} \equiv e_{ilm} \nabla_l \alpha_{km} = e_{klm} \nabla_l \alpha_{im},$$

and the second satisfies the conditions

$$e_{klm} \nabla_l \beta_{im} = -e_{ilm} \nabla_l \beta_{km}, \quad \nabla_l \beta_{lk} = 0,$$

then Eq. (1) can be split into two equations, one for the symmetric part of the tensor u_{ik} (for the elastic deformation tensor $\hat{\epsilon}$):

$$\operatorname{Rot} \hat{\varepsilon} = \hat{\alpha} \tag{33}$$

and one for its antisymmetric part, which is not associated with the elastic deformation and the elastic stresses.

Further, if the tensor

$$j_{ik} = -I_{ik}^S - \tfrac{1}{2} \nabla_k v_l, \tag{34}$$

is introduced, Eq. (8) becomes

$$\tfrac{1}{2} \operatorname{Grad} \mathbf{v} - \partial \hat{\varepsilon} / \partial t = \hat{\mathbf{j}}. \tag{35}$$

Equations (33) and (35), together with the equation of motion (6) written in terms of the tensor $\hat{\epsilon}$

$$2\mu \operatorname{Div} \hat{\varepsilon} + \lambda \operatorname{grad} \operatorname{Sp} \hat{\varepsilon} - \rho \, \partial \mathbf{v} / \partial t = 0,$$

comprise Holländer's system of equations.[5] Naturally, (33) and (35) imply a continuity equation of the type (9) which relates the tensors $\hat{\alpha}$ and $\hat{\mathbf{j}}$.[5] With regard to this system, it is assumed that the tensors $\hat{\alpha}$ and $\hat{\mathbf{j}}$ are given. However, as written here these relations are only formal, since it is the Burgers vector $\mathbf{b}$ and the tensor densities directly connected with it that are physically determined.

3. THE DEFORMATION FIELD AT A LARGE DISTANCE FROM A SYSTEM OF DISLOCATIONS

We consider the deformation of an infinite medium caused by a system of moving dislocation loops at large distances, i.e., at distances R_0 significantly greater than the dimensions of the system L. We assume that the dislocations move with velocities $\mathbf{V}$ small compared with the velocity of sound c in the solid body. Then the ratios L/R_0 and V/c are small parameters, in powers of which we can expand the quantities characterizing the deformation field. We limit ourselves to the first nonvanishing terms of such an expansion, considering the smallness of both parameters to be of the same order of magnitude.

We start by analyzing the dislocation deformation which is described by the tensor potentials $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$, which are in turn subject to Eqs. (32). We express the solution of these equations in terms of retarded potentials

$$\hat{\mathbf{A}}(\mathbf{r}, t) = \frac{1}{4\pi} \int \frac{dV'}{R} \hat{\mathbf{D}}\left(\mathbf{r}', t - \frac{R}{c}\right), \quad R = |\mathbf{r} - \mathbf{r}'|, \tag{36}$$

$$\hat{\mathbf{B}}(\mathbf{r}, t) = \frac{1}{4\pi\mu}\int \frac{dV'}{R}\,\hat{\mathbf{I}}\left(\mathbf{r}', t - \frac{R}{c}\right), \quad (37)$$

and note that the potential $\hat{\mathbf{B}}$ is proportional to the parameter V/c.

The expansions of $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are similar to those of the scalar and vector potentials of an electromagnetic field at a large distance from a system of charges and currents (see, for example, [12]). The "total charge" of our system, i.e., the total Burgers vector of all the dislocation loops, is zero:

$$\int D_{ik}\, dV = \sum b_k \oint dx_i = 0, \quad (38)$$

since all the dislocation loops are closed.

We locate the origin of coordinates anywhere inside the dislocation system and denote by R_0 the distance from the origin to the point of observation ($\mathbf{n}$ is a unit vector in corresponding direction). Then, because the dimensions of the system are small, $R \approx R_0 - \mathbf{r}'\cdot\mathbf{n}$ and elementary manipulation leads to the following expressions for the first terms of the expansions of $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$:

$$A_{ik} = \frac{e_{ilm} n_m}{4\pi R_0}\left[\frac{1}{R_0} M_{lk} + \frac{1}{c}\frac{d}{dt} M_{lk}\right],$$

$$B_{ik} = -\frac{1}{4\pi\mu R_0}\left[\frac{d}{dt} M_{ik}\right], \quad (39)$$

where the tensor $\hat{\mathbf{M}}$ is given by

$$M_{ik} = \frac{1}{2}\int e_{ilm} x_l D_{mk}\, dV, \quad (40)$$

and the quantities in the square brackets are taken at the moment of time $(t - R_0/c)$.

The physical significance of the tensor $\hat{\mathbf{M}}$ is easily clarified by going over in (40) to summation over the dislocation loops $\mathscr{D}$:

$$M_{ik} = \frac{1}{2} e_{ilm} \sum b_k \oint_{\mathscr{D}} x_l dx_m = \sum s_i b_k; \quad (41)$$

$\mathbf{s}$ is an axial vector with components equal to the areas included by the projections of the dislocation loops $\mathscr{D}$ on planes perpendicular to the corresponding coordinate axes:

$$s_i = \frac{1}{2} e_{ikl} \oint_{\mathscr{D}} x_k dx_l, \qquad \oint_{\mathscr{D}} x_i dx_k = e_{ikl} s_l.$$

Thus, the tensor $\hat{\mathbf{M}}$ is directly expressed in terms of the dislocation loops and their Burgers vectors. By analogy with the similar definition in electrodynamics, we shall call the tensor $\hat{\mathbf{M}}$ the tensor of the dislocation moment of the system.*

*The dislocation moment of a closed loop does not depend on the choice of the origin of coordinates.

Since

$$ds_i/dt = \oint_{\mathscr{D}} e_{ikm} V_k dx_m,$$

where $\mathbf{V}$ is the velocity of an element of the dislocation loop, it follows from (41) and (12) that

$$dM_{ik}/dt = -\int I_{ik} dV. \quad (42)$$

Relation (42) was utilized in writing down (39).

Introducing the symbols of differentiation with respect to the coordinates and the time at the point of observation, formulae (39) can be simplified:

$$\hat{\mathbf{A}} = \frac{1}{4\pi}\,\mathrm{Rot}\,\frac{\hat{\mathbf{M}}(t - R_0/c)}{R_0}, \qquad \hat{\mathbf{B}} = -\frac{1}{4\pi\mu}\frac{\partial}{\partial t}\frac{\hat{\mathbf{M}}(t - R_0/c)}{R_0}. \quad (43)$$

All the quantities on the right-hand side of (43) are taken at the time $(t - R_0/c)$.

Knowing the potentials $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$, which at great distances are completely determined by (39) or (43), we can easily calculate the tensor $\hat{\mathbf{g}}$ and the vector $\mathbf{w}$ in the same approximation. Since the dislocation deformation is considered far from the dislocations, the tensor $\hat{\mathbf{g}}$ can be expressed as the gradient of some vector. In fact, performing an identity transformation, we can write

$$\left(\mathrm{Rot}\,\hat{\mathbf{A}} - \rho\frac{\partial \hat{\mathbf{B}}}{\partial t}\right)_{ik} = \nabla_i \frac{1}{4\pi}\nabla_l \frac{M_{lk}}{R_0}, \quad (44)$$

where the right-hand side, naturally, is taken at the time $(t - R_0/c)$.

We substitute (39), (40), and (44) in (30), and, remembering (16), verify that

$$\hat{\mathbf{g}} = -\mathrm{Grad}\,\mathbf{g}, \quad \mathbf{w} = \partial \mathbf{g}/\partial t,$$

where the vector $\mathbf{g}$ is equal to

$$g_i = \frac{1}{4\pi}\left[\nabla_k \frac{M_{ik} - M_{ki}}{R_0} + n_i \frac{M_{kk}}{R_0^2}\right]. \quad (45)$$

We now proceed to find the vector $\mathbf{U}$. The solution of (28) for this vector in an unbounded medium has the form [8]

$$U_i(\mathbf{r}, t) = -\frac{c^2}{2\pi}\int \nabla_i \nabla_k \left(\frac{1}{R}\right)\left\{\int_{R/a}^{R/c} \tau D_k(\mathbf{r}', t - \tau)\, d\tau\right\} dV'$$

$$+ \frac{1}{2\pi}\int \frac{X_i X_k - \delta_{ik} R^2}{R^3} D_k\left(\mathbf{r}', t - \frac{R}{c}\right) dV'$$

$$- \frac{\gamma^2}{2\pi}\int \frac{X_i X_k}{R^3} D_k\left(\mathbf{r}', t - \frac{R}{a}\right) dV', \quad (46)$$

where

$$X_i = x_i - x_i', \quad c^2 = \mu/\rho, \quad a^2 = (\lambda + 2\mu)/\rho, \quad \gamma = c/a.$$

The second and third integrals in (46) are similar to the retarded potentials (36) and (37), and imply the separation of the displacement vector field into transverse and longitudinal elastic waves; the first integral has, however, a more complicated structure.

We expand the vector **U** in powers of the small parameters specified earlier and, to simplify the notation, we introduce

$$M^*_{ik} = \int D_i x_k dV \equiv \delta_{ik} M_{ll} - M_{ik}.$$

Simple calculation then shows that the vector **U** splits into two parts:

$$\mathbf{U} = \mathbf{U}_0 + \mathbf{U}_1,$$

one of which depends on the behavior of the dislocation moment in the time interval $(t - R_0/c, t - R_0/a)$:

$$U^0_i = \frac{c^2}{2\pi}\nabla_i\nabla_k\nabla_l\left(\frac{1}{R_0}\right)\int_{R_0/a}^{R_0/c}\tau M^*_{kl}(t-\tau)\,d\tau, \tag{47}$$

whereas the second is determined by the dislocation moment at the times $(t - R_0/c)$ and $(t - R_0/a)$ only—these times are associated, respectively, with the delay of the shear and compression waves in the solid body:

$$\mathbf{U}_1 = \mathbf{U}_t + \mathbf{U}_l,$$

$$U^t_i = \frac{1}{2\pi}\left[R_0\nabla_i\nabla_k\left(\frac{1}{R_0}\right)n_m M^*_{km} - \nabla_m\left\{\frac{n_i n_k - \delta_{ik}}{R_0}M^*_{km}\right\}\right], \tag{48}$$

$$U^l_i = -\frac{\gamma^2}{2\pi}\left[R_0\nabla_i\nabla_k\left(\frac{1}{R_0}\right)n_m M^*_{km} - \nabla_m\left\{\frac{n_i n_k}{R_0}M^*_{km}\right\}\right]_a. \tag{49}$$

In all these formulae, differentiation is with respect to the coordinates of the point of observation, and the square brackets with index a signify that the values of the quantities included in them are taken at the time $(t - R_0/a)$.

It is easy to verify that div $\mathbf{U}_t = 0$ and curl $\mathbf{U}_l = 0$; therefore $\mathbf{U}_t$ and $\mathbf{U}_l$ contribute to the transverse and longitudinal parts of the displacement field, respectively.

We introduce the tensor $t_{ik} = \delta_{ik} - n_i n_k$, which, when multiplied by any vector, gives the "transverse" component of this vector relative to the direction **n**, and use the notation:

$$N_{ikl} = t_{ik}n_l + n_i\delta_{lk} + n_k\delta_{li} - 4n_i n_k n_l.$$

Then (48) and (49) can be reduced to

$$U^t_i = -\frac{1}{2\pi}\left[t_{ik}\nabla_l\frac{M_{kl}}{R_0} - N_{ikl}\frac{M_{kl}}{R_0^2}\right], \tag{50}$$

$$U^l_i = -\frac{\gamma^2}{2\pi}\left[n_i n_k\nabla_l\frac{M_{kl}}{R_0} + (N_{ikl} + n_i\delta_{kl})\frac{M_{kl}}{R_0^2}\right]_a. \tag{51}$$

Formulae (50) and (51), together with (47) and (45), determine the elastic displacement vector $\mathbf{u} = \mathbf{U} + \mathbf{g}$.

In the static case, when the tensor $\hat{\mathbf{M}}$ is independent of the time, the value of **u** agrees with that obtained from the Burgers formula.[13] At distances R_0 that are large compared with the wavelength of the elastic waves emitted ($R_0 \gg cL/V$), the expressions obtained determine the elastic displacement vector of the radiation field. In the latter case all formulae are considerably simplified, and we have for the vector displacement in the radiated sound wave

$$\mathbf{u} = (2\pi c R_0)^{-1}\{\gamma^3\mathbf{n}\,[(\mathbf{n}\dot{\mathbf{M}})]_a + [\dot{\mathbf{M}} - \mathbf{n}\,(\mathbf{n}\dot{\mathbf{M}})]\}. \tag{52}$$

The vector **M** in (52) denotes the "projection" of the symmetrical part of the tensor $\hat{\mathbf{M}}$ on the direction **n**:

$$M_l = n_k M^S_{lk} \equiv \tfrac{1}{2}n_k(M_{ik} + M_{ki}),$$

and the dot over the letter signifies differentiation with respect to the time.

With the aid of (52) we calculate, using the well known formula for the flux density of acoustic energy,[8] the differential intensity of radiation dJ (the radiation in the element of solid angle dO):

$$dJ = \frac{\rho}{4\pi^2 c}\{\gamma^5\overline{[(\mathbf{n}\ddot{\mathbf{M}})]^2_a} + \overline{[\ddot{\mathbf{M}}]^2} - \overline{[(\mathbf{n}\ddot{\mathbf{M}})]^2}\}dO. \tag{53}$$

The bar in (53) denotes averaging over a region in space with dimensions greater than the wavelength of the radiation.

The total energy of the elastic waves radiated by a system of moving dislocations is

$$J = \frac{\rho}{5\pi c}\{\tfrac{2}{3}\gamma^5\overline{[\ddot{M}^S_{ik}]^2_a} + \overline{[\ddot{M}^S_{ik}]^2}\}.$$

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