MACROSCOPIC MASS RENORMALIZATION AND ENERGY LOSSES OF CHARGED PARTICLES IN A MEDIUM

V. N. TSYTOVICH

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

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We have used Green's-function methods to find a quantum-mechanical expression for the macroscopic mass renormalization and energy losses of charged particles in absorbing media with spatial dispersion. We have considered the influence of the spatial dispersion of the dielectric constant on the magnitude of the macroscopic mass renormalization in the classical limit and we have found the first quantum corrections. We have considered the quantum corrections to the characteristic collective losses of electrons in thin films.

1. INTRODUCTION

A free electron in a medium possesses a different mass from that in a vacuum.[1,2] In first-order perturbation theory this effect is a result of the emission and absorption of a virtual photon propagating in the medium (less the effect in a vacuum) and can be called a macroscopic mass renormalization (m.m.r.). The physical effect of m.m.r. is perfectly clear since a fast electron in the medium corresponds to an excitation with an energy spectrum bounded at high velocities. It is clear that the magnitude of the m.m.r. depends on the velocity of the electron relative to the medium.* In earlier papers, either incorrect expressions[2] were used for the magnitude of the m.m.r. or m.m.r. was taken to be a quantity which is not the changed particle mass.[4]

The m.m.r. problem is of interest in connection with papers on transition radiation[4,5] in which it was shown that the work done by the forces on the particle when it passes the dividing boundary between two media is not the same as the energy loss in the transition radiation. The difference between these quantities is the additional work ΔW spent on m.m.r., and ΔW is the difference in energy for the same velocities, $\Delta W = \Delta m/\sqrt{1-v^2}$. In what follows we find the difference in energy for the same momentum, which in first approximation in Δm is $\Delta E = \Delta m\sqrt{1-v^2}$, i.e., $\Delta W = \Delta E/(1-v^2)$. One must bear this last formula in mind in applications.

When there is strong absorption in the medium, there is no unambiguous classical definition of the energy in the electromagnetic field and it is impossible to define the m.m.r. in the classical limit as the difference between the electromagnetic energy of the particle in the medium and in a vacuum. Even in transparent media, the field energy comprises, along with the self-field, also the radiation field, since the particle emits radiation continuously and it is difficult to split off the field energy component connected with the m.m.r. for arbitrary velocities, except for the case where Cerenkov and polarization losses are impossible.[6]

In the following we use a quantum-mechanical derivation of the m.m.r. which enables us to find it for media which have a strong absorption and, moreover, taking spatial dispersion into account. We find, together with the classical expressions, also the quantum corrections to the m.m.r. They increase at high energies as $\ln^2[1/(1-v^2)]$. We analyze also the role of the spatial dispersion in the m.m.r.

We must draw attention to the interesting case of a "smeared out" separating boundary (for instance, the boundary of a plasma) when the density of the medium changes smoothly from zero to a finite value over distances which are appreciably larger than the zone where the transition radiation is formed. We showed in [6] that there is no transition radiation then, but the work done by the forces when the particle traverses the separating boundary does not vanish and is the same as the work spent on the m.m.r.

We shall use for our calculations the Green's function method.[7] Doubt is cast on the results obtained by means of phenomenological quantum

*In[3] it was stated incorrectly that the magnitude of the m.m.r. decreases when the particle energy increases.

electrodynamics.[3,8] The Green's function method enables us to consider energy losses of particles in media with strong absorption, taking spatial dispersion into account, when phenomenological quantum electrodynamics does not apply. The classical theory of energy losses, which was first developed by Tamm and Frank[9] and by Fermi,[10] was considered by Agranovich, Rukhadze, and Silin[11,12] and applied to media with spatial dispersion. Ginzburg[13] and Sokolov[14] considered quantum effects for Cerenkov losses.

2. GENERAL RELATIONS

1. We consider a spin-$\frac{1}{2}$ particle moving in a medium with spatial dispersion with an energy E and momentum **p**.* Because of losses, E is a complex quantity

$$E = E' + iE'', \qquad E'' = -\gamma/2. \tag{1}$$

The real part of E is connected with **p** by a relation which takes the m.m.r. into account and differs from the energy of the free particle $\epsilon_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$. The quantity γ is the probability for energy loss of the particle per unit time (emission of transverse and longitudinal quanta in an isotropic medium).

The connection between E and **p** is given by the poles of the electron Green's function which satisfies, when the system has translational symmetry, the equation

$$[i\hat{p} + m + \delta\hat{M}(p)]\, G(p) = 1, \tag{2}$$

where $\hat{p} = \gamma_\nu p_\nu$, $p_\nu = \{\mathbf{p}, iE\}$, γ_ν are the Dirac matrices and $\delta\hat{M}(p)$ is the mass operator; we have omitted the spinor indices in (2). For the case of electromagnetic interactions we can, if we take terms of first order in e^2 into account, write $\delta\hat{M}$ in the form

$$\delta\hat{M}(E, \mathbf{p}) = i\gamma_\nu \delta p_\nu (E, \mathbf{p}) + \delta m (E, \mathbf{p}), \tag{3}$$

$$\delta p_\nu = \delta p'_\nu + i\delta p''_\nu, \qquad \delta m = \delta m' + i\delta m''. \tag{4}$$

Substituting (4) in (2) and introducing

$$P_\nu = p_\nu + \delta p_\nu, \qquad \mu = m + \delta m, \tag{5}$$

we get a relation between E and **p**:

$$|i\hat{P} + \mu| = 0. \tag{6}$$

This determinant corresponds to the Dirac equation and we have thus

$$E = \pm\sqrt{P^2 + \mu^2} - \delta E, \tag{7}$$

where $\delta E = -i\delta p_4(E, \mathbf{p})$.

*In quantum-mechanical analysis we are dealing with a quantum state with a well-defined energy and momentum.

We shall be interested henceforth only in positive E and retain only the plus sign in (7). Substituting (1) in (7), assuming the losses to be small, and retaining only terms linear in e^2 we get, by equating the real and imaginary parts in (7),

$$\Delta E = E' - \varepsilon_{\mathbf{p}} = -\delta E' + \mathbf{p}\delta\mathbf{p}'/\varepsilon_{\mathbf{p}} + m\delta m'/\varepsilon_{\mathbf{p}}, \tag{8}$$

$$\gamma/2 = \delta E'' - \mathbf{p}\delta\mathbf{p}''/\varepsilon_{\mathbf{p}} - m\delta m''/\varepsilon_{\mathbf{p}}, \qquad \varepsilon_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}. \tag{9}$$

Equation (8) can be written in terms of the magnitude of the m.m.r., which is introduced by means of the equation

$$\Delta m = \sqrt{E'^2 - \mathbf{p}^2} - m; \tag{10}$$

up to terms of order e^2 we get

$$\Delta m = \delta m' + \mathbf{p}\delta\mathbf{p}'/m - \varepsilon_{\mathbf{p}}\delta E'/m. \tag{11}$$

2. To construct the mass operator we need know the photon Green's function. In an isotropic medium the photon Green's function satisfies the equation

$$(k^2\delta_{\mu\nu} - k_\mu k_\nu - 4\pi\Pi_{\mu\nu})\, D_{\nu\sigma} = 4\pi\alpha_{\mu\sigma}, \qquad k^2 = \mathbf{k}^2 - \omega^2;$$
$$\mu, \nu = 1, 2, 3, 4; \qquad i, j = 1, 2, 3; \qquad k_\mu = \{\mathbf{k}, i\omega\}; \tag{12}$$

the components of the polarization operator are connected with the dielectric constant tensor

$$\varepsilon_{ij} = \varepsilon^t(\omega, \mathbf{k})(\delta_{ij} - k_i k_j/\mathbf{k}^2) + \varepsilon^l(\omega, \mathbf{k})\, k_i k_j/\mathbf{k}^2$$

through the relations (see [15])

$$k_4^2\delta_{ij} - 4\pi\Pi_{ij} = k_4^2\varepsilon_{ij}, \qquad k_i k_4 + 4\pi\Pi_{i4} = k_i k_4 \varepsilon^l,$$
$$\mathbf{k}^2 - 4\pi\Pi_{44} = \mathbf{k}^2\varepsilon^l. \tag{13}$$

The tensor $\alpha_{\mu\sigma}$ must satisfy the condition

$$\alpha_{\mu\sigma} j_\sigma = j_\mu, \tag{14}$$

where $j_\mu(\omega, \mathbf{k})$ are the external current components. In a vacuum ($\epsilon = 1$) it is convenient to assume that

$$\alpha_{\mu\sigma} = \delta_{\mu\sigma} - k_\mu k_\sigma/k^2,$$

which satisfies (14) because of the current conservation law $k_\mu j_\mu = 0$. When there is a medium present we must, of course, work in the rest frame of the medium, and a choice such as the above one is inconvenient and is not the only one possible. As there is a unique vector **k**, the components $\alpha_{\mu\sigma}$ can be constructed from the k_i only as follows:

$$\alpha_{ij} = \alpha^t(\delta_{ij} - k_i k_j/\mathbf{k}^2) + \alpha^l k_i k_j/\mathbf{k}^2,$$
$$\alpha_{i4} = \alpha_1 k_i k_4/\mathbf{k}^2, \qquad \alpha_{4i} = \alpha_2 k_i k_4/\mathbf{k}^2, \qquad \alpha_{44} = \alpha_3. \tag{15}$$

Substituting (15) in (14) we get

$$\alpha^t = 1, \qquad \alpha^l - \alpha_1 - 1 = 0, \qquad \alpha_3 - \alpha_2 k_4^2/\mathbf{k}^2 = 1. \tag{16}$$

It is convenient to choose $\alpha^l = \alpha_2 = 0$, $\alpha^t = \alpha_3 = 1$, $\alpha_1 = -1$.

From similar considerations it follows that we can write for $D_{\nu\sigma}$ relations such as (15) which, however, we shall not write down. Substituting these in (12) together with (15) for our choice of α, and using (13), we get a simple set of equations for the different D. There are fewer such equations than unknowns. Using gauge invariance we can put several of the unknowns equal to zero. We have then

$$D_{j4} = D_{4j} = 0, \qquad D_{44} = D^l, \qquad D_{ij} = D^t(\delta_{ij} - k_ik_j/\mathbf{k}^2); \tag{17}$$

$$D^t(\omega, \mathbf{k}) = \frac{4\pi}{\mathbf{k}^2 - \omega^2\varepsilon^t(\omega, \mathbf{k})}, \qquad D^l(\omega, \mathbf{k}) = \frac{4\pi}{\mathbf{k}^2\varepsilon^l(\omega, k)}. \tag{18}$$

It is well known [16] that we have for the causal $D_c^{t,l}$ the spectral formula*

$$D_c^{t,l}(\omega, \mathbf{k}) = \frac{1}{\pi}\int_0^\infty A(\omega', \mathbf{k}) \times \left\{\frac{1}{\omega + \omega' - i\delta} - \frac{1}{\omega - \omega' + i\delta}\right\} d\omega', \quad \delta \to +0, \tag{19}$$

while for the retarded $D_{ret}^{t,l}$ we have†

$$D_{ret}^{t,l}(\omega, \mathbf{k}) = \frac{1}{\pi}\int_0^\infty A(\omega', \mathbf{k}) \times \left\{\frac{1}{\omega + \omega' + i\delta} - \frac{1}{\omega - \omega' + i\delta}\right\} d\omega', \delta \to +0. \tag{20}$$

If we evaluate the imaginary part of $D_{ret}^{t,l}$ from (20) we verify readily that $A(\omega, \mathbf{k}) = \operatorname{Im} D_{ret}^{t,l}(\omega, \mathbf{k})$ when $\omega > 0$. Equation (19) expresses thus the causal Green's function in terms of the imaginary part of the retarded one.

3. The causal mass operator has in first order in the square of the charge the form

$$\delta\hat{M}_c(E, \mathbf{p}) = -\frac{ie^2}{(2\pi)^4}\int \gamma_\mu G_c(E - \omega, \mathbf{p} - \mathbf{k})\gamma_\nu D_{c,\mu\nu}(\omega, \mathbf{k})\, d\omega\, d\mathbf{k}, \tag{21}$$

where G_c is the free electron Green's function, which can conveniently be written in the form [15]

$$G_c(E, \mathbf{p}) = \frac{1}{\varepsilon_\mathbf{p} - E - i\delta}\frac{m - i\hat{p}^+}{2\varepsilon_\mathbf{p}} + \frac{1}{\varepsilon_\mathbf{p} + E - i\delta}\frac{m - i\hat{p}^-}{2\varepsilon_\mathbf{p}}, \tag{22}$$

where $p_\mu^\pm = \{\mathbf{p}, \pm i\epsilon_\mathbf{p}\}$.

Substituting (19) and (22) in (21) we can easily integrate over ω, taking it into account that only products of factors containing poles in different planes of the complex variable ω give a contribution to the integral. As a result we get

$$\begin{aligned}\delta\hat{M}(E, \mathbf{p}) = -\frac{e^2}{8\pi^4}\int_0^\infty d\omega \int d\mathbf{k}\, \{&(E - \varepsilon_{\mathbf{p}-\mathbf{k}} - \omega + i\delta)^{-1} \times [\operatorname{Im} D_{ret}^t(\omega, \mathbf{k})\Lambda_{+,\mathbf{p}-\mathbf{k}}^t + \operatorname{Im} D_{ret}^l(\omega, \mathbf{k})\Lambda_{+,\mathbf{p}-\mathbf{k}}^l] \\ &- (E + \varepsilon_{\mathbf{p}-\mathbf{k}} + \omega + i\delta)^{-1}[\operatorname{Im} D_{ret}^t(\omega, \mathbf{k})\Lambda_{-,\mathbf{p}-\mathbf{k}}^t + \operatorname{Im} D_{ret}^l(\omega, \mathbf{k})\Lambda_{-,\mathbf{p}-\mathbf{k}}^l]\};\end{aligned} \tag{23}$$

$$\Lambda_{\pm,\mathbf{p}}^t = \gamma_i\frac{m - i\hat{p}^\pm}{2\varepsilon_\mathbf{p}}\gamma_j\left(\delta_{ij} - \frac{k_ik_j}{\mathbf{k}^2}\right), \qquad \Lambda_{\pm,\mathbf{p}}^l = \gamma_4\frac{m - i\hat{p}^\pm}{2\varepsilon_\mathbf{p}}\gamma_4. \tag{24}$$

We easily obtain the classical formula for losses in media with spatial dispersion [12] if we neglect the effect connected with the macroscopic mass renormalization* i.e., if we put $E' = \epsilon_\mathbf{p}$ and neglect quantum corrections (recoil), $|\mathbf{k}| \ll |\mathbf{p}|$, putting $\Lambda_{\mathbf{p}-\mathbf{k}} \approx \Lambda_\mathbf{p}$ and $\epsilon_{\mathbf{p}-\mathbf{k}} - \epsilon_\mathbf{p} \approx -\mathbf{k}\partial\epsilon_\mathbf{p}/\partial\mathbf{p} = -\mathbf{k}\cdot\mathbf{v}$ where $v = \partial\epsilon_\mathbf{p}/\partial\mathbf{p}$ is the particle velocity. Assuming that $D^{t,l}(\omega, \mathbf{k})$ depends on $|\mathbf{k}|$ only in the anti-Hermitian part of $\delta\hat{M}$, we can integrate over $k_\parallel = \mathbf{k}\cdot\mathbf{v}/v \equiv \omega'/v$ using a δ-function to replace the energy denominator in (23). Performing also the averaging in the dispersion relation we get

$$\gamma = -2\langle\delta\hat{M}''(\varepsilon_\mathbf{p}, \mathbf{p})\rangle = -\frac{e^2}{2\pi^2 v}\int_0^\infty d\omega\int_0^\infty \varkappa\, d\varkappa \times [\operatorname{Im} D_{ret}^t(\omega, \sqrt{\omega^2/v^2 + \varkappa^2})\langle\Lambda_{+,\mathbf{p}}^t\rangle + \operatorname{Im} D_{ret}^l(\omega, \sqrt{\omega^2/v^2 + \varkappa^2})\langle\Lambda_{+,\mathbf{p}}^l\rangle], \tag{25}$$

where $\langle\hat{A}\rangle = \langle\bar{u}\hat{A}u\rangle$ and u is the Dirac spinor of a free particle. Averaging the operators (24) we get

$$\langle\Lambda_{+,\mathbf{p}}^t\rangle = -v^2 + \frac{(\mathbf{v}\mathbf{k})^2}{k^2} = -\frac{v^2\varkappa^2}{\varkappa^2 + \omega^2/v^2}, \qquad \langle\Lambda_{+,\mathbf{p}}^l\rangle = 1. \tag{26}$$

Substituting (18) and (26) into (25) we get the well-known classical result (see [12]).

Moreover, we find from (23) the required δp_ν and δm, which occur in the dispersion relations (8) and (9):

$$\delta m = -\frac{e^2}{(2\pi)^4}\int_0^\infty d\omega\int d\mathbf{k}\,\frac{m}{\varepsilon_{\mathbf{p}-\mathbf{k}}}[2\operatorname{Im} D_{ret}^t(\omega, \mathbf{k}) + \operatorname{Im} D_{ret}^l(\omega, \mathbf{k})]\,[(E' - \varepsilon_{\mathbf{p}-\mathbf{k}} - \omega + i\delta)^{-1} - (E' + \varepsilon_{\mathbf{p}-\mathbf{k}} + \omega + i\delta)^{-1}], \tag{27}$$

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*These dispersion relations are true when $\omega \gg T$, where T is the temperature. In the opposite case there occur the well-known factors coth $(\omega/2T)$ (see [17]).

†One sees easily from the definition of D_c and D_{ret} that their real parts are the same while the dispersion relations easily give the imaginary parts.

*Taking this effect into account means in practice that we consider loss terms proportional to e^4. It is therefore meaningful to consider this only if we consider at the same time other terms in e^4 which arise from higher approximations to the mass operator.

$$\delta E = -\frac{e^2}{(2\pi)^4}\int_0^\infty d\omega \int d\mathbf{k}\, [2\,\mathrm{Im}\, D^t_{ret}(\omega, \mathbf{k}) - \mathrm{Im}\, D^l_{ret}(\omega, \mathbf{k})]$$
$$\times [(E' - \varepsilon_{\mathbf{p}-\mathbf{k}} - \omega + i\delta)^{-1} + (E' + \varepsilon_{\mathbf{p}-\mathbf{k}} + \omega + i\delta)^{-1}], \quad (28)$$

$$\delta \mathbf{p} = -\frac{e^2}{(2\pi)^4}\int_0^\infty d\omega \int d\mathbf{k}\Big[\frac{2\mathbf{k}\,(\mathbf{p}-\mathbf{k},\, \mathbf{k})}{\mathbf{k}^2}\,\mathrm{Im}\, D^t_{ret}(\omega, \mathbf{k})$$
$$+ (\mathbf{p}-\mathbf{k})\,\mathrm{Im}\, D^l_{ret}(\omega, \mathbf{k})\Big][(E' - \varepsilon_{\mathbf{p}-\mathbf{k}} - \omega + i\delta)^{-1} - (E' + \varepsilon_{\mathbf{p}-\mathbf{k}} + \omega + i\delta)^{-1}]. \quad (29)$$

We can get simpler expressions for the imaginary parts of (27) to (29) if we integrate over $k_{\parallel} = \mathbf{k}\cdot\mathbf{p}/p$ using δ-functions ($E > 0$):

$$\delta m'' = \frac{e^2}{16\pi^2}\int_0^\infty d\omega \int_0^\infty d\varkappa^2 \frac{m}{F}\,[2\,\mathrm{Im}\, D^t_{ret}(\omega, \sqrt{k_\parallel^2 + \varkappa^2}) + \mathrm{Im}\, D^l_{ret}(\omega, \sqrt{k_\parallel^2 + \varkappa^2})], \quad (30)$$

$$\delta E'' = \frac{e^2}{16\pi^2}\int_0^\infty d\omega \int_0^\infty d\varkappa^2 \frac{(E'-\omega)}{F}\,[2\,\mathrm{Im}\, D^t_{ret}(\omega, \sqrt{k_\parallel^2 + \varkappa^2}) - \mathrm{Im}\, D^l_{ret}(\omega, \sqrt{k_\parallel^2 + \varkappa^2})], \quad (31)$$

$$\frac{\mathbf{p}\delta\mathbf{p}''}{p} = \frac{e^2}{16\pi^2}\int_0^\infty d\omega \int_0^\infty d\varkappa^2 \frac{1}{F}\Big[\frac{2k_\parallel\,(pk_\parallel - k_\parallel^2 - \varkappa^2)}{k_\parallel^2 + \varkappa^2} \times \mathrm{Im}\, D^t_{ret}(\omega, \sqrt{k_\parallel^2 + \varkappa^2}) + (p - k_\parallel)\,\mathrm{Im}\, D^l_{ret}(\omega, \sqrt{k_\parallel^2 + \varkappa^2})\Big]; \quad (32)$$

$$k_\parallel = p - F, \qquad p = |\mathbf{p}|,$$
$$F = \sqrt{p^2 + (E'-\omega)^2 - \varepsilon_{\mathbf{p}}^2 - \varkappa^2}. \quad (33)$$

We can obtain as a particular case, from Eqs. (30)—(33), the quantum-mechanical expressions for the Cerenkov losses in transparent media without taking spatial dispersion into account, if we neglect corrections connected with the macroscopic mass renormalization, i.e., if we assume $E' \approx \epsilon_{\mathbf{p}}$. Retaining in (30)—(33) only terms connected with transverse losses ($\sim \mathrm{Im}\, D^t_{ret}$) and substituting from (18)

$$\mathrm{Im}\, D^t_{ret}(\omega, \mathbf{k}) = \mathrm{Im}\frac{4\pi}{\varkappa^2 + k_\parallel^2 - \omega^2\varepsilon(\omega)} = 4\pi^2\delta(\varkappa^2 + k_\parallel^2 - \omega^2 n^2(\omega)), \quad (34)$$

$n^2(\omega) \equiv \epsilon(\omega)$, we can integrate (30)—(32) over κ^2 in an elementary manner, using the δ-function (34). Substituting the result in (9) we get finally

$$\gamma = e^2 v\int d\omega\left\{1 - \frac{1}{n^2v^2} - \frac{\omega}{vp}\left(1 - \frac{1}{n^2}\right) + \frac{\omega^2 n^2}{4p^2}\left(1 - \frac{1}{n^4}\right)\right\}, \quad (35)$$

which is the same as the well-known result obtained with phenomenological quantum electrodynamics. [3,13,14]

3. MACROSCOPIC MASS RENORMALIZATION

1. Let us consider the classical limit for the m.m.r. We limit ourselves solely to the case when the energy difference ΔE connected with the renormalization is small compared with the particle energy E'. We can then substitute $E' = \epsilon_{\mathbf{p}}$ in the real parts of δm, δE, and $\delta \mathbf{p}$. Moreover, in the classical limit we can replace $\epsilon_{\mathbf{p}-\mathbf{k}}$ by $\epsilon_{\mathbf{p}}$, except in the denominator in the first term within the square brackets in (27) and (28), where we must substitute $\epsilon_{\mathbf{p}-\mathbf{k}} - \epsilon_{\mathbf{p}} \approx -\mathbf{k}\cdot\mathbf{v}$; the second term in the square brackets in (27)—(29) gives quantum corrections of order $h/2\epsilon_{\mathbf{p}}$ when $\omega \ll 2\epsilon_{\mathbf{p}}$, and must also be discarded. Finally, since the integration with respect to $\mathbf{k}$ is over an infinite domain, we can symmetrize the integrand obtained with respect to $\mathbf{k}\cdot\mathbf{v}$ [$f(\mathbf{k}\cdot\mathbf{v}) \to f(\mathbf{k}\cdot\mathbf{v})/2 + f(-\mathbf{k}\cdot\mathbf{v})/2$]. One sees easily that there will then be in front of $D^{t,l}$ factors $(\mathbf{k}\cdot\mathbf{v} - \omega)^{-1} - (\mathbf{k}\cdot\mathbf{v} + \omega)^{-1}$ and the integration over ω will lead, by virtue of the dispersion equation (20), to $\mathrm{Re}\, D^{t,l}_{ret}(\mathbf{k}\cdot\mathbf{v}, \mathbf{k})$. Finally, substituting the expressions obtained into (8) we have

$$\Delta E = \frac{e^2}{16\pi^3}\int d\mathbf{k}\Big[\mathrm{Re}\, D^l_{ret}(\mathbf{kv}, k) - \left(v^2 - \frac{(\mathbf{vk})^2}{k^2}\right)\mathrm{Re}\, D^t_{ret}(\mathbf{kv}, k)\Big]. \quad (36)$$

The difference of the magnitudes of ΔE in the medium and in a vacuum has a physical meaning. Using the fact that the quantity $\mathrm{Re}\, D^{l,t}$ is an even function of ω, one sees easily that (25) and (36) are the real and imaginary parts of the complex quantity

$$\Delta U = \Delta E - \frac{i\gamma}{2} = \frac{e^2}{4\pi^2 v}\int_0^\infty d\omega \int_0^\infty \varkappa\, d\varkappa\Big[D^l_{ret}(\omega, \sqrt{\omega^2/v^2 + \varkappa^2}) - \frac{\varkappa^2 v^2}{\varkappa^2 + \omega^2/v^2}\,D^t_{ret}(\omega, \sqrt{\omega^2/v^2 + \varkappa^2})\Big]. \quad (37)$$

One can verify that ΔU is none other than

$$\Delta U = \frac{1}{2}\int(\rho\varphi - \mathbf{A}\,\mathbf{j})\,dV = \frac{1}{8\pi}\int(\mathbf{E}\mathbf{D}' - \mathbf{B}^2)\,dV, \quad (38)$$

where $D'_i = \epsilon_{ij}E_j$ (see [18]). The easiest way to verify this is by substituting directly the field of a uniformly moving charge into (38). After subtracting the vacuum value, ΔU has thus the obvious meaning of the difference in the self-field in the medium and in a vacuum.*

*Lindhard's statement[2] that the quantity which in our notation is of the form

$$\frac{1}{8\pi}\int\left\{\mathbf{E}\varepsilon^l\mathbf{E} + \mathbf{B}\left[1 - \frac{\omega^2}{k^2}(\varepsilon^t - \varepsilon^l)\right]\mathbf{B}\right\}dV$$

plays the role of the classical analog of the m.m.r. is erroneous.

2. We consider ΔE in more detail in the case where the medium is transparent and one can neglect the spatial dispersion: $\epsilon^t = \epsilon^l = \epsilon = n^2(\omega)$. Integrating in (36) over κ we get then for the transverse ΔE^t and the longitudinal ΔE^l of the renormalizations:

$$\Delta E^t = -\frac{e^2 v}{4\pi}\int_{-\infty}^{\infty} d\omega \left\{\left(\frac{1}{n^2 v^2} - 1\right)\ln|1 - n^2 v^2| - \left(\frac{1}{v^2} - 1\right)\ln(1 - v^2)\right\}, \quad (39)$$

$$\Delta E^l = \frac{e^2}{4\pi v}\int_{-\infty}^{\infty} d\omega \left(\frac{1}{n^2} - 1\right)\ln\frac{\kappa_{max}^2 v^2}{\omega^2}. \quad (40)$$

The expression for the longitudinal renormalization is practically independent of the cutoff parameter κ_{max}^2, which was introduced in the usual way, because the integration contour can be deformed into the upper half-plane of ω and because there are no singularities of $n^2(\omega)$ in the upper half-plane of the complex ω

$$\int_{-\infty}^{\infty} d\omega \left(\frac{1}{n^2} - 1\right) = 0.$$

In (40) there can thus occur instead of κ_{max}^2 any number having the same dimensionality.

Bearing in mind that we wish to obtain a qualitative result, we evaluate ΔE^t and ΔE^l for a system of oscillators (as was done in [10] for Cerenkov losses):*

$$\Delta E^t = \frac{e^2\sqrt{\omega_0^2 + \omega_s^2}}{4v}\left[(1 - v_0^2)\left(\frac{\pi}{2} - \arcsin\frac{1 + v_0^2 - 2v^2}{1 - v_0^2}\right) - 2\sqrt{(1 - v^2)(v^2 - v_0^2)}\right] \text{ when } v^2 > v_0^2,$$

$$\Delta E^t = 0 \qquad \text{when } v^2 < v_0^2; \quad (41)$$

$$\Delta E^l = -\frac{\pi}{4}\frac{e^2}{v}\frac{\omega_0^2}{\sqrt{\omega_s^2 + \omega_0^2}};$$

$$v_0^2 = \omega_s^2(\omega_0^2 + \omega_s^2)^{-1}, \quad \varepsilon(\omega) = 1 - \omega_0^2(\omega^2 - \omega_s^2)^{-1}. \quad (42)$$

In the ultrarelativistic limit $\gamma \gg 1$, there appears in the transverse renormalization a term cancelling the longitudinal renormalization and the result obtained is independent of ω_s, if $\gamma^2 \gg 1/(1 - v_0^2)$:

$$\Delta E_{\gamma \gg 1} \approx -e^2\omega_0/\gamma, \quad \Delta m_{\gamma \gg 1} = -e^2\omega_0, \quad \gamma = 1/\sqrt{1 - v^2}. \quad (43)$$

This means that the plasma approximation for an arbitrary medium is a good one when $\gamma \gg 1$. In the non-relativistic limit $v \ll v_0$ and $v \ll 1$, the renormalization reduces to the longitudinal one. One must bear in mind that in the non-relativistic limit, $v \ll 1$, the work ΔW connected with the m.m.r. ($\sim e^2\omega_s/v$) is appreciably greater than the energy losses in the transition radiation ($\sim e^2\omega_s v^2$). It is easy to estimate that bremsstrahlung due to acceleration or deceleration in m.m.r. exceeds the transition radiation when $v^6 < e^4(\omega_s^2/m^2)$ ($v < 10^{-3}$).

3. To study the role of spatial dispersion in the m.m.r. we rewrite (36) slightly

$$\Delta E^l = \frac{e^2}{2\pi}\,\mathrm{Re}\int_0^{\infty} dk \int_{-1}^{1} dx \left[\frac{1}{\varepsilon^l(kvx, k)} - 1\right],$$

$$\Delta E^t = -\frac{e^2 v^2}{2\pi}\,\mathrm{Re}\int_0^{\infty} dk \int_{-1}^{1} dx\,(1 - x^2) \times \left[\frac{1}{1 - v^2x^2\varepsilon^t(kvx, k)} - \frac{1}{1 - v^2x^2}\right]. \quad (44)$$

For a free electron gas (either degenerate or a Boltzmann gas) we have (see [2,18])

$$\varepsilon^l(kvx, k) = 1 + f^2(x)/k^2d^2,$$

$$\varepsilon^t(kvx, k) = 1 + \omega_0^2\varphi(x)/v^2k^2, \quad (45)$$

where d is the Debye radius and $f^2(x)$ and $\varphi(x)$ have the following form for a degenerate gas:

$$f^2(x) = 1 + \frac{v}{2v_0x}\ln\left|\frac{vx/v_0 - 1}{vx/v_0 + 1}\right| \approx 1 \text{ when } v \ll v_0,$$

$$\varphi(x) = 3\pi i v/4v_0x \qquad \text{when } v \ll v_0; \quad (46)$$

v_0 is the electron velocity at the Fermi surface. By virtue of (44), the integration of (43) over k is elementary:

$$\Delta E^l \approx -\frac{e^2}{2d}, \qquad \Delta E^t \approx \frac{1}{7}\sqrt{\frac{\pi}{3}}\,v^2\sqrt{\frac{v}{v_0}}\,\omega_0, \quad (47)$$

where $v \ll v_0$ and $v^2 \ll 1$; ω_0 is the Langmuir frequency. In other words, when $v \ll v_0$ it is important to take spatial dispersion into account and this leads to an appreciably smaller value of ΔE than (42). In the opposite case, $v \gg v_0$ spatial dispersion has very little effect on the result:

$$\Delta E^l \approx -\frac{\pi e^2}{4v}\omega_0\left(1 + v\left(\frac{v_0^2}{v^2}\right)\right).$$

Similar relations are valid for the Boltzmann gas. The thermal velocity of the electrons plays in that case the role of v_0.*

4. We consider now the quantum corrections to the classical expression obtained here for the m.m.r. We are interested in the case of ultrarelativistic energies, when the plasma approximation is applicable. If we expand the integrands for

*arc sin = $\sin^{-1}$.

*One must stipulate that the equations obtained are true for a plasma only if, roughly speaking, $T \ll kvx$, i.e., the region of small k and x must be excluded from considerations. We shall consider this problem in more detail in a separate paper.

ΔE^t in powers of ω/ϵ_p, taking only the terms of first order in $\hbar$ into account in the expansion, the resulting integral diverges logarithmically at the upper limit, due to the inadmissibility of the expansion when $\omega \sim \epsilon_p$ if one neglects spatial dispersion, or to the inadmissibility of the neglect of the spatial dispersion when $k/\omega_0 > k_{max}/\omega_0 \sim 1/v_0 \gg 1$. Since logarithms of large numbers occur in both cases, the numerical value of the coefficient under the logarithm sign is unimportant. We give the result of the calculation:

$$\Delta E^t \approx \Delta E^t_{cl} - \frac{e^2\omega_0^2}{2\pi\varepsilon_p}\left\{\frac{\pi^2}{12} + \ln^2 2 + \frac{7}{3}\ln 2 + \ln^2\frac{\varepsilon_p}{m} - \left(\ln\frac{\varepsilon_p}{m}\right)\left(1 + 2\ln\frac{k_{max}}{\omega_0}\right) + \frac{10}{3}\ln\frac{k_{max}}{\omega_0} + 2\ln\frac{k_{max}}{\omega_0}\ln\frac{2k_{max}}{\omega_0}\right\}. \quad (48)$$

It follows from (48) that if we assume that the logarithmic terms, which depend only weakly on the energy, are approximately constant, the ratio of the quantum corrections to the classical result remains, in contradistinction to the case of Cerenkov losses, for instance, approximately constant. This is understandable, for at ultrarelativistic velocities the main role in the spectral composition of the transverse field of a particle in the medium is played by the frequencies $\omega_0\epsilon_p/m$ (compare this with transition radiation). The ratio ω/ϵ_p is thus roughly speaking independent of the energy.

The quantum corrections to the longitudinal renormalization depend essentially on the spatial dispersion if, as usual, $md \gg 1$, where d is the Debye radius. It follows from (27)—(29) that only values k smaller than d contribute to the longitudinal renormalization. We can therefore make the following estimate. We neglect spatial dispersion and put $\operatorname{Im} D^l = -2\pi^2\omega_0\delta(\omega-\omega_0)$, restricting the integration to the limit $k_{max} \approx 1/d$. Retaining only the terms linear in $\hbar$ we get

$$\Delta E^l \approx \Delta E^l_{cl} - \frac{e^2}{d}\frac{\omega_0}{\pi\varepsilon_p}. \quad (49)$$

One should emphasize that the spatial dispersion is unimportant in the classical limit (when $v \gg v_0$).

4. QUANTUM CORRECTIONS TO THE CHARACTERISTIC LOSSES OF ELECTRONS IN THIN FILMS

Quantum corrections to longitudinal losses can, in contradistinction to quantum corrections to transverse losses, be very important near threshold. We consider longitudinal losses without neglecting quantum corrections. Silin[12] has considered this problem in great detail in the classical limit in connection with a consideration of the characteristic losses of electrons in thin films. One can use the general relations found in the foregoing to find the probability for the scattering of an electron accompanied by the creation of a plasmon (longitudinal quantum) at arbitrary angles, and not merely at small angles as was done in [12]. Using (9) and substituting the longitudinal part of (30) to (33) we get

$$\gamma_l = -\frac{e^2}{8\pi^2}\int_0^\infty d\omega\int_0^\infty\frac{d\varkappa^2}{F}\left\{(E'-\omega) + \frac{p(p+k_\parallel)}{\varepsilon_p} + \frac{m^2}{\varepsilon_p}\right\} \times \operatorname{Im} D^l_{ret}\left(\omega, \sqrt{k_\parallel^2 + \varkappa^2}\right), \quad (50)$$

with $F^2 > 0$. We can connect the scattering angle θ with κ^2 and ω:

$$\cos\theta \equiv \frac{(\mathbf{pp}')^2}{p^2p'^2} = \frac{(p-k_\parallel)^2}{(p-k_\parallel)^2+\varkappa^2} = \frac{F^2}{F^2+\varkappa^2}, \quad \sin^2\theta = \frac{\varkappa^2}{F^2+\varkappa^2}. \quad (51)$$

Neglecting ΔE, since we need to consider higher approximations in e^2 if we take it into account, we get the following expression for the differential scattering probability:

$$\gamma_l = \int_0^\infty d\omega\int\gamma_l(\omega, \theta)\, 2\pi\sin\theta\, d\theta; \quad (52)$$

$$\gamma_l(\omega, \theta) = -\frac{e^2}{2\pi^2}\frac{1}{\sqrt{p^2+\omega^2-2\varepsilon_p\omega}}\left\{\varepsilon_p - \omega + \frac{m^2}{\varepsilon_p} + \frac{p}{\varepsilon_p}\sqrt{p^2+\omega^2-2\varepsilon_p\omega}\cos\theta\right\}\frac{1}{k^2}\operatorname{Im}\varepsilon_l^{-1}(\omega, k), \quad (53)$$

where

$$k^2 = 1 - \frac{2p\cos\theta}{\sqrt{p^2+\omega^2-2\varepsilon_p\omega}} + \frac{p^2}{p^2+\omega^2-2\varepsilon_p\omega}. \quad (54)$$

Here $\operatorname{Im}\epsilon_l^{-1} < 0$. In the transparent region $\operatorname{Im}\epsilon_l^{-1} = -\pi\delta(\epsilon_l)$.

At small scattering angles the angular dependence of the discrete loss lines is mainly determined by the angular dependence of the quantity k^2, since the numerator of (53) depends weakly on a small angle. Here k^2 is of the form

$$k^2 = \frac{p}{\sqrt{p^2+\omega^2-2\varepsilon_p\omega}}(\theta^2 + \theta_0^2); \quad (55)$$

$$\theta_0^2 = \frac{\sqrt{p^2+\omega^2-2\varepsilon_p\omega}}{p}\left(1 - \frac{p}{\sqrt{p^2+\omega^2-2\varepsilon_p\omega}}\right)^2. \quad (56)$$

For small ω/vp

$$\theta_0^2 = \theta_{0,cl}^2 + \theta_{0,qu}^2; \quad \theta_{0,cl}^2 = \omega^2/v^2p^2;$$

$$\theta_{0,qu}^2 = \theta_{0,cl}^2\,\omega\,(2 - v^2)/vp. \quad (57)$$

When ω approaches $pv/2$ the angle θ_0 increases. When $\omega > p^2/2m$ the scattering probability (53) tends to zero.

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