PROPERTIES OF THE SOLUTION OF THE LOW EQUATION FOR A MODEL OF A LOCAL FIELD THEORY

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Submitted to JETP editor August 12, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **42**, 520-525 (February, 1962)

We obtain and study the solution of the Low equation for the scattering amplitude for scalar mesons and a fixed nucleon which can exist in two states with different masses. The solution is valid only with a definite restriction on the coupling constant g_r. A comparison is made between the Low amplitude and the solution found on the basis of the Hamiltonian formalism. It turns out that for energies $\omega < 2\mu$ the contributions to the scattering amplitude from many-particle states are unimportant. The problem of the connection of the resonance solution with a nonphysical pole of the amplitude is discussed.

INTRODUCTION

THE question of the ambiguity of the solution of the Low equation for the scattering amplitude has been examined in many papers.[1-3] By examples of exactly soluble models it has been shown that to a whole class of Hamiltonians which differ in the set of states in which the scatterer can exist there corresponds a single Low equation with a multiple-valued solution. Furthermore it has been shown[1] that for charged mesons with a fixed nucleon the solutions of the Low equation contain restrictions on the coupling constant g_r. Recently attention has been called to the fact that similar restrictions on the coupling constant can arise not only for special models, but also in the exact theory, from an analysis of the dispersion relations for πN scattering.[4] An interesting question in this connection is whether these restrictions are the result of the general principles used in the derivation of the dispersion relations and the Low equation or of the approximations which are made in the process. These approximations are:

1) replacement of the exact unitarity relation by an approximate one which takes into account only two-particle intermediate states (two-particle unitarity), and

2) restriction to a finite number of partial waves in the scattering amplitude.

In a paper by Khalfin[5] it is asserted that the second assumption, together with the assumption that the coupling constant is arbitrary, can lead to inconsistency of the dispersion relations and the unitarity relation. In the soluble models, where the scattering usually involves only one wave (the S wave or the P wave), restrictions on g_r arise from the exact solution of the Low equation. Precisely this case is considered in the present paper.

Furthermore, in this paper we use the solution of the Schrödinger equation for our model[6] to estimate the importance of the many-particle contributions to the scattering amplitude, and show that for energies less than the threshold for inelastic processes, $\omega < 2\mu$, the contribution of the higher states does not exceed 15 percent.

1. THE MODEL AND THE EQUATION FOR THE SCATTERING AMPLITUDE

Here we shall consider the Low equation for the scattering amplitude in a simple model of a field theory with a fixed nucleon.[6] The Hamiltonian of the system is

$$H = m_0(\psi^+\psi) + \frac{1}{2}\int d\mathbf{x}\,[\pi^2(\mathbf{x}) + (\nabla\varphi(\mathbf{x}))^2 + \mu^2\varphi^2(\mathbf{x})] + g(\psi^+\tau_1\psi)\int d\mathbf{x}\,\varphi(\mathbf{x})\,\delta(\mathbf{x}) + \Delta m_0(\psi^+\tau_3\psi). \quad (1)$$

The stationary nucleon can exist in two states (proton, neutron) which have different masses: $m_{0p} = m_0 + \Delta m_0$, $m_{0n} = m_0 - \Delta m_0$ (the zero index denotes the mass of the bare nucleon). Because of the existence of these degrees of freedom of the nucleon there can be processes of elastic and inelastic scattering of mesons by the nucleon. Starting from the Hamiltonian formalism, we have previously[6] obtained the expression for the elastic-scattering amplitude.

In what follows we shall use this expression for comparison with the amplitude obtained from the Low equation. In the derivation of this equation we shall start from the dispersion relation for the scattering amplitude (we note that in this model only D scattering is possible):

$$M_N(\omega)=\frac{\delta_N g_r^2}{(2\pi)^3 2\omega}\left[\frac{1}{\omega-\Delta}-\frac{1}{\omega+\Delta}\right]$$
$$+\frac{1}{\pi\omega}\int_{\mu}^{\infty}d\omega'\omega'\,\mathrm{Im}\,M_N(\omega')\left[\frac{1}{\omega'-\omega-i\varepsilon}+\frac{1}{\omega'+\omega}\right];\quad(2)$$

$M_N(\omega)$ is the amplitude for scattering of a meson of energy $\omega=(k^2+\mu^2)^{1/2}$ by the nucleon $(N=p,n)$; $\Delta=m_p-m_n$ is the difference of the observed masses of the "proton" and "neutron" and determines the position of the one-nucleon pole; and g_r is the observed (renormalized) interaction constant. The one-nucleon term can be of either sign, depending on whether the scattering is by a proton ($N=p$ and $\delta_p=1$) or by a neutron ($N=n$ and $\delta_n=-1$). We assume that $\Delta<\mu$, since in the opposite case the nucleon would have an unstable state $m_p>m_n+\mu$, which would decay into a "neutron" and a meson.

Using the unitarity relation*

$$\mathrm{Im}\,M_N(\omega)=(2\pi)^2k\omega\,|M_N(\omega)|^2+a_N(\omega),\quad(3)$$

where $a_N(\omega)$ is the contribution from inelastic processes, and substituting Eq. (3) in Eq. (2), we get the Low equations for the amplitude $M_N(\omega)$ with neglect of the many-particle contribution $a_N(\omega)$. It must be noted here that the two-particle unitarity relation so obtained, which is valid for $\omega<2\mu$, if considered in the entire range $\mu\le\omega\le\infty$, imposes rather strong restrictions on the amplitude $M_N(\omega)$: $M_N(\omega)$ must not decrease at infinity more slowly than $1/\omega^2$. From Eqs. (2) and (3) we get the Low equation

$$M_N(\omega)=\frac{\delta_N g_r^2}{(2\pi)^3 2\omega}\left[\frac{1}{\omega-\Delta}-\frac{1}{\omega+\Delta}\right]$$
$$+\frac{4\pi}{\omega}\int_{\mu}^{\infty}d\omega'k'\omega'^2\,|M_N(\omega')|^2\left[\frac{1}{\omega'-\omega-i\varepsilon}+\frac{1}{\omega+\omega'}\right].\quad(4)$$

This equation can be solved by the well known method of Castillejo, Dalitz, and Dyson.[1] We shall not make the calculations here, and only give the result:

$$M_N(\omega)=\frac{2\delta_N g_r^2}{(2\pi)^3}\frac{\Delta}{\omega(\omega^2-\Delta^2)}$$
$$\times\left\{1-\frac{g_r^2\delta_N\Delta}{4\pi\sqrt{\mu^2-\Delta^2}}\frac{\sqrt{\mu^2-\Delta^2}-\sqrt{\mu^2-\omega^2}}{\sqrt{\mu^2-\Delta^2}+\sqrt{\mu^2-\omega^2}}\right\}^{-1},\quad(5)$$

where $(\mu^2-\omega^2)^{1/2}$ is taken positive for $-\mu<\omega<\mu$.

We note only the following facts which are essential in the construction of the solution. First, in the solution (5) we have dropped from the denominator the function

$$S(\omega)=\sum_i R_i\left[\frac{1}{\omega_i-\omega}+\frac{1}{\omega_i+\omega}\right].$$

As Dyson has shown,[3] this function describes the contributions to the scattering amplitude that are due to unstable states of the scatterer. In our present model, however, the nucleon and meson cannot form a bound system.[6] Second, the undetermined constants that arise in the solution by the method of Castillejo et al[1] are in our case exactly determined, since one knows the position of the one-nucleon pole and the residue at this pole.

2. PROPERTIES OF THE SOLUTION OF THE INTEGRAL EQUATION (4) AND RESTRICTIONS ON THE COUPLING CONSTANT

It is obvious from Eq. (5) that $M_N(\omega)$ has poles at the points $\pm\Delta$ and cuts along $(-\infty,-\mu]$ and $[\mu,\infty)$, but in addition to this the function can also have a pole in the interval $[-\mu,\mu]$ when

$$-\frac{g_r^2\delta_N\Delta}{4\pi\sqrt{\mu^2-\Delta^2}}\frac{\sqrt{\mu^2-\Delta^2}-\sqrt{\mu^2-\omega^2}}{\sqrt{\mu^2-\Delta^2}+\sqrt{\mu^2-\omega^2}}=1.\quad(6)$$

An additional pole in the amplitude $M_N(\omega)$ would be in contradiction with the previously assumed analytical properties of $M_N(\omega)$, on the basis of which (4) was obtained. This pole can be excluded by imposing restrictions on the observed coupling constant g_r. For this purpose we rewrite Eq. (6) in a different form:

$$\sqrt{\mu^2-\omega^2}=-\sqrt{\mu^2-\Delta^2}$$
$$\times\left[1-\delta_N\frac{g_r^2\Delta}{4\pi\sqrt{\mu^2-\Delta^2}}\right]\Big/\left[1+\delta_N\frac{g_r^2\Delta}{4\pi\sqrt{\mu^2-\Delta^2}}\right].\quad(7)$$

If we now set*

$$g^2/4\pi<\sqrt{\mu^2-\Delta^2}/\Delta,\quad(8)$$

then the root of Eq. (7) will lie on the other sheet of the Riemann surface, since $(\mu^2-\omega^2)^{1/2}<0$ and

*The expression for the total elastic cross section in terms of $M_N(\omega)$ is

$$\sigma^N(\omega)=\pi^{-1}\omega^2\,|M_N(\omega)|^2.$$

If we introduce the amplitude $f_N(\omega)=(2\pi)^{-4}\omega M_N(\omega)$, the relation (3) takes the more usual form

$$\mathrm{Im}\,f_N(\omega)=\frac{k}{4\pi}\sigma_{el}^N(\omega)+\frac{k}{4\pi}\sigma_{in}^N(\omega).$$

*The inequality (8) is outwardly similar to a restriction on the coupling constant obtained by Grivob, Zel'dovich, and Perelomov[8] for a case in which there is a bound state.

we had defined the root $(\mu^2 - \omega^2)^{1/2}$ as having the positive sign when $\omega < \mu$. Thus the restriction (7) on the coupling constant arises from the solution of the Low equation (4) and the one-particle unitarity relation. The same sort of situation has been discussed by a number of authors.[4,7] Khalfin[5] has shown in the general case that such restrictions can arise when the amplitude contains a finite number of scattering phase shifts; in our model only the S phase shift is present.

The assumptions that the quantity g_r is arbitrary, that the number of scattering phase shifts is finite, and that the energy is unbounded are actually internally contradictory. To support this assertion we can present, in addition to Khalfin's arguments, the following considerations. By introducing a momentum cut off $v(k) = L^2/(k^2 + L^2)$ into our model we get in the way described above a scattering amplitude $M_N(\omega, L)$, which will now be a function of the cutoff momentum L. Instead of Eq. (6) we now have as the relation determining the position of the positive pole the equation

$$1 = \frac{\delta_N g_r^2 \Delta}{4\pi \sqrt{\mu^2 - \Delta^2}} \frac{\sqrt{\mu^2 - \Delta^2} - \sqrt{\mu^2 - \omega^2}}{\sqrt{\mu^2 - \Delta^2} + \sqrt{\mu^2 - \omega^2}} \times \frac{L^3 [(\sqrt{\mu^2 - \omega^2} + \sqrt{\mu^2 - \Delta^2} + L)^2 + L\sqrt{\mu^2 - \Delta^2}]}{\mu(\sqrt{\mu^2 - \omega^2} + L)^2 (\sqrt{\mu^2 - \Delta^2} + L)^3}. \quad (9)$$

It can be seen from this that a limitation on the energy (introduction of the momentum cutoff $k_{max} = L$) broadens the range of allowable values of g_r.

It is interesting to consider the question of the existence of a resonance in the solutions of the Low equation:

$$\operatorname{Re} M_N(\omega_{res}) = 0, \qquad \operatorname{Im} M_N(\omega_{res}) \neq 0.$$

It follows from an examination of the solution (5) that the condition $\operatorname{Re} M_N(\omega) = 0$ is satisfied for

$$k_{res}^2 = (\mu^2 - \Delta^2)\left[\frac{g_r^2 \delta_N \Delta}{4\pi\sqrt{\mu^2 - \Delta^2}} - 1\right] \Big/ \left[\frac{g_r^2 \delta_N \Delta}{4\pi\sqrt{\mu^2 - \Delta^2}} + 1\right].$$

From this it follows that if the quantity g_r^2 is restricted by the inequality (8) (the condition that there be no nonphysical pole in the amplitude) there is no resonance, since k_{res} is then an imaginary quantity. Thus in our case there is a resonance when the amplitude has a nonphysical pole. The same situation is realized in another exactly soluble model[1] (scalar charged mesons, fixed nucleon). There the solution is correct for $g_r^2/2\pi < 1$ (absence of a nonphysical pole), and the resonance energy is $\omega_{res} = g_r^2\mu/2\pi$; consequently again the resonance exists only when there is a nonphysical pole in the amplitude.

As is well known, when Chew and Low[9] analyzed their equation for πN scattering (pseudoscalar mesons, fixed nucleon) they found a resonance in the P wave. According to Khalfin's paper[5] there should be a restriction on g_r^2 in this model also, as in the two preceding ones. Chew and Low, however, do not indicate any such restriction, and consequently the question arises: is not the resonance in the Chew-Low equation, as in the examples given above, due to the existence of a nonphysical pole in the amplitude for the "resonance" values of g_r^2 and the cutoff momentum L?

It is hard to answer this question, however, without knowing the exact solution.

3. COMPARISON WITH THE SOLUTION OF THE SCHRÖDINGER EQUATION

Starting from the Schrödinger equation with the Hamiltonian (1) we have previously[6] obtained the meson-nucleon scattering amplitude in the form of a power series in the parameter $\Delta m = \Delta m_0 \times \exp\{-g^2 \sum_k \omega_k^{-3}\}$

$$\widetilde{M}_N(\omega) = \frac{g^2 \delta_N}{(2\pi)^3 \omega^2} \frac{2\Delta m}{\omega}\Big\{1 - i\delta_N \Delta m \int_0^\infty dx\,(1 - \cos \omega x) \times \Big[\exp\Big\{2g^2 \sum_k \omega_k^{-3} e^{-i\omega_k x}\Big\} - 1\Big] + \ldots\Big\}, \quad (10)$$

where g is the unrenormalized (bare) coupling constant. In the same paper it was shown that the observed constant g_r and the mass difference $\Delta = m_p - m_n$ of the "proton" and "neutron" can also be expressed in the form of series in Δm, in which each term is finite:

$$\frac{g_r}{g} = 1 - 2\Delta m^2 \int_0^\infty x\,dx \Big[\exp\Big\{2g^2 \sum_k e^{-\omega_k x}\omega_k^{-3}\Big\} - 1\Big] - \ldots \quad (11)$$

$$\Delta = 2\Delta m \Big\{1 + \Delta m^2 \int_0^\infty dx_1 \int_0^\infty dx_2\, x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2} \times \exp\Big\{2g^2 \sum_k \frac{1}{\omega_k^3}(e^{-\omega_k x_1} - e^{-\omega_k(x_1+x_2)} + e^{-\omega_k x_2})\Big\} + \ldots\Big\} \quad (12)$$

[for more details see [6], Eq. (11)].

To be able to compare the amplitudes M_N and $\widetilde{M}_N$ it is necessary to use Eqs. (11) and (12) to express the renormalized quantities g_r and Δ in the expression (5) in terms of the constants g and Δm; in doing this we shall assume $\Delta m \ll \mu$, and therefore shall everywhere confine ourselves to second-order terms in expansions in $\Delta m/\mu$. The resulting expression for the amplitude $M_N(\omega)$ is

$$M_N(\omega)=\frac{g^2\delta_N}{(2\pi)^3\omega^2}\frac{2\Delta m}{\omega}\left\{1+\frac{\delta_N g^2}{\pi}\frac{\Delta m}{\mu}\left(\frac{\mu^2-k^2}{2\omega^2}+\frac{ik\mu}{\omega^2}\right)\right\}. \quad (13)$$

To compare the expressions (10) and (13) we now have still to separate the real and imaginary parts in Eq. (10). For energies $\omega < 2\mu$ (2μ is the threshold for inelastic processes) we have

$$\widetilde{M}_N(\omega)=\frac{g^2\delta_N}{(2\pi)^3\omega^2}\frac{2\Delta m}{\omega}\left\{1-\delta_N\frac{\Delta m}{\mu}I_1(\omega,g)+\frac{g^2\delta_N}{\pi}\frac{\Delta m}{\mu}\left(\frac{\mu^2-k^2}{2\omega^2}+\frac{ik\mu}{\omega^2}\right)\right\}, \quad (14)$$

where*

$$I_1(\omega,g)=\mu\int_0^\infty dx\,(2-2\operatorname{ch}\omega x)\times\left[\exp\left\{2g^2\sum_k\frac{e^{-\omega_k x}}{\omega_k^3}\right\}-1-2g^2\sum_k\frac{e^{-\omega_k x}}{\omega_k^3}\right]. \quad (15)$$

Thus to second order in $\Delta m/\mu$ the exact amplitude $\widetilde{M}_N$ of Eq. (14) differs from the Low amplitude $M_N(\omega)$ of Eq. (13) by a term $-\delta_N\Delta m I_1/\mu$, which takes account of the contributions from the higher states to the real part of the amplitude. The imaginary parts of M_N and $\widetilde{M}_N$ are equal in this energy range. When $\mu \le \omega \le 2\mu$, we have

$$0\leqslant I_1(\omega,g)\leqslant 0.13,\qquad 1.5\geqslant\frac{g^2}{\pi^2}\frac{\mu^2-k^2}{2\omega^2}\geqslant-0.78$$

(here we have taken $g^2/\pi^2 = 1$, since $I_1(\omega, g)$ is a maximum for this value and, as has been shown earlier,[6] a value $g^2/\pi^2>1$ has no meaning in this model). It can be seen from this that the contribution of the many-particle states to the real part of the amplitude does not exceed 15 percent, which is quite satisfactory from the point of view of the influence of many-particle states on low-energy processes.

Let us now consider the region $2\mu \le \omega \le 3\mu$. In this case the exact amplitude is

$$\widetilde{M}_N(\omega)=\frac{\delta_N g^2}{(2\pi)^3\omega^2}\frac{2\Delta m}{\omega}\left\{1+\frac{g^2\delta_N}{\pi}\frac{\Delta m}{\mu}\left(\frac{\mu^2-k^2}{2\omega^2}+\frac{ik\mu}{\omega^2}\right)-\delta_N\frac{\Delta m}{\mu}I_2(\omega,g)+\frac{\delta_N g^4}{2\pi^2}\frac{\Delta m}{\mu}\int_{\omega-\mu}^{\omega^2/4\mu}\frac{dx}{x^2}\sqrt{\frac{(x+\mu)^2-\omega^2}{\omega^2-4\mu x}}\right\}, \quad (16)$$

where $I_2(\omega, g)$ is a real function analogous to the expression (15) but of more complicated form. It follows from a comparison of the one-meson amplitude (13) and the exact amplitude (16) that the real and imaginary parts of the amplitudes are both different in this range of energies. When $2\mu \le \omega \le 3\mu$ we have

$$0.13\leqslant I_2(\omega,g)\leqslant 0.92,\qquad -0.8\geqslant\frac{g^2}{\pi}\frac{\mu^2-k^2}{2\omega^2}\geqslant-0.98,$$

i.e., the real parts already differ by 100 percent, and for the imaginary parts we have

$$0\leqslant\frac{g^2}{2\pi^2}\int_{\omega-\mu}^{\omega^2/4\mu}\frac{dx}{x^2}\sqrt{\frac{(x+\mu)^2-\omega^2}{\omega^2-4\mu x}}\leqslant 0.15,\quad 1.3\geqslant\frac{g^2}{\pi}\frac{k\mu}{\omega^2}\geqslant 0.8,$$

i.e., there is a difference of 20 percent. Thus in the range $2\mu \le \omega \le 3\mu$ the contribution from the higher states is important, and the one-particle amplitude M_N differs from the exact amplitude $\widetilde{M}_N$ by a factor of two.

The example of this model gives good confirmation of the assumption on which the dispersion approach in present field theory is based—the assumption that at low energies (below the threshold of inelastic processes) the contributions to the scattering amplitude from higher states are unimportant. Of course one cannot get information from this example about the actual scattering of relativistic particles.

We now consider the question: does the relation (8) between the coupling constant and Δ, which comes from the solution of the integral equation (4), also hold for the g_r and Δ obtained on the basis of renormalization procedures [6] and given by the series (11) and (12)?

In our previous paper [6] we showed that a necessary condition for the convergence of the series (11) and (12) is

$$\Delta m/(1-g^2/\pi^2)<1. \quad (17)$$

If in the relation (8) we substitute for g_r and Δ the corresponding series, it is found that in the first orders in Δm the inequality (8) is satisfied under the condition (17) on the quantities g and Δm. Without knowing the exact sums of the series (11) and (12), however, we cannot talk about Eq. (8) holding rigorously for the renormalized quantities g_r and Δ obtained on the basis of the solution of the Hamiltonian (1).

*ch = cosh.

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Translated by W. H. Furry
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