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FINITE AMPLITUDE WAVES IN MAGNETOHYDRODYNAMICS

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One-dimensional waves moving perpendicularly to a magnetic field in a viscous heat- and electricity-conducting medium are considered. Exact solutions are obtained for the magnetohydrodynamic equations and terms linear and quadratic in the Mach number are retained. It is demonstrated that all results of ordinary hydrodynamics apply to the case under consideration.

1. Kaplan and Stanyukovich[1] obtained an exact Riemann solution for one-dimensional traveling waves propagating transversely to a magnetic field in the absence of energy dissipation. This solution, however, becomes invalid as soon as discontinuities are formed in the wave. When dissipative terms are taken into account in the initial equations, the exact solutions are also suitable after the discontinuities are formed in the wave. Such solutions are known for several hydrodynamic problems[2-5].

In the present paper we obtain analogous solutions for magnetohydrodynamic problems, particularly for a low intensity shock wave and for a wave radiated by a harmonically oscillating plane. These solutions enable us, first, to trace the transition of the initially specified zero-width shock wave into a shock wave with stationary width and to estimate the time of this transition; second, we can analyze the subsequent deformation of a propagating velocity wave that is sinusoidal in a certain plane.

2. The magnetohydrodynamic equations for our one-dimensional case have, under the customary assumptions, the following form (see the book by Landau and Lifshitz[6])

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0, \tag{1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\left(\frac{4}{3}\eta + \zeta\right)\frac{\partial^2 v}{\partial x^2} - \frac{1}{4\pi\rho} H\frac{\partial H}{\partial x}, \tag{2}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial (vH)}{\partial x} + \frac{c^2}{4\pi\sigma}\frac{\partial^2 H}{\partial x^2}, \tag{3}$$

$$\rho T\left(\frac{\partial s}{\partial t} + v\frac{\partial s}{\partial x}\right) = \sigma_{11}'\frac{\partial v}{\partial x} + \varkappa\frac{\partial^2 T}{\partial x^2} + \frac{c^2}{16\pi^2\sigma}\left(\frac{\partial H}{\partial x}\right)^2, \tag{4}$$

$$p = p(\rho, s). \tag{5}$$

We retain the Landau and Lifshitz notation and use $\mathbf{v} = (v, 0, 0)$ and $\mathbf{H} = (0, H, 0)$.

We retain in the basic equations the terms linear and quadratic in the amplitudes and only the linear terms with dissipative coefficients, i.e., we assume the linear dissipative terms to be of second order of smallness. It can be shown that this corresponds to Reynolds numbers of the order of unity. In this approximation we can rewrite Eqs. (1) — (5) in the Lagrange variables a and t:

$$\rho_0 da = \rho dx, \tag{6}$$

$$\frac{\partial v}{\partial t} - \frac{1}{\rho_0}\left(\frac{4}{3}\eta + \zeta\right)\frac{\partial^2 v}{\partial a^2} + \frac{1}{\rho_0}\frac{\partial p'}{\partial a} + \frac{1}{4\pi\rho_0} H\frac{\partial h}{\partial a} = 0, \tag{7}$$

$$\frac{\partial h}{\partial t} + \frac{\rho H}{\rho_0}\frac{\partial v}{\partial a} - \frac{c^2}{4\pi\sigma}\frac{\partial^2 h}{\partial a^2} = 0, \tag{8}$$

$$\rho_0 T_0\frac{\partial s'}{\partial t} = \varkappa\frac{\partial^2 T}{\partial a^2}, \tag{9}$$

$$p' = u_0^2\rho' + \frac{1}{2}\left(\frac{\partial^2 p}{\partial \rho^2}\right)_s \rho'^2 + \left(\frac{\partial p}{\partial s}\right)_\rho s'. \tag{10}$$

Here $u_0^2 = (\partial p/\partial \rho)_s$ and $H = H_0 + h$; the subscript 0 denotes the constant equilibrium values of the quantities, while h, ρ', p', and s' are the small changes that these quantities experience in the wave.

Since we disregard third-order terms, we can transform the second-order terms by using the equations of the linear theory of isotropic wave motion[6]

$$h = \frac{H}{u}v, \qquad T' = \left(\frac{\partial T}{\partial p}\right)_s \rho' u_0^2, \qquad \frac{\partial}{\partial t} = -u\frac{\partial}{\partial a}; \qquad u = \sqrt{u_0^2 + \frac{H_0^2}{4\pi\rho_0}}. \tag{11}$$

We transform h and p' into functions of v and ρ' and their derivatives by means of (8) — (11) and substitute in (7), obtaining an equation in v and ρ', which is reduced by the relations

$$v = \partial z/\partial t, \quad \rho' = -\rho_0 \partial z/\partial a + \rho_0 (\partial z/\partial a)^2,$$

(z is the displacement) to the form

$$\frac{\partial^2 z}{\partial t^2} - u^2 \frac{\partial^2 z}{\partial a^2} - \frac{b}{\rho_0}\frac{\partial^3 z}{\partial t \partial a^2} + 2u_0^2 \varepsilon \frac{\partial z}{\partial a}\frac{\partial^2 z}{\partial a^2} = 0; \tag{12}$$

$$b = \frac{4}{3}\eta + \zeta + \varkappa\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\frac{u_0^2}{u^2} + \frac{c_0^2 H_0^2}{16\pi^2 \sigma u^2},$$

$$\varepsilon = 1 + \rho_0 (\partial^2 p/\partial \rho^2)_s / 2u_0^2 + 3H_0^2/8\pi\rho_0 u_0^2.$$

When $H_0 = 0$ Eq. (12) has the same form but the coefficients u, b, and ϵ are replaced by their values u_0, b_0, and ϵ_0 for $H_0 = 0$. Therefore all the results obtained for hydrodynamic waves in this approximation (see [5] for details) can be generalized to our case by replacing in the final formulas the coefficients u_0, b_0, and ϵ_0 by u, b, and $u_0^2\epsilon/u^2$ respectively.*

3. We specify a wave with

$$\partial z(a, 0)/\partial a \equiv \partial z_0(a)/\partial a \tag{13}$$

at the instant t = 0. Rewriting (12) in the dimensionless variables (see [3])

$$\tilde{t} = t/\tau, \quad \tilde{\xi} = a/u\tau - t/\tau, \quad \tilde{z} = \varepsilon u_0^2 z/\tau u^3,$$

where $\tau = b/2\rho_0 u^2$, we obtain

$$\frac{\partial^2 \tilde{z}}{\partial \tilde{t}^2} - 2\frac{\partial^2 \tilde{z}}{\partial \tilde{t}\partial \tilde{\xi}} - 2\frac{\partial^3 \tilde{z}}{\partial \tilde{t}\partial \tilde{\xi}^2} + 2\frac{\partial^3 \tilde{z}}{\partial \tilde{\xi}^3} + 2\frac{\partial \tilde{z}}{\partial \tilde{\xi}}\frac{\partial^2 \tilde{z}}{\partial \tilde{\xi}^2} = 0. \tag{14}$$

We estimate the terms of (14) by using the expression for a low-amplitude damped wave, since the nonlinear and dissipative terms in (12), which distort the waveform, are in our case of the same order of magnitude. As a result we obtain

$$\tilde{z} \sim \frac{\varepsilon u_0^2}{u^2}\frac{v}{u}\frac{k}{\gamma} \sim 1, \quad \frac{\partial \tilde{z}}{\partial \tilde{t}} \sim \left(\frac{\gamma}{k}\right)^2 \tilde{z}, \quad \frac{\partial \tilde{z}}{\partial \tilde{\xi}} \sim \frac{\gamma}{k}\tilde{z}.$$

Neglecting in (14) terms of order $(\gamma/k)^4$ we obtain for the quantity $\tilde{\nu} = -\partial\tilde{z}/\partial\tilde{\xi}$ the expression

$$\frac{\partial^2 \tilde{\nu}}{\partial \tilde{\xi}^2} = \frac{\partial \tilde{\nu}}{\partial \tilde{t}} + \tilde{\nu}\frac{\partial \tilde{\nu}}{\partial \tilde{\xi}}. \tag{15}$$

The substitution (see [8,9])

$$\tilde{\nu} = -2\partial \ln \tilde{\varphi}/\partial \tilde{\xi}$$

reduces (15) to the well known heat-conduction equation.

We thus obtain a solution that satisfies (15) and (13)

$$\tilde{\nu}(\tilde{\xi}, \tilde{t}) = -2\frac{\partial}{\partial \tilde{\xi}} \ln\left\{\int_{-\infty}^{+\infty} \exp\frac{1}{2}\left[-\int_{\alpha}^{0}\frac{\partial \tilde{z}_0}{\partial \tilde{\xi}}d\tilde{\xi} - \frac{(\tilde{\xi}-\alpha)^2}{2\tilde{t}}\right]d\alpha\right\}. \tag{16}$$

*We note that from the linearlized equation (12) we can obtain in the usual manner the known expression for the absorption coefficient of a low-amplitude wave. (see[7]), $\gamma = b\omega^2/2\rho_0 u^3$.

In the case of a shock wave with a pressure distribution specified at the instant t = 0

$$p' = \begin{cases} \text{const} = p_1 & (a < 0), \\ 0 & (a > 0), \end{cases} \tag{17}$$

expression (16) assumes after suitable transformation the form

$$p'(y, t) = \frac{\rho_0 u^2}{\varepsilon}\tilde{\nu} = p_1\left[1 + \exp\left(\frac{\varepsilon p_1}{bu}y\right)\frac{\int_{-\alpha t - y}^{\infty}\exp\left(-\frac{z^2\rho_0}{2bt}\right)dz}{\int_{-\alpha t + y}^{\infty}\exp\left(-\frac{z^2\rho_0}{2bt}\right)dz}\right]^{-1} \tag{18}$$

Here

$$\alpha = \varepsilon p_1/2\rho_0 u, \quad y = a - (u + \alpha)t.$$

When y = 0 we get $p' = p_1/2$. This means that the point of the wave profile with $p' = p_1/2$ propagates with constant velocity $u + \alpha$, which is simultaneously the velocity of propagation of the shock wave. The velocities of the other parts of the profile depend on y and t, so that the wave profile becomes deformed. As $t \to \infty$, however, the ratio of the integrals in (18) tends to unity for any finite y and the wave assumes the stationary form

$$p'(y, t) = \frac{p_1}{2}\left(1 - \text{th}\,\frac{y}{\delta}\right), \tag{19*}$$

where $\delta = 2bu/\epsilon p_1$ is called the width of the shock wave.† The time required to establish an essentially stationary waveform (t_1) can be estimated from the relation [see (18)]

$$\alpha t_1 - \delta \approx \sqrt{2bt_1/\rho_0},$$

which yields $t_1 \approx 4\delta/\alpha$.

The figure shows the distribution of p'/p_1 along the shock wave at the instants $0.01t_1$, $0.1t_1$, and t_1 (curves 1 – 3, respectively). The values of p'/p_1 at the instant t_1 differ from the stationary values for $|y| < \delta$ by less than 5%. It can be shown that when $\sqrt{t/t_1} \ll 1$ we have at the point y = 0

$$\frac{d(p'/p_1)}{d(y/\delta)} = \frac{1}{5}\sqrt{\frac{t_1}{t}}.$$

4. We now consider a wave with a velocity field specified in the plane a = 0:

$$\partial z(0, t)/\partial t = \partial z_0(t)/\partial t. \tag{20}$$

We introduce the dimensionless variables

$$\tilde{a} = a/U_a, \quad \tilde{\tau} = t/U_t - \tilde{a}, \quad \tilde{z} = z/U_z;$$

$$U_a = b/2\rho_0 u_0, \quad U_t = U_a/u_0, \quad U_z = U_a/\varepsilon.$$

*th = tanh

†Sirotina and Syrovat-skii [7] obtained this value of δ by a different method.

Estimating as before and neglecting terms of order $(\gamma/k)^4$ in (12) we obtain for the quantity

$$\tilde{v} = -\partial\tilde{z}/\partial\tilde{t} = -\partial\tilde{z}/\partial\tilde{\tau}$$

the following equation

$$\frac{\partial\tilde{v}}{\partial\tilde{a}} + \tilde{v}\,\frac{\partial\tilde{v}}{\partial\tilde{\tau}} = \frac{\partial^2\tilde{v}}{\partial\tilde{\tau}^2}. \qquad (21)$$

Equation (21) and the boundary condition (20) are similar in form to (15) and to the initial condition (13). Therefore if we replace $\tilde{\xi}$ by $\tilde{\tau}$ and $\tilde{t}$ by $\tilde{a}$ in the right half of (16) we obtain a function $\tilde{v}(\tilde{\tau}, \tilde{a})$ that satisfies Eq. (21) and condition (20). At the particular boundary condition

$$v_0(t) = z_0\omega \sin \omega t \qquad (22)$$

integration and use of the formula

$$e^{z\cos\varphi} = I_0(z) + 2\sum_{n=1}^{\infty} I_n(z)\cos n\varphi$$

result in a solution of the form

$$v(a, t) = \frac{bu}{\rho_0 u_0^2 \varepsilon} \times \frac{\partial}{\partial t}\ln\left[1 + 2\sum_{n=1}^{\infty}(-1)^n \frac{I_n(R)}{I_0(R)} e^{-n^2\gamma a}\cos n\omega\left(t - \frac{a}{u}\right)\right]. \qquad (23)$$

Here $R = \epsilon z_0 \rho_0 u_0^2/bu$.

At large values of R away from the radiating plane this solution can be represented in simpler form. At large R

$$I_n(R) = \frac{e^R}{\sqrt{2\pi R}}\left[1 - \frac{4n^2 - 1}{8R} + O\left(\frac{1}{R^2}\right)\right]. \qquad (24)$$

Therefore at distances

$$a \gg 1/2R\gamma \qquad (25)$$

we can write approximately

$$\sum_{n}^{\infty}(-1)^n \frac{I_n(R)}{I_0(R)} e^{-n^2\gamma a}\cos n\omega\left(t - \frac{a}{u}\right) = \sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos n\omega\left(t - \frac{a}{u}\right), \qquad (26)$$

where $q = \exp(-\gamma a - 1/2R)$. Actually, if (25) holds true the main contribution to the sum (26) is made by the n_1 first terms, for which

$$I_n(R)/I_0(R) \approx e^{-n^2/2R},$$

for when $n_1^2 \sim 2R$ or more the factor

$$e^{-n_1^2\gamma a} \sim e^{-2R\gamma a}$$

causes the remainder of the sum to vanish even when $a > a_0 = 2/R\gamma$. We note that $a_0/4$ is the distance over which a discontinuity is accumulated in an initially sinusoidal wave in a lossless medium.

Transforming (23) further by means of the known formula for the fourth ϑ-function[10]

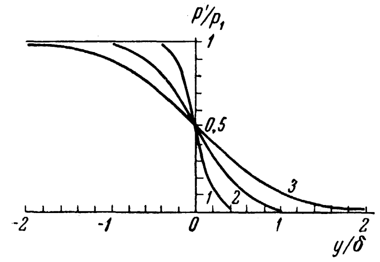
$$\vartheta_4(z, q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nz,$$

$$\frac{\partial}{\partial t}\ln\vartheta_4(z, q) = 4\sum_{n=1}^{\infty}\frac{q^n \sin 2nz}{1 - q^{2n}},$$

we obtain

$$v(a, t) = \frac{b\omega u}{\rho_0 u_0^2\,\varepsilon}\sum_{n=1}^{\infty}\frac{\sin n\omega(t - a/u)}{\operatorname{sh} n(\gamma a + 1/2R)}. \qquad (27)^*$$

A result corresponding to (27) was obtained for H = 0 by Fay[2] as an exact periodic solution of (12) without any initial or boundary conditions.



When R is large we have for a certain integer n′

$$\operatorname{sh} n'\gamma a \approx n'\gamma a,$$

and (27) coincides, accurate to the n′-th harmonic, with the Fourier expansion of a sawtooth curve. At distances $a > 2/\gamma$ we can confine ourselves in the sum (27) to the first terms only. Thus, an initially sinusoidal wave is converted at distances on the order of a_0 into a sawtooth wave, which then goes over into a damped sinusoidal wave as it propagates.

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*sh = sinh