SOVIET PHYSICS JETP VOLUME 15, NUMBER 2 AUGUST, 1962

ON THE THEORY OF TRANSITION RADIATION

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Submitted to JETP editor July 2, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **42**, 471-478 (February, 1962)

The problem of the transition radiation of a charged particle in a plasma is solved in the kinetic approximation, with spatial dispersion taken into account. Mirror reflection of the electrons from the plasma-vacuum boundary is assumed. The limiting cases of weak and strong spatial dispersion are treated for both nonrelativistic and relativistic plasmas. With weak dispersion the radiative energy loss of a particle is composed of the transition radiation and Cerenkov radiation from longitudinal waves which emerges into the vacuum. The latter radiation has a comparatively narrow spectrum. With strong spatial dispersion the expressions for the radiation field and the energy loss of the particle can be obtained in the surface-impedance approximation. The radiative energy loss of a particle moving along the axis of a gyrotropic substance without spatial dispersion is calculated. The problem of the features of the transition and Cerenkov radiations of an electron in a transparent medium are discussed.

1. In 1946 it was shown by Ginzburg and Frank [1] that the passage of a charged particle through the boundary separating two media is accompanied by a specific radiation, which has been named transition radiation. Transition radiation is analogous to the radiation from collisions, since at the instant of passage from vacuum into the medium there is an "annihilation" of the electron and its image.

The theory of transition radiation in a medium without spatial dispersion has been developed in a series of papers by Garibyan, [2-5] Pafomov, [6] and others, in which it is shown that if the speed v of the particle is sufficiently large there is Cerenkov radiation of electromagnetic waves in addition to the transition radiation. In a medium with spatial dispersion longitudinal waves (plasmons) can be excited and transformed under certain conditions into an electromagnetic wave at the bounding surface. The effect of spatial dispersion on the radiative energy loss of a particle passing through a bounding surface has been considered by Zhelnov [7] and by one of the present writers. [8]*

*There is a mistake in Eq. (8) of [8]. We take this opportunity to present the correct expression for A:

$$A=\frac{(\varepsilon_1-\varepsilon_2)(1-\varepsilon_2\beta^2+\beta\sqrt{\varepsilon_1-\sin^2\theta})}{(1-\beta^2\varepsilon_2\cos^2\theta)(1+\beta\sqrt{\varepsilon_1-\varepsilon_2\sin^2\theta})}$$

$$-\frac{v_{02}(1-\varepsilon_2)}{v\sqrt{\varepsilon_2}(1+\sqrt{\varepsilon_2\rho_{02}/\rho_{01}})}\left\{\left[1-\frac{v}{v_{02}}\sqrt{\varepsilon_2\left(1-\frac{v_{02}^2}{c^2}\sin^2\theta\right)}\right]^{-1}\right.$$

$$\left.-\frac{1-\beta^2\varepsilon_2}{1-\beta^2\varepsilon_2\cos^2\theta}\right\},\quad \varepsilon_2=1-\omega_0^2/\omega(\omega-i\nu),$$

where ν is the frequency of collisions of the plasma electrons.

In the same paper, [8] the energy losses of the particle to transition radiation are calculated in the hydrodynamical approximation. This approximation corresponds to weak spatial dispersion. In the limiting case of weak spatial dispersion the kinetic theory which will be developed here leads to a natural result, namely: the difference between the kinetic and hydrodynamical theories amounts to a factor 3 in the expression for the parameter α that characterizes the spatial dispersion: $\alpha_k = 3\alpha_h = 3(T/mc^2)(\omega_0/\omega)^2$ [see Eq. (16)].

Zhelnov [7] took into account the presence of longitudinal waves in a phenomenological way by expanding the dielectric constant of the medium in a power series in the wave vector; this obviously also corresponds to the case of weak spatial dispersion. Longitudinal waves appear here, too, and the order of the Maxwell equations is raised, and consequently an additional boundary condition is necessary. It is clear that there will be emergence into the vacuum of the Cerenkov radiation produced from the longitudinal waves in the medium only if the longitudinal and transverse waves get "intermingled" in the boundary conditions. It is by this fact that the choice of the boundary conditions in Zhelnov's paper was dictated.

Although such phenomenological boundary conditions do make it possible to obtain the qualitative features of the transition radiation, these conditions are not entirely consistent.

The physical model to which the additional condition $\Delta E_n = 0$ on the bounding surface corresponds is unclear. Furthermore, in writing the fundamental relation $\mathbf{D} = \epsilon\mathbf{E} + \delta\Delta\mathbf{E}$ the term δ_1 grad div $\mathbf{E}$, which is of the same order as $\delta\Delta\mathbf{E}$, has been omitted. Finally, one cannot determine the limits of applicability of this condition within the framework of the phenomenological theory.

In the present paper the problem of the transition radiation at a plasma-vacuum boundary is treated in the kinetic approximation. Then the question of the additional condition for the electromagnetic field does not even arise, since the condition results in a natural way from the boundary condition for the distribution function of the plasma electrons at the boundary with vacuum. This allows us to consider not only the limiting case of weak spatial dispersion, but also that of strong spatial dispersion. Assuming specular reflection of the electrons from the surface of the plasma, one can solve the problem exactly even when there is a uniform external magnetic field perpendicular to the bounding surface. In the case of weak spatial dispersion the final formulas differ considerably from those obtained by Zhelnov.[7] The differences are due both to the different form of the additional boundary condition (which for us is of the form $j_n = 0$ on the boundary) and to the fact that the term in grad div $\mathbf{E}$ is omitted in [7].

In the case of strong spatial dispersion it is shown that the transition radiation is determined only by the surface impedance of the plasma (impedance approximation).

In the last sections of the paper we consider the radiative energy loss of a particle in gyrotropic and transparent media without taking spatial dispersion into account. In the investigation of the transition radiation emitted forward in the case of a transparent medium bounded by vacuum difficulties arise owing to the diverging of certain integrals over the frequency of the Cerenkov waves. Mathematically these difficulties are due to the fact that in the integrals from which the radiation field is determined the pole and the saddle point can lie close together, and the asymptotic form of the field depends strongly on the distance between them.

In this connection there are contradictory views in the literature concerning the generation of cylindrical Cerenkov waves.[2,6] By using the method of Van der Waerden[9] we have succeeded in obtaining an asymptotic form for the field which is useful for any distance between the pole and the saddle point. This also makes it possible to settle the question of the features of the transition and Cerenkov radiations in transparent media.

2. Let the electron move along the normal to the surface $z = 0$ of a plasma which occupies the half-space $z > 0$. The complete system of equations consists of Maxwell's equations and the linearized kinetic equation for the electrons in the plasma (the motion of the ions is neglected):

$$\operatorname{rot}\mathbf{H} = \frac{i\omega}{c}\mathbf{E} + \frac{4\pi}{c}[\mathbf{j} + e\mathbf{v}_0\delta(\mathbf{r} - \mathbf{v}_0 t)], \qquad \mathbf{j} = -e\int \mathbf{v} f d^3\mathbf{v},$$
$$\operatorname{rot}\mathbf{E} = -\frac{i\omega}{c}\mathbf{H}, \qquad (i\omega + \nu) f + \mathbf{v}\nabla f = e\mathbf{E}\mathbf{v}\frac{\partial f_0}{\partial\varepsilon}. \quad (1)^*$$

Here $\mathbf{E}$ and $\mathbf{H}$ ($\sim e^{i\omega t}$) are the electric and magnetic fields; f is the term added to the equilibrium distribution function f_0; ϵ, $\mathbf{v}$, and ν are the energy, speed, and effective collision frequency of the plasma electrons; $\mathbf{v}_0 = (0, 0, -v_0)$ is the velocity of the particle. The boundary conditions for the fields are the usual ones: the tangential components of $\mathbf{E}$ and $\mathbf{H}$ are continuous. For the distribution function we use the condition of mirror reflection:

$$f(0, \mathbf{w}, -v_z) = f(0, \mathbf{w}, v_z). \qquad (2)$$

The system (1) is most simply solved by the Fourier method. Using the condition (2), one can show that with the even continuation of the tangential components of the electric field $[E_T(-z) = E_T(z)]$ and the odd continuation of the normal components $[E_z(-z) = E_z(z)]$ into the region $z < 0$ the Maxwell equations expressed in terms of the Fourier components become the algebraic equations

$$L_{ik}(\omega, \varkappa, k)\,\mathscr{E}_k(\omega, \varkappa, k) = N_i(\omega, \varkappa, k), \qquad (3)$$

where

$$L_{ik} = \begin{pmatrix} (\omega/c)^2\varepsilon_{xx} - \varkappa_y^2 - k^2 & (\omega/c)^2\varepsilon_{xy} + \varkappa_x\varkappa_y & (\omega/c)^2\varepsilon_{xz} + ik\varkappa_x \\ (\omega/c)^2\varepsilon_{yx} + \varkappa_x\varkappa_y & (\omega/c)^2\varepsilon_{yy} - \varkappa_x^2 - k^2 & (\omega/c)^2\varepsilon_{yz} + ik\varkappa_y \\ (\omega/c)^2\varepsilon_{zx} - ik\varkappa_x & (\omega/c)^2\varepsilon_{zy} - ik\varkappa_y & (\omega/c)^2\varepsilon_{zz} - \varkappa^2 \end{pmatrix},$$

$$N_\alpha = \overline{E'_\alpha(0, \varkappa, \omega) + i\varkappa_\alpha E_z(0, \varkappa, \omega)} \qquad (\alpha = x, y);$$
$$N_z = \frac{ev_0\omega}{4\pi c^2}[\delta_+(\omega + kv_0) - \delta_+(\omega - kv_0)],$$
$$\mathscr{E}_i(\varkappa, k) = \int d^3\mathbf{r}\, e^{i\varkappa\rho} \, {}^{\cos}_{\sin}\, kz E_i(\rho, z).$$

The cosine Fourier expansion is used for the tangential field components, and the sine expansion for the normal component; Greek letters denote the tangential components; repeated indices are summed from 1 to 3;

$$\delta_+(x) = \frac{1}{\pi}\int_0^\infty e^{i\alpha x}\, d\alpha = \delta(x) + \frac{i}{\pi}\mathrm{P}\frac{1}{x}.$$

The dielectric constant tensor is

$$\varepsilon_{ik}(\varkappa, k, \omega) = \delta_{ik} + (4\pi/i\omega)\,\sigma_{ik}(\varkappa, k, \omega);$$
$$\sigma_{\alpha\beta} = A\int \frac{w_\alpha w_\beta e^{-\varepsilon/T}\, d^3\mathbf{v}}{i(\omega - \varkappa\mathbf{w} - kv_z) + \nu}, \qquad \sigma_{zz} = A\int \frac{v_z^2 e^{-\varepsilon/T}\, d^3\mathbf{v}}{i(\omega - \varkappa\mathbf{w} - kv_z) + \nu},$$
$$\sigma_{\alpha z} = A\int \frac{w_\alpha v_z e^{-\varepsilon/T}\, d^3\mathbf{v}}{(\omega - \varkappa\mathbf{w} - kv_z) - i\nu}, \qquad \sigma_{\alpha\beta} = \sigma_{\beta\alpha}, \qquad \sigma_{\alpha z} = -\sigma_{z\alpha},$$
$$A = (ne^2/T)\,(m/2\pi T)^{3/2}, \qquad (4)$$

*rot = curl.

n and T are the equilibrium density and temperature (in energy units) of the plasma.

We shall calculate the radiation field in the vacuum. First we must find the general solution of the Maxwell equations in plasma and vacuum, and then by using the radiation condition at infinity and the continuity of the tangential components at the bounding surface we must find the connection between the undetermined constants in the solutions of the homogeneous equations.

The procedure for the calculations is described in detail in a paper by Garibyan.[2] Therefore we give directly the final result of the calculations for the spectral density of the radiation in the vacuum, when the motion of the electron is from the plasma to the bounding surface:*

$$\frac{dW}{d\omega d\Omega}=\frac{e^2\beta^2\sin^2\theta\cos^2\theta}{\pi^2 c(1-\beta^2\cos^2\theta)^2}|\eta(\varepsilon,\theta)|^2. \tag{5}$$

Here $\beta = v_0/c$; θ is the angle between the z axis and the direction of observation; and $d\Omega$ is an element of solid angle in the direction θ. This formula gives both the transition radiation and the Cerenkov radiation emerging into the vacuum from the longitudinal waves in the plasma, in the presence of a finite damping of the waves in the plasma.

3. In the case of weak spatial dispersion the quantity η is of the form

$$\eta=\frac{1+\beta\zeta+\beta B(1-\beta^2\cos^2\theta)}{\cos\theta+\zeta}, \tag{6}$$

where

$$\zeta=\frac{(\varepsilon_0-\sin^2\theta)^{1/2}+\alpha^{1/2}(\varepsilon_0-\alpha\sin^2\theta)^{1/2}}{\alpha(\varepsilon_0/\alpha-\sin^2\theta)^{1/2}[(\varepsilon_0-\sin^2\theta)^{1/2}+(\varepsilon_0/\alpha-\sin^2\theta)^{1/2}]},$$

$$1/B=\alpha[1-\beta(\varepsilon_0-\sin^2\theta)^{1/2}][1-\beta(\varepsilon_0\alpha^{-1}-\sin^2\theta)^{1/2}]$$

$$\times[(\varepsilon_0-\sin^2\theta)^{1/2}+(\varepsilon_0/\alpha-\sin^2\theta)^{1/2}]; \tag{7}$$

$$\varepsilon_0=1-\frac{\omega_0^2}{\omega(\omega-i\nu)},\quad \alpha=\frac{3T}{mc^2}\frac{\omega_0^2}{\omega^2},\quad \omega_0^2=\frac{4\pi e^2 n}{m}. \tag{8}$$

In the formulas (7) and (8) the radicals have the arithmetical value for positive radicand, and for negative radicand the imaginary part of the root is positive.

The formulas that have been obtained are valid if the phase velocity $v_{ph} = \omega/q$ of the waves in the plasma is large in comparison with the mean thermal velocity $s = (3T/m)^{1/2}$. For frequencies ω far from ω_0 this condition is practically always satisfied if $|\epsilon_0| \ll c^2/s^2$. In this case the spatial dispersion is unimportant; the parameter α can be set equal to zero, and the formula (6) goes over into the well known formula of Ginzburg and Frank.[1]

*For the opposite direction of motion one has only to change the sign of β in the final formulas.

Near resonance $\omega = \omega_0$ the formulas are valid, since

$$|\varepsilon_0|\approx 2|\omega-\omega_0|/\omega_0\ll 1.$$

The frequency region in which spatial dispersion is of importance is determined from the condition $|\epsilon_0/\alpha| \sim \sin^2\theta$ and is of the order of $(3T\omega_0/mc^2)\sin^2\theta$.

It is also not hard to get analogous expressions for the intensity of the radiation from a particle traveling through a relativistic plasma in which the dispersion law of the electrons is of the form $\epsilon = cp$. All of the formulas are of the same form, and now*

$$\varepsilon_0=1-\frac{4\pi ne^2}{\omega(\omega-i\nu)}\frac{c^2}{3T},\qquad \alpha=\frac{4\pi ne^2c^2}{5\omega^2 T}.$$

For a relativistic plasma the frequency region in which the formulas (5)—(8) can be applied is limited by the condition

$$|\varepsilon_0|\ll 1,\text{ i.e., } |\omega-\omega_0^*|\ll\omega_0^*=\left(\frac{4\pi ne^2c^2}{3T}\right)^{1/2}.$$

4. Let us now turn to the study of the features of the radiation in the range of frequencies in which the condition for strong spatial dispersion, $v_{ph} \ll s$, is satisfied. This inequality is identical with $|n| \gg c/s \gg 1$, where n is the effective index of refraction. Since the effective dielectric constant is large, it is obvious that in calculating the spectral density of the radiation we can describe the plasma by means of the surface impedance $\zeta = E/H$, where E, H are the tangential components of the alternating fields at the surface of the plasma. As is well known, to find the radiation in the impedance approximation we can confine ourselves to the solution of the external electrodynamic problem. In this approximation there are no longitudinal waves in the plasma, and there remains only the transition radiation. Its spectral density is given by the previous formula (5), where now the quantity η is of the form

$$\eta=\frac{1+\beta\zeta}{\cos\theta+\zeta}, \tag{9}$$

$$\zeta=\left(\frac{2}{27\pi}\right)^{1/2}\left(\frac{\omega^2\sqrt{mT}}{ne^2c}\right)^{1/3}(1+i\sqrt{3}). \tag{10}$$

In this approximation $|\zeta| \ll 1$, since Eq. (9) is valid for $|\epsilon_0|s^2/c^2 \sim \omega_0^2 s^2/\omega^2c^2 \gg 1$.

Consequently, in the entire range of angles for which $|\cos\theta| > |\zeta|$ the transition radiation is just the same as for an ideally conducting medium:

*In the calculation of the dielectric constant of the relativistic plasma we have not taken into account processes of production and annihilation of electron-positron pairs in the plasma.

$$dW/d\omega d\Omega = e^2\beta^2\sin^2\theta/\pi^2 c(1-\beta^2\cos^2\theta)^2. \qquad (11)$$

For a relativistic plasma the condition for strong spatial dispersion is realized for $\pi^2 ne^2c^2/\omega^2 T \gg 1$, and the expression for ζ is just Eq. (10), if in this formula we replace the mass m by $8T/\pi c^2$.

It must be noted that the formula (9) is valid in all cases in which the medium can be described by means of an isotropic surface impedance, and not only for small ζ. In particular, Eqs. (9)—(11) keep the same form in the case of strong spatial dispersion when there is a constant magnetic field perpendicular to the bounding surface, since ζ does not depend on the magnetic field.

The expressions (5) and (9) can also be used to estimate the intensity of the transition radiation in metals or in sufficiently dense plasma in a magnetic field parallel to the bounding surface.

With strong spatial dispersion the wavelength in the medium is small in comparison with the radius of curvature for the electron, and therefore we can neglect the bending of the particle path and assume that it has a rectilinear motion near the surface. To get the transition radiation in this case we have only to substitute the known expressions for the surface impedance of the medium [10,11] in the definition of η, Eq. (9).

In the case in which the medium has a sufficiently large permeability μ the conditions for strong spatial dispersion are satisfied because of the large value of μ, and the impendance ζ, proportional to $\mu^{1/2}$, may not be small in comparison with unity.

5. Let us consider the transition radiation in the case of a gyrotropic medium without spatial dispersion, when the particle moves along the axis of gyrotropy. In such a medium the tensor ϵ_{ik} is of the form

$$\varepsilon_{ik} = \begin{pmatrix} \varepsilon_1 & -ig & 0 \\ ig & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_0 \end{pmatrix}. \qquad (12)$$

An example of a gyrotropic medium is a cold plasma in a perpendicular magnetic field, for which

$$\varepsilon_1 = 1 - \frac{\omega_0^2(\omega - i\nu)}{\omega[(\omega-i\nu)^2 - \Omega^2]}, \quad \varepsilon_0 = 1 - \frac{\omega_0^2}{\omega(\omega - i\nu)},$$

$$g = \frac{\omega_0^2\Omega}{\omega[(\omega-i\nu)^2-\Omega^2]};$$

$\Omega = |e|H/mc$ is the electron cyclotron frequency. The particle moves along the magnetic field. In this case the spectral intensity of the radiation in the vacuum is given by the general formula (5), in which η is now of the form

$$\eta = [(\cos\theta + P)(1+Q\cos\theta) - R^2\cos\theta]^{-1}\Big\{(1+Q\cos\theta)(1+\beta P) - \beta R^2\cos\theta + \frac{1-\beta^2\cos^2\theta}{\varepsilon_0(\mu_2^2-\mu_1^2)}\Big[\frac{(\mu_1^2-s_1^2)(1+Q\cos\theta)+qR\cos\theta}{1-\beta\mu_1} - \frac{(\mu_2^2-s_1^2)(1+Q\cos\theta)+qR\cos\theta}{1-\beta\mu_2}\Big]\Big\}, \qquad (13)$$

where

$$P = \frac{s_0^2(s_1^2+\mu_1\mu_2)}{\varepsilon_0\mu_1\mu_2(\mu_1+\mu_2)}, \quad Q = \frac{\varepsilon_1 s_0^2 + \varepsilon_0\mu_1\mu_2}{\varepsilon_0\mu_1\mu_2(\mu_1+\mu_2)},$$

$$R = \frac{gs_0^2}{\varepsilon_0\mu_1\mu_2(\mu_1+\mu_2)}, \quad s_0^2 = \varepsilon_0 - \sin^2\theta, \quad s_1^2 = \varepsilon_1 - \sin^2\theta; \qquad (14)$$

and μ_1, μ_2 are the roots of the equation

$$\varepsilon_0\mu^4 - \mu^2(\varepsilon_1 s_0^2 + \varepsilon_0 s_1^2) + s_0^2(\varepsilon_1 s_1^2 - g^2) = 0,$$

$$\mu_{1,2}^2 = (2\varepsilon_0)^{-1}\{\varepsilon_1 s_0^2 + \varepsilon_0 s_1^2 \pm [(\varepsilon_1 s_0^2 - \varepsilon_0 s_1^2)^2 + 4\varepsilon_0 s_0^2 g^2]^{1/2}\}, \qquad (15)$$

with positive real parts and negative imaginary parts. For g = 0 the formula (13) goes over into the well known formula obtained in the paper of Pafomov [6] for the radiation loss of a particle at the boundary between a uniaxial crystal and vacuum. Equation (13) shows that under certain conditions besides the transition radiation in a gyrotropic medium there is also an energy loss of the particle owing to the generation of Cerenkov radiation from the extraordinary waves. The frequency of this radiation is determined by the condition $1 - \beta\mu_1 = 0$.

6. In conclusion let us examine the question of the radiation energy loss of a particle in an isotropic transparent medium without spatial dispersion.

The expression for the radial component of the electromagnetic field in the vacuum obtained in Garibyan's paper [2] [Eq. (23)] is of the form

$$\mathbf{E}(R, \theta, t) = \int_{-\infty}^{\infty} d\omega\, \mathbf{E}_\omega(R, \theta)\, e^{-i\omega t}. \qquad (16)$$

The Fourier component E_ω of the field is determined by the integral

$$E_\omega(R, \theta) = \int_0^\infty dx \exp\Big\{i\omega\frac{R}{c}f(x)\Big\} W(x), \qquad (17)$$

where

$$W(x) = \frac{e}{\pi v}\Big(\frac{\omega}{2\pi Rc\sin\theta}\Big)^{1/2} e^{-3\pi i/4}\frac{x^{3/2}(1-x^2)^{1/2}}{\varepsilon(1-x^2)^{1/2}+(\varepsilon-x^2)^{1/2}} \times \Big\{\frac{1+\beta(\varepsilon-x^2)^{1/2}}{x^2-(\varepsilon-\beta^{-2})} - \frac{\varepsilon+\beta(\varepsilon-x^2)^{1/2}}{x^2-1+\beta^{-2}}\Big\},$$

$$f(x) = x\sin\theta + \sqrt{1-x^2}\cos\theta. \qquad (18)$$

The formula (17) differs from the corresponding expression (23) in Garibyan's paper only by our having replaced κ (the radial component of the wave vector) by the quantity $x = c\kappa/\omega$.

The function $W(x)$ has a pole at the point $x_p = (\epsilon - \beta^{-2})^{1/2}$ (the other poles are of no importance). Since ϵ is complex ($\epsilon = \epsilon' + i\epsilon''$), the pole lies above the x axis ($\mathrm{Re}\, x_p > 0$). The saddle point is $x_s = \sin\theta$. We are interested in the asymptotic behavior of $E_\omega(R,\theta)$ for large values of $\omega R/c$ (for simplicity we take $\omega > 0$; all the calculations are analogous for $\omega < 0$). Since ϵ' depends on ω, for a certain value $\omega = \omega'$ the pole (or more exactly $\mathrm{Re}\, x_p$) can coincide with x_s; ω' is the frequency of the Cerenkov waves emerging into the vacuum.[2]

In order to obtain an asymptotic form which is valid for arbitrary distance between the pole and the saddle point we use the method of Van der Waerden.[9] For this purpose we add to and subtract from the function $W(x)$ a term $W_0/(x - x_p)$, where W_0 is the residue of $W(x)$ at the pole $x = x_p$. Then

$$E_\omega(R,\theta) = \int_0^\infty dx\left[W(x) - \frac{W_0}{x - x_p}\right]\exp\left\{i\frac{\omega R}{c}f(x)\right\} + W_0\int_0^\infty \frac{dx}{x - x_p}\exp\left\{i\frac{\omega R}{c}f(x)\right\}. \tag{19}$$

In the first integral in Eq. (19) the pole $x = x_p$ is absent, and therefore its asymptotic behavior is determined by the saddle point only $[f''(x_s) = -\cos^{-2}\theta]$

$$E_{1\omega}(R,\theta) \sim \left[W(x_s) - \frac{W_0}{x_s - x_p}\right]\exp\left\{i\frac{\omega R}{c} - \frac{\pi i}{4}\right\}\sqrt{\frac{2\pi c}{\omega R}}\,|\cos\theta|. \tag{20}$$

In the second integral we deform the path into the curve of steepest descent, which passes through $x = x_s$ at the angle 45° with the real axis of x.

For $0 \le (\epsilon' - \beta^{-2})^{1/2} \le \sin\theta$ and sufficiently small ϵ'' the pole x_p lies between $\mathrm{Im}\, x = 0$ and the curve of steepest descent. Therefore

$${}_{2\omega}(R,\theta) = 2\pi i W_0 \exp\left\{i\frac{\omega}{c}Rf(x_p)\right\} - W_0\int_{-\infty\exp(3\pi i/4)}^{\infty\exp(3\pi i/4)}\frac{dx}{x - x_p}\exp\left\{i\frac{\omega}{c}Rf(x)\right\}. \tag{21}$$

In the last integral it must be noted that the coefficient of the exponential cannot be expanded in powers of $x - x_s$, since the radius of convergence of this series is small. The exponent can be expanded near the saddle point. After obvious transformations we get the following final asymptotic formula:

$$E_\omega(R,\theta) \sim E_{1\omega}(R,\theta) + 2\pi i W_0 \exp\left(\frac{i\omega}{c}Rf(x_p)\right) + E_{3\omega}(R,\theta), \tag{22}$$

$$E_{3\omega}(R,\theta) = W_0\exp\left(i\frac{\omega R}{c}\right)\int_{-\infty}^{\infty}\frac{d\xi\, e^{-\xi^2}}{\xi - \Delta};$$

$$\Delta = (x_s - x_p)\, e^{-3\pi i/4}\sqrt{\omega R/2c}\,|\sec\theta|. \tag{23}$$

Equations (22) and (23) give the correct limiting results for small and large values of $|\Delta|$. When the saddle point and the pole are located sufficiently far apart ($|\Delta| \gg 1$), $E_{3\omega}$ is cancelled by the corresponding term in $E_{1\omega}$ and one gets the same result as in Garibyan's paper.[2] When $|\Delta| \to 0$ (the pole and the saddle point coincide), $f(x_p) \to 1$ so that

$$E_{\omega_0} = E_{1\omega}(R,\theta) + \pi i W_0 \exp(i\omega R/c).$$

Thus for $|\Delta| \ll 1$, $E_{3\omega}$ reaches its maximum and behaves like a cylindrical wave, and for $|\Delta| \gg 1$ it behaves like a spherical wave. Since for a transparent medium $\epsilon''(\omega R/c)^{1/2} \ll 1$, and the effective width of the frequency interval in which $E_{3\omega}$ reaches its maximum is proportional to $R^{-1/2}$, in the expression (24) for $E(R,t,\theta)$ the term $E_{3\omega}$ gives a contribution of the spherical-wave type, just as $E_{1\omega}$ does.

In the case of a nontransparent medium, for $\epsilon''(\omega R/c)^{1/2} \gg 1$ (i.e., $|\Delta| \gg 1$) we cannot let the damping go to zero, there is no cylindrical wave, and there remains only the spherical wave from the saddle point

$$E_\omega \sim W(x_s)\sqrt{\frac{2\pi c}{\omega R}}\cos\theta\exp i\left(\frac{\omega}{c}R - \frac{\pi}{4}\right). \tag{24}$$

In Garibyan's paper[2] two limiting cases were treated separately, namely the case when the saddle point and the pole are far apart ($|\Delta| \gg 1$) and that when they coincide ($\Delta = 0$). The general case, in which the saddle point and the pole are an arbitrary distance apart, was not treated. As can be seen from the results obtained here, although the frequency interval $\Delta\omega'/\omega' \sim (\lambda\!\!\!^{-}/R)^{1/2}$ is indeed small, its contribution to the total energy loss is by no means small. The remark of Pafomov[6] about the inaccuracy of the interpretation of Garibyan's result actually does not apply to this case, since he considers a nontransparent medium with sufficiently large damping, so that there is no pole between the real axis and the curve of steepest descent and there is no cylindrical wave.

It is obvious that the situation is quite analogous in a plasma, in which Cerenkov radiation from the longitudinal waves arises at the frequency $\omega = \omega_0[1 + (s/v)^2(1 + \beta^2\sin^2\theta)]^{1/2}$.

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Translated by W. H. Furry
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