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SHOCK WAVE STRUCTURE IN A DENSE HIGH-TEMPERATURE PLASMA

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The structure of a plasma shock wave of arbitrary intensity is considered in the two-temperature hydrodynamic approximation with the electronic thermal conductivity and the energy exchange between ions and electrons taken into account. Radiation effects are also included. Two kinds of solutions are obtained: a continuous solution, and a discontinuous solution with an isothermal electron jump. The nature of the solution depends on the wave intensity and the fractional radiation pressure in the initial state of the plasma in front of the shock. The appearance of the discontinuity also depends on the plasma parameters (ion charge Z, adiabaticity index γ) and has a simple physical meaning.

THE structure of shock waves in a plasma has been studied earlier by Zel'dovich,[1] who described the basic effects qualitatively, and by Shafranov,[2] who obtained quantitative results, in particular, for the case of a shock wave of limiting intensity. In the present article we analyze the structure of a shock wave of arbitrary intensity in a plasma in the absence of external fields but with radiation taken into account. The basic result is the determination of the conditions characteristic of a viscous isothermal electron jump.[1,2] This condition has a simple physical significance and, in the limiting case, becomes the Rayleigh condition, well known in the theory of the isothermal jump ([3], page 423).

We consider the case in which the radiation does not attain thermodynamic equilibrium with the electron component of the plasma and in which the density is appreciably smaller than the equilibrium value; under these conditions the radiation can, in general, be neglected in the analysis of the structure of the shock wave. This description applies when the system dimensions are small enough.[2] In this case energy transfer is due primarily to the electronic thermal conductivity of the plasma.

In conjunction with two other plasma characteristics, i.e., the almost complete electrical neutrality over distances of the order of an electron mean free path l_e and the slow rate of energy exchange between electrons and ions, the high value of the electronic thermal conductivity ($\mathrm{Pr} \ll 1$) is more or less responsible for the unique features of the structure of a shock wave in a plasma. These structural features are essentially the following. First, for shocks of limiting intensity the width of the shock front is approximately $l_e\sqrt{M/m}$ and not $\sim l_e$ or $\sim l_i$, as would be expected by analogy with the case of a neutral gas[3] ($l_i \approx l_e Z^{-2}$ is the ion mean free path, where Z is the ion charge; M and m are the masses of the ion and electron respectively. Second, there is a difference in the electron and ion temperatures in the transition layer, characteristic of all high-Z plasmas. Third, at a sufficiently high shock intensity $K = p_2/p_1$ the continuous solution obtained by considering electronic thermal conductivity and neglecting the viscosity and ion thermal conductivity goes over to a discontinuous solution with a viscous isothermal electron jump.

The analysis of a shock wave in a plasma becomes more complicated when radiation is taken into account. If the conditions for local thermodynamic equilibrium apply this problem can be reduced to the preceding problem by taking account of the radiant thermal conductivity, the pressure, and the radiation energy. However, it has been shown[1,4] that equilibrium between the radiation and matter does not obtain in the region of the viscous jump. For this reason the kinetic equation must be used to describe radiation effects.

Actually, for the conditions of interest to us the production of a viscous isothermal electron jump can be obtained without considering the corresponding kinetic equation for the general case. The important result here is the establishment of a second upper limit. Specifically, when a shock wave with radiation reaches a sufficiently high intensity it is found that continuous flow (without the jump) is restored. This property has been noted earlier by Belokon',[5] who used the one-temperature approximation. We may note that in this respect a viscous isothermal jump is not compatible with the kinetic radiation equation[4] in a one-temperature theory.

The notion of a viscous isothermal electron jump is an approximation. If the plasma viscosity and the ion thermal conductivity were introduced the jump would extend over a distance $\sim l_i$. Nevertheless there are a number of valid reasons for neglecting these factors: 1) except in the jump itself, these factors are not important in the transition region; 2) the jump structure as determined with these factors taken into account is still not accurate because the hydrodynamic approximation is used; 3) introducing these factors into the equations for the shock-wave structure makes the mathematical aspects of the problem extremely complicated; 4) the error due to neglecting these factors is very small since variation in electron temperature over the width of the jump can be neglected. This variation is due to energy exchange with the ions, the finite ratio of ion to electron thermal conductivities, and the variation in the radiation intensity that arises because the radiation mean free path is usually appreciably greater than l_i.

Thus, the structure of the shock wave in a plasma can be described satisfactorily in the two-temperature hydrodynamic approximation by taking account of the electron thermal conductivity and the energy exchange between the ions and electrons. When radiation is introduced supplementary terms must be added in the hydrodynamic equations and in the kinetic radiation equation.

Finally, we wish to emphasize that when the linear dimensions of the plasma are limited the present analysis applies only if the plasma is sufficiently dense. In this case the mean free paths for the electrons and ions (l_e and l_i) are appreciably smaller than the linear dimension of the plasma.

1. EQUATIONS FOR SHOCK WAVE STRUCTURE IN A PLASMA WITH RADIATION

We present a phenomenological derivation of the equations for the structure of a shock wave in a plasma which, in the general case, consists of ions of mass M and charge Z, and electrons. Electrical neutrality is assumed for all plasma particles*

$$n = ZN, \tag{1.1}$$

where n and N are the numbers of electrons and ions per unit volume. We assume further that the plasma is ideal ([6], page 249)

$$D_Z \gg d_Z,$$

$$d_Z = [(Z+1)N]^{-1/3}, \qquad D_Z = [kT/4\pi Ne^2 Z(Z+T/\Theta)]^{1/2}, \tag{1.2}$$

where the ion temperature T and the electron temperature Θ can be different. If (1.2) holds the equations of state for an ideal gas apply:

$$p = p_i + p_e, \quad E = E_i + E_e, \quad p_i = NkT,$$

$$p_e = NZk\Theta, \quad E_i = p_i/(\gamma - 1)\rho, \quad E_e = p_e/(\gamma-1)\rho. \tag{1.3}$$

In (1.3) $\rho = NM$ while E_i and E_e are computed per unit mass of the ion gas. The adiabaticity index γ can differ from 5/3 in the general case. In this case we can treat a partially ionized material with high Z approximately. The quantity γ is then called the effective adiabaticity index, in contrast with the Poisson adiabaticity index.*

We now write the equation of motion for the plasma particles taking account of viscosity ([3], page 65):

$$\left(1+\frac{Zm}{M}\right)\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial \Pi'_{ik}}{\partial x_k}, \quad \Pi'_{ik} = \Pi_{ik} + \pi_{ik},$$

$$\Pi_{ik} = p\delta_{ik} + \left(1+\frac{Zm}{M}\right)\rho v_i v_k - \sigma'_{ik}. \tag{1.4}$$

The momentum density flow tensor for the radiation π_{ik} is expressed in terms of the integrated radiation intensity ([8], page 218):

$$\pi_{ik} = \frac{1}{c}\int I\alpha_i\alpha_k d\Omega, \quad I = \int_0^\infty I_\nu d\nu; \tag{1.5}$$

The quantity I in (1.5) is a function of the direction cosines α_i ($i = 1, 2, 3$).

The radiation intensity I_ν is determined by the kinetic equation, in which we must take account of the motion of matter. We do not write this equation here since it is not used below. We also introduce the conservation of energy in the plasma ([3], page 226):

$$\frac{\partial}{\partial t}\left[\rho\left(1+\frac{Zm}{M}\right)\frac{v^2}{2} + \rho(E_i + E_e) + u_r\right]$$
$$= -\operatorname{div}\left\{\rho \mathbf{v}\left[\left(1+\frac{Zm}{M}\right)\frac{v^2}{2} + w_i + w_e\right] + \mathbf{F}_i + \mathbf{F}_e - (\sigma'_i \mathbf{v}) - (\sigma'_e \mathbf{v}) + \mathbf{F}_r\right\}. \tag{1.6}$$

Because the ratio m/M is small, in (1.6) we assume that the viscosity tensor $\sigma' = \sigma'_i + \sigma'_e$ and the

*The terms "plasma particle," "electron gas particle," "ion gas particle" are used in the usual hydrodynamic sense[2]. The electrical neutrality condition is satisfied with high accuracy when (1.2) is satisfied.

*$\gamma - 1 \approx p_e/\rho E_e$ where the right side is computed by means of the generalized Fermi-Thomas equation (cf. [7]).

thermal conductivity flow $\mathbf{F} = \mathbf{F}_i + \mathbf{F}_e$ [9,10] can be added; $\mathbf{F}_r$ is the radiation energy flux and u_r is the radiation energy density:

$$F_{ri} = \int I\alpha_i \, d\Omega, \qquad u_r = \frac{1}{c}\int I d\Omega. \tag{1.7}$$

Equations (1.4) and (1.6) are supplemented by the equation of continuity

$$\partial\rho/\partial t + \operatorname{div} \rho\mathbf{v} = 0. \tag{1.8}$$

When $\Theta \neq T$ we must add an additional equation for the electron temperature Θ. We can write an energy conservation relation for the electron gas and the radiation that interacts with it:

$$\frac{\partial}{\partial t}\left(\frac{Zm}{M}\frac{\rho v^2}{2} + \rho E_e + u_r\right) = -\operatorname{div}\left[\rho\mathbf{v}\left(\frac{Zm}{M}\frac{v^2}{2} + w_e\right) + \mathbf{F}_e - (\sigma_e'\mathbf{v}) + \mathbf{F}_r\right] + c(\Theta, T, \rho) + (\mathbf{f}_{er}\mathbf{v}), \tag{1.9}$$

where $c(\Theta, T, \rho)$ is the rate of transfer of energy from the ions to the electrons.

The expression in the right side of (1.9) is not complete without a supplementary term $\mathbf{f}_{er}\mathbf{v}$ to describe the work of the "external forces" $\mathbf{f}_{er}$ done by the ion gas in the presence of radiation. The force $\mathbf{f}_{er}$, which allows (1.1) to be satisfied is analogous to the coupling reaction in ordinary mechanics. We can find an expression for this force by utilizing this analogy. The electron gas exerts on the ion gas a force given by

$$-\operatorname{grad} p_e + \operatorname{Div}\sigma_e' - \operatorname{Div}\pi - (Zm/M)\rho\, d\mathbf{v}/dt,$$

where the operation Div a applied to a tensor means $\partial a_{ik}/\partial x_k$.

This relation is derived by writing the equation of motion of the electron gas in Lagrangian form taking account of the radiation interaction. The reaction produced by the ion gas, or the reaction coupling, is found by d'Alembert's principle

$$\mathbf{f}_{er} = -(-\operatorname{grad} p_e + \operatorname{Div}\sigma_e' - \operatorname{Div}\pi - (Zm/M)\rho\, d\mathbf{v}/dt). \tag{1.10}$$

It is evident that this force is electrical in nature.

When the force $\mathbf{f}_{er}$ is introduced the equation of motion of the electron gas obviously becomes an identity while the equation of motion of the ion gas coincides with (1.4). Thus, introducing $\mathbf{f}_{er}$ and using (1.10) we have $\mathbf{v}_e = \mathbf{v}_i = \mathbf{v}$; this condition is sufficient to satisfy (1.1).

Substituting (1.10) in (1.9) and using (1.8) together with the thermodynamic identity we obtain the entropy equation for the electron gas with radiation taken into account:

$$\rho\Theta\, dS_e/dt = -\operatorname{div}(\mathbf{F}_e + \mathbf{F}_r) + \operatorname{div}(\sigma_e'\mathbf{v}) - \mathbf{v}\operatorname{Div}(\sigma_e' - \pi) + c(\Theta, T, \rho) - \partial u_r/\partial t. \tag{1.11}$$

In deriving (1.11) we have been concerned primarily with the nonequilibrium behavior of the electrons and ions. This apparent nonequilibrium situation can be easily avoided if we use the coupling reaction exerted by the electron gas on the ion gas in place of (1.10):

$$\mathbf{f}_i = -(-\operatorname{grad} p_i + \operatorname{Div}\sigma_i' - \rho\, d\mathbf{v}/dt). \tag{1.12}$$

Furthermore, if the quantity $\mathbf{f}_i$ (1.12) is used in the energy conservation relation for the ion gas (there are, obviously, no ion radiation terms since the radiation interacts only with the electron component of the plasma), by subtracting the last term of (1.6) and using the transformations indicated above we again obtain (1.11).

Thus, if (1.1) and (1.2) are satisfied the plasma-radiation system is described by (1.3), (1.4), (1.6), and (1.11). We write the equation for a plane stationary shock wave in the coordinate system fixed in the front (the time derivatives vanish), having integrated (1.3), (1.4), and (1.6) with respect to x and taking $\mathbf{F}_i \equiv \sigma_i' \equiv \sigma_e' \equiv 0$:

$$\rho v = \rho_1 v_1 = j, \tag{1.13}$$

$$p + \rho v^2 + \pi_{xx} = p_1 + \rho_1 v_1^2 + \pi_{xx1}, \tag{1.14}$$

$$v^2/2 + w - j^{-1}\varkappa_e d\Theta/dx + j^{-1}F_{rx} = v_1^2/2 + w_1 + j^{-1}F_{rx1}. \tag{1.15}$$

For convenience we take $\rho = (mZ + M)N$; $j > 0$, i.e., the x axis is in the direction of the final plasma state.

Furthermore, using (1.11), the thermodynamic identity (1.13), and the stationarity condition, we have

$$\frac{dE_e}{dx} + p_e\frac{d}{dx}\left(\frac{1}{\rho}\right) = \frac{1}{j}\frac{d}{dx}\left(\varkappa_e\frac{d\Theta}{dx}\right) - \frac{1}{j}\frac{dF_{rx}}{dx} + \frac{v}{j}\frac{d\pi_{xx}}{dx} + \frac{c(\Theta, T, \rho)}{j}. \tag{1.16}$$

Equations (1.13)—(1.16) coincide with the corresponding equations given by Shafranov and Braginskii [2,9] if we set F_{rx} and π_{xx} equal to zero and substitute the expressions for κ_e and $c(\Theta, T, \rho)$.

In the case of the discontinuous solution with a viscous isothermal electron jump (1.16) must be integrated over the vanishingly small thickness of the jump, subject to the condition $\Theta = \Theta_p = \text{const}$ with (1.3) taken into account; we then have from (1.16) [1,2]

$$\frac{Zk}{M + Zm}\Theta_p \ln\frac{v_{p2}}{v_{p1}} = \frac{1}{j}\left[\left(\varkappa_e\frac{d\Theta}{dx}\right) - F_{rx}\right]_{p1}^{p2} + \frac{1}{j}\int_{(p1)}^{(p2)} v\, d\pi_{xx} = \frac{1}{j}\left[\left(\varkappa_e\frac{d\Theta}{dx}\right)_{p2} - \left(\varkappa_e\frac{d\Theta}{dx}\right)_{p1}\right]. \tag{1.17}$$

The radiation terms in (1.17) vanish because the infinitesimally thin plasma layer cannot produce a finite change in the energy and momentum flux of the radiation.* The subscript p1 refers to quantities in front of the jump while the subscript p2 refers to quantities behind the jump.

When supplemented by the kinetic radiation equation Eqs. (1.13)—(1.17) describe completely the structure of a shock wave in a plasma with radiation taken into account.

2. DETERMINATION OF THE BOUNDARY FOR CONTINUOUS AND DISCONTINUOUS SOLUTIONS AND ITS PHYSICAL SIGNIFICANCE

An analysis of the kinematic stability of a shock wave with respect to spontaneous radiation of small (sound) perturbations yields the necessary and sufficient condition for stability ([3], page 406)

$$c_1 < v_1, \quad c_2 > v_2, \tag{2.1}$$

where c_1 and c_2 are the propagation velocities for small perturbations with respect to the matter ahead of and behind the front while v_1 and v_2 are the material velocities in the coordinate system fixed in the front. The quantities c_1 and c_2 in (2.1) are usually the adiabatic sound velocities (c_{ad}). However, it can easily be shown that in a medium characterized by high thermal conductivity ([3], page 421) perturbations of wavelength $\lambda < \lambda_0 \approx \chi/c_{ad}$ (χ is the thermal conductivity) are of isothermal nature and propagate with the isothermal sound velocity (c_{is}).

We now note that the propagation velocity of the perturbations is always smaller in the isothermal case than in the adiabatic case ([6], page 69):

$$c_{ad}^2 = c_{is}^2 + \frac{T}{C_v \rho^2}\left(\frac{\partial p}{\partial T}\right)_\rho^2 \tag{2.2}$$

(C_v is the heat capacity per unit mass). Hence, the case

$$c_{1is} < v_1, \quad c_{1ad} < v_1, \quad c_{2ad} > v_2, \quad c_{2is} < v_2 \tag{2.3}$$

is not trivial and can be realized.

However, (2.3) does not always mean that the shock is unstable against spontaneous radiation of isothermally propagating perturbations characterized by $\lambda < \lambda_0$. First, $\lambda_0 \sim \delta$, where δ is the widths of the transition region of the shock wave [when $\mathrm{Pr} \ll 1$ we have $\lambda_0 \sim (v_T l_0/c_{ad})/\mathrm{Pr} \sim l_0/\mathrm{Pr} \sim \delta$ where l_0 and v_T are the mean free path and the thermal velocity of the heavy particles responsible for viscosity]; second, there are no entropy perturbation waves at these wavelengths. Landau and Lifshitz,[3] on the other hand, assume that $\lambda \gg \delta$ and derive (2.1) taking account of the entropy wave. If the structure of a shock wave is considered, the condition in (2.3) has an important consequence. It means that there cannot be an isothermal jump inside the transition layer.

Repeating our analysis for the case of an isothermal jump we note that when (2.3) is satisfied an arbitrary perturbation arising in the displacement of the jump is characterized by three parameters (two parameters pertain to waves with velocity $v_2 \pm c_{2is}$ and one to the displacement of the jump). On the other hand, only two conditions must be satisfied by a perturbation at the surface of the jump (the energy equation drops out of the continuity relation since it is transformed into a boundary condition for the thermal-conductivity equation). Hence, in the case described by (2.3) an isothermal jump will be unstable and the solution in the transition layer must be continuous. On the other hand, if the following condition is satisfied:

$$c_{1is} < v_1, \quad c_{1ad} < v_1, \quad c_{2ad} > v_2, \quad c_{2is} > v_2, \tag{2.4}$$

then the perturbation is characterized by two parameters (the $v_2 - c_2$ wave no longer exists) and the jump becomes stable.

It follows from (2.3) and (2.4) that the boundary of the discontinuous solutions is given by the condition

$$c_{2is} = v_2. \tag{2.5}$$

We now apply these considerations to a plasma, in which case it is necessary to distinguish between T and Θ. The electronic thermal conductivity of the plasma must be considered and the definition of c_{is} must be modified. We introduce the notion of a velocity of propagation for perturbations that are isothermal with respect to the electrons and adiabatic with respect to the ions:

$$c_{is}^* = [(\partial p_e/\partial \rho)_\Theta + (\partial p_i/\partial \rho)_{S_i}]^{1/2}. \tag{2.6}$$

The introduction of perturbations with velocity c_{is}^* in a plasma is obviously an approximation, but this approximation is as good as the one used for the isothermal electron jump. The wavelength of these perturbations is limited from below by the ion mean free path l_i, i.e., the jump width. The wavelength is limited from above by the quantity

$$\lambda_0 \approx l_e v_{Te}/v_{Ti} \approx \sqrt{M/m}\, l_e,$$

i.e., the width of the transition region of the shock wave in the plasma when radiation is neglected. With radiation present the wavelength range for

*We emphasize that the usual transfer equation cannot be used here because this equation does not take account of the variable velocity of the medium (in our case, the plasma).

the perturbations $\lambda_0 < \lambda < \delta$ is not important because the simple isothermal jump (width $\sim \lambda_0$) is incompatible with the kinetic radiation equation. [4]

Thus, by analogy with (2.5) the boundary separating the continuous and discontinuous solutions of the system of equations in (1.13)—(1.16) is described by the condition

$$c^*_{2\mathrm{is}} = v_2, \qquad (2.7)$$

where the region of continuous solutions is obtained for the inequality $c^*_{2\mathrm{is}} < v_2$ by analogy with (2.3), and the region of continuous solutions from $c^*_{2\mathrm{is}} > v_2$ by analogy with (2.4).

In using these analogies we must keep the following situation in mind. The number of parameters that characterize an arbitrary perturbation in displacement of an isothermal electron jump is one greater than for a simple isothermal case, since the ion entropy wave remains. However, the number of conditions at the jump is increased by virtue of the electron entropy equation (1.17).

We can obtain quantitative results from (2.7). For simplicity, we first treat the case in which there is no radiation but in which γ and Z in the equations of state (1.3) are arbitrary. We can then compute the critical values of the compression $\rho_2/\rho_1 = \delta^*$, the wave force $p_2/p_1 = K^*$, and the Mach number $v_1/c_{1\,\mathrm{ad}} = M^*$ $(c_{1\,\mathrm{ad}} = [\gamma k(Z+1)T_1/M]^{1/2})$ above which the solution becomes discontinuous:

$$\delta^* = \left(\frac{1}{2Z}\frac{\gamma+1}{\gamma-1} - \frac{1}{Z+1}\frac{\gamma-1}{2\gamma}\right) \Big/ \left(\frac{1}{2Z}\frac{\gamma+1}{\gamma-1} - \frac{1}{Z+1}\frac{\gamma+1}{2\gamma}\right),$$

$$K^* = \frac{\gamma+1}{\gamma+1-2(\gamma-1)Z/(Z+1)},$$

$$M^* = \left[\frac{1}{\gamma}\frac{\gamma^2+(3Z+1)\gamma-Z}{(3Z+1)-\gamma(Z-1)}\right]^{1/2}. \qquad (2.8)$$

When $Z \to \infty$ the well-known Rayleigh result is obtained from (2.8) ([3], page 423). As before, the one-temperature approximation should not be valid for the structure of a shock wave; however, because the contribution of the ion component to the pressure p, the internal energy E etc. is so small, the critical values in (2.8) should coincide with those obtained in the one-temperature approximation. For any $\gamma > 1$, it is easily verified that δ^*, K^*, and M^* in (2.8) increase with increasing Z. In the particularly important case $\gamma = 5/3$, we have from (2.8)

	δ^*	K^*	M^*
$Z = 1$	$\frac{19}{16}$	$\frac{4}{3}$	$(\frac{19}{15})^{1/2}$,
$Z \gg 1$	$\frac{3}{2}$	2	$(\frac{9}{5})^{1/2}$.

Formally, it would be more rigorous to derive (2.7) by qualitative analysis of the original system of equations. In the absence of radiation the entire system (1.13)—(1.16) can be reduced to a single differential equation for the Θv diagram. It is then convenient to introduce the dimensionless variables

$$u = v/v_1, \quad t = kZ\Theta/(M + Zm)\, v_1^2.$$

Thus, when radiation is neglected the basic equation for the structure of the shock wave in the plasma becomes

$$\frac{dt}{du} = (1-u)(u-u_2)u^2\left[\frac{1}{2}\frac{\gamma+1}{\gamma-1}(1+u_2) - \frac{\gamma+1}{\gamma-1}u - \frac{t}{u}\right]$$
$$\times\left\{\frac{1}{\gamma-1}(1-u)(u-u_2)u^2 + 6\frac{\gamma-1}{\gamma+1}\varphi_0\frac{M+Zm}{M}t\left[\frac{Z+1}{Z}t + u^2 - \frac{\gamma+1}{2\gamma}u\right]\right\}^{-1}. \qquad (2.9)$$

The quantity φ_0 in (2.9) is a monotonic function of Z. For example, when Z = 1 we find $\varphi_0 = 3.20$ and when $Z \gg 1$, $\varphi_0 = 12.8$. [10] The parameter u_2 is the reciprocal of the shock-wave compression and determines the problem completely.

If the function $t(u)$ is known the remaining quantities characterizing the structure of the transition region are easily found by quadratures or simple algebraic relations. The derivation of (2.7) given above contains one assumption that is not explicitly stated: at the instant it is created the jump is located at the very end of the transition region so that the quantities in (2.7) should appear with the subscript "2." This assumption is rigorously verified by a qualitative analysis of Eq. (2.9). [11] According to the qualitative analysis the boundary between the continuous and discontinuous solutions is given by the value $u_2^* = (\delta^*)^{-1}$, at which the zero isocline [the numerator of the fraction in (2.9)] passes through the point denoting the final state (u_2, t_2). It is easily shown that this gives a condition identical with (2.7).

The critical values δ^*, K^*, and M^* do not depend on κ_e and $c(\Theta, T, \rho)$. However, the position and magnitude of the isothermal electron jump naturally depend on these plasma properties. These quantities have been computed by Shafranov [2] (cf. his Table 2) for Mach numbers greater than M^* although the value of M itself is given incorrectly for $\gamma = 5/3$ and Z = 1.

3. CONDITIONS FOR THE FORMATION OF AN ISOTHERMAL ELECTRON JUMP WITH RADIATION TAKEN INTO ACCOUNT

Let us now consider the case in which radiation must be taken into account. Equation (2.7) shows that the results in (2.8) apply as long as the radiation does not make a contribution to the pressure and internal energy of the plasma, but acts only as a supplementary mechanism for the transfer of energy in the transition region. To obtain the result analogous to (2.8) when taking account of radiation one must consider the Hugoniot condition and (2.7) simultaneously. If the kinetic radiation equation is solved in the coordinate system fixed in the moving plasma, then when $v/c \ll 1$ the Galilean transformation for the energy-momentum tensor is

$$u_r = u_r', \quad \pi = \pi',$$
$$\mathbf{F}_r = \mathbf{F}_r' + (\pi' \mathbf{v}) + \mathbf{v} u_r', \tag{3.1}$$

where $\mathbf{v}$ is the plasma velocity.

The radiation is in equilibrium in the initial and final plasma states. Thus

$$F_r' = 0, \quad u_r' = \sigma T^4, \quad \pi_{xx}' = \tfrac{1}{3}\sigma T^4, \quad \sigma = \pi^2 k^4 / 15 h^3 c^3.$$

Substituting these expressions in (3.1) and F_{rx2}, F_{rx1}, π_{xx2}, π_{xx1} from (3.1) in (1.14) and (1.15), using (1.3) and (1.13) we obtain the Hugoniot relation, which is written in dimensionless form:

$$a t_2 / u_2 + \tfrac{1}{3}\alpha t_2^4 + u_2 = a t_1 + \tfrac{1}{3}\alpha t_1^4 + 1, \tag{3.2}$$
$$\tfrac{1}{2}u_2^2 + b t_2 + \tfrac{4}{3}\alpha u_2 t_2^4 = \tfrac{1}{2} + b t_1 + \tfrac{4}{3}\alpha t_1^4;$$

$$a = 1 + Z, \quad b = \frac{\gamma}{\gamma - 1}(1 + Z), \quad \alpha = \left[\frac{(M + Zm)\, v_1^2}{k}\right]^4 \frac{\sigma}{j v_1}; \tag{3.3}$$

Here, in contrast with Section 2 $t = kT/(M + Zm)r_1^2$.

In solving (3.2) and (3.3) it is convenient to introduce dimensionless parameters for the radiation pressure ratio and the material pressure ratio:

$$\beta = \alpha t_1^3 / 3a, \quad \beta' = \alpha u_2 t_2^3 / 3a, \tag{3.4}$$

which are obviously related by the expression

$$\beta' = (t_2/t_1)^3 u_2 \beta. \tag{3.5}$$

When β and β' are substituted in (3.2) and (3.3), the following expressions are obtained for t_1 and t_2:

$$t_1 = \frac{\varkappa u_2 - (1 + u_2)/2}{\nu - \varkappa u_2} \, \frac{1 - u_2}{a(1 + \beta)}, \tag{3.6}$$

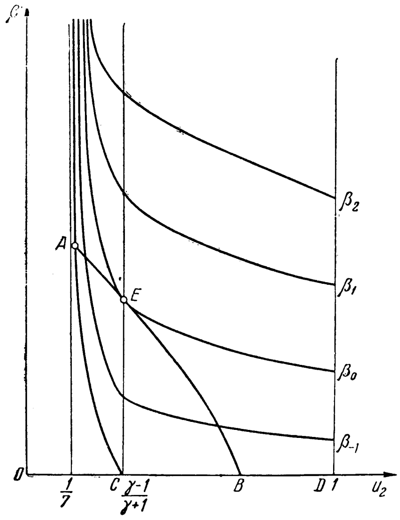
$$t_2 = \frac{\nu - (1 + u_2)/2}{\nu - \varkappa u_2} \, \frac{(1 - u_2)\, u_2}{a(1 + \beta')}; \tag{3.7}$$

$$\varkappa = \frac{4a\beta' + b}{a(1 + \beta')}, \quad \nu = \frac{4a\beta + b}{a(1 + \beta)}. \tag{3.8}$$

An equation relating β and β' is found by substituting t_1 and t_2 from (3.6) and (3.7) in (3.5) and using (3.8):

$$\beta' = \beta u_2^4 \left[\frac{(4a\beta + b) - a(1 + \beta)(1 + u_2)/2}{(4a\beta' + b) - a(1 + \beta')(1 + u_2)/2}\right]^3. \tag{3.9}$$

This equation is solved numerically for a given value of γ and the desired value of Z and yields the function $\beta' = f(\beta, u_2)$ (cf. the figure). The quantities t_1 and t_2 are then found by means of (3.6)—(3.8); finally, (3.4) is used to compute the parameter α, which is expressed in terms of M,



The curve AB represents the boundary between the continuous and discontinuous solutions. The point A corresponds to β_h' from (3.12) (a shock wave of maximum intensity with radiation). The point B corresponds to $\beta' = 0$, $u_2 = u_2^* = (\delta^*)^{-1}$ from (2.8) (shock wave without radiation). The remaining curves represent the Hugoniot adiabat with $\beta_3 > \beta_2 > \beta_1 > \beta_0 > \beta_{-1}$; on the line ACBD for $\beta = 0$ there is a nodal point C at which

$$d\beta'/du_2|_{u_2=\alpha} = -1/\alpha(7\alpha - 1)$$
$$(\alpha = (\gamma - 1)/(\gamma + 1)).$$

The portion of the curve marked AE corresponds to the upper boundary while the portion EB corresponds to the lower boundary (cf Table I).

Z, ρ_1 and v_1. Thus, all quantities can be expressed in terms of β and u_2 when M, Z, ρ_1 and γ are given. We can thus form the Hugoniot adiabat taking account of radiation. We may note that it is a simple matter to verify that the Zemplen condition $(\partial^2 V/\partial p^2)_S > 0$ is satisfied ([3], page 402) whence it follows that the inequality in (2.1) also holds.

Equation (2.7), giving the boundary between the continuous and discontinuous solutions, can be expressed in terms of the dimensionless variables through the use of (1.3) and $\gamma = 5/3$; it assumes the form

$$u_2 = \left[\left(a + \tfrac{2}{3}\right) t_2\right]^{1/2}. \tag{3.10}$$

The radiation pressure does not appear in (3.10) since it is independent of ρ.

We substitute t_2 from (3.7) and (3.8) in (3.10) and find an expression for β':

$$\beta' = \frac{4(4\beta + 5/2)(a + 1/3)u_2 - (1+\beta)(6a + 2/3)u_2^2 + (a + 2/3)(4 + 7\beta)}{2au_2[4u_2(1+\beta) - 4\beta - 5/2]}. \qquad (3.11)$$

In addition to the trivial root $\beta' = \beta$, $u_2 = 1$, (3.9) and (3.11) have two other roots u_{2h} and u_{2l} when $\beta < \beta_0$. It is evident from (3.7) and (3.8) that as β increases t_2 diminishes without limit ($\beta' > \beta$) while $u_2 \geq 1/7$ ($\gamma > 4/3$). Thus, at some sufficiently high value $\beta = \beta_0$ the relation in (3.10) cannot hold for any value of u_2. An equality such as that given by (2.3) now holds, i.e., there is no discontinuous solution for $\beta > \beta_0$. The system (3.9) and (3.11) has been solved numerically for the important case Z = 1 and the solutions are given in Table I.

Table I.
The quantities δ_{2l}, δ_{2h} ($\gamma = 5/3$, Z = 1)

β	δ_{2l}	δ_{2h}	β	δ_{2l}	δ_{2h}
0.0	1.19	6.58	1.2	2.26	5.97
0.2	1.31	6.47	1.4	2.48	5.83
0.4	1.48	6.39	1.6	2.72	5.67
0.6	1.66	6.30	1.8	3.00	5.47
0.8	1.85	6.20	2.0	3.44	5.22
1.0	2.05	6.09	2.2774	4.33	4.33

In this case $\beta_0 = 2.2774$ ($\beta_0' = 3.8249$). We give an additional result for the case $Z \gg 1$:

$$\beta_0 = 1.4055 \ (\beta_0' = 2.5840) \text{ and } \delta_{2l} = \delta_{2h} = 4.30.$$

When $\beta = 0$, the root δ_{2h} can be given in explicit form. We write $\beta = 0$ in (3.9) and (3.11). Then,

$$u_2 = (1 + \beta')/(4 + 7\beta'), \qquad (3.9')$$

$$\beta' = \frac{5(a + 1/3)u_2 - (3a + 1/3)u_2^2 - 2(a + 2/3)}{au_2(4u_2 - 5/2)}. \qquad (3.11')$$

If we substitute (3.9′) in (3.11′) and eliminate from the resulting cubic equation in β' the spurious root $\beta' = -5/9$, we can find an expression for β_h':

$$\beta_h' = 2\frac{a+1}{a} + \left[4\left(\frac{a+1}{a}\right)^2 + 2\frac{a+1}{a}\right]^{1/2}. \qquad (3.12)$$

Using (3.9′) and (3.12) we have compiled Table II for the critical values of the compression δ_{2h} as a function of Z in the case of a limitingly strong shock wave. The appearance of upper boundaries for the discontinuous solution can be understood if we refer to (2.7). At high values of β', the quantity $c_{is2}^* \sim T_2^{1/2} \sim v_1^{1/4}$, while $v_2 \sim v_1$, i.e., the left side of (2.7) becomes smaller than the right and this means, as shown in Sec. 2, that the jump is unstable.

Table II.
The quantities δ_{2h}, β_h' ($\gamma = 5/3$, shock wave of maximum intensity)

Z	β_h'	δ_{2h}
1	6.46	6.58
2	5.79	6.56
3	5.46	6.53
∞	4.45	6.46

By analogy with Eq. (2.9) we can obtain an equation for the structure of the shock wave in a plasma taking account of radiation by transforming (1.13) —(1.16):

$$\frac{d\Theta}{dv} = \left(w_e + w_i + \frac{v^2}{2} + \frac{F_{rx}}{j} - w_1 - \frac{F_{rx1}}{j} - \frac{v_1^2}{2}\right)\left(\frac{A\Theta}{v} + \frac{\gamma+1}{\gamma-1}v + \frac{1}{\gamma-1}\frac{v}{j}\frac{d\pi_{xx}}{dv} + \frac{\gamma}{\gamma-1}\frac{\pi_{xx}}{j} - \frac{\gamma}{\gamma-1}\frac{p_1}{j} - \frac{\gamma}{\gamma-1}\frac{\pi_{xx1}}{j} - \frac{\gamma}{\gamma-1}v_1\right)\left\{\frac{A}{\gamma-1}\left(w_1 + \frac{F_{rx1}}{j} + \frac{v_1^2}{2} - w_e - w_i - \frac{v^2}{2} - \frac{F_{rx}}{j}\right) + \frac{c(\Theta, T, \rho)\varkappa_e}{j^2}\right\}^{-1}. \qquad (3.13)$$

Here, A = kZ/(M+mZ) and the numerator becomes the total derivative of the radiation momentum flow $d\pi_{xx}/dv$. Its value at every point as well as the values of π_{xx} and F_{rx} can be found from the kinetic radiation equation taking account of the movement of matter.

The vanishing of the zero isocline [the second bracket in (3.13)] at the point corresponding to the final state behind the wave front, and the Hugoniot condition can be easily shown to give a condition identical to (3.10). We note that in this case it is first necessary to remove from

$$\left(\frac{d\pi_{xx}}{dv}\right)_2 = \left(\frac{\partial\pi_{xx}}{\partial v}\right)_2 + \left(\frac{\partial\pi_{xx}}{\partial\Theta}\right)_2\left(\frac{d\Theta}{dv}\right)_2$$

the second term and carry it over to the left side of (3.13).

The curve $\beta' = f(u_2)$, which forms the boundary between the continuous and discontinuous solutions, can be conveniently plotted on one curve with the Hugoniot adiabat (see the figure). The entire region under the curve AEB corresponds to the discontinuous solution while the region of continuous solutions lies above.

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