ON A PHENOMENOLOGICAL DESCRIPTION OF THE PION-PION INTERACTION

L. D. SOLOV'EV and CH'EN TS'UNG-MO

Joint Institute for Nuclear Research

Submitted to JETP editor August 12, 1961, resubmitted January 13, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **42**, 526-537 (February, 1962)

A relation between various low-energy processes is established by assuming that resonances occur in the two-pion P state with isotopic spin 1 and the three-pion P state with isotopic spin 0. The nucleon form factors, elastic pion-nucleon scattering, S state pion-pion scattering, the pion form factors, photoproduction, and inelastic pion-nucleon scattering are discussed.

1. "BIPION" AND "TRIPION"

It is well known that the method of dispersion relations becomes most effective if the dispersion integrals contain resonance amplitudes. Neglecting the non-resonance amplitudes and knowing the resonance parameters, we can in this case easily express the amplitudes for many physical processes in terms of these parameters. In the present paper we shall assume that resonances occur in the two-pion P state with isotopic spin 1 and the three-pion P state with isotopic spin 0.

There is direct experimental evidence for the existence of the two-pion resonance.[1] The three-pion resonance[2] is rather problematic. Its existence is only supported by the data on the isoscalar form factors of the nucleons and on the form factor of the π^0 meson,[3,4] and by the theoretical consideration that it could be a direct consequence of the two-pion resonance. Our estimates for processes connected with three-pion exchange will therefore be less trustworthy than the estimates for processes in which three-pion exchange is forbidden.

All of the following discussion could be presented in the language of dispersion relations. However, we shall use the language of Feynman diagrams. To this end it is sufficient to consider the field $B^n_\alpha(x)$ of a "particle" with ordinary and isotopic spin 1 and "mass" m_B (we shall call this "particle" the bipion) and the field $T^n(x)$ of a "particle" with ordinary spin 1, isotopic spin 0, and "mass" m_T (which we shall call the tripion). (Both "particles" may have only formal meaning.) In order to take into account the $\pi\pi$ interaction in various low-energy processes, we must then consider the simplest diagrams corresponding to the exchange of a virtual bipion or tripion.

We have the following "Lagrangians," describing the interaction of the bipion and the tripion with a photon,*

$$\Lambda_1 : A_n(x) B^n_3(x) :, \quad \eta_1 : A_n(x) T^n(x) :;$$

with two pions,

$$\Lambda_2 \varepsilon_{\alpha\beta\gamma} : \varphi_\alpha(x) \frac{\partial \varphi_\beta(x)}{\partial x^n} B^n_\gamma(x) :, \qquad 0;$$

with a photon and a pion,

$$\Lambda_3 \varepsilon_{lmns} : \frac{\partial A^l(x)}{\partial x_m} \frac{\partial B^n_\alpha(x)}{\partial x_s} \varphi_\alpha(x) :,$$

$$\eta_3 \varepsilon_{lmns} : \frac{\partial A^l(x)}{\partial x_m} \frac{\partial T^n(x)}{\partial x_s} \varphi_3(x) : ;$$

and finally, with two nucleons,

$$E : \bar{\psi}(x) \gamma_n \tau_\alpha \psi(x) B^n_\alpha(x) :$$

$$+ M : \bar{\psi}(x) \frac{i}{2} [\gamma_l, \gamma_n] \tau_\alpha \psi(x) \partial B^n_\alpha(x)/\partial x_l :;$$

$$E_1 : \bar{\psi}(x) \gamma_n \psi(x) T^n(x) :$$

$$+ M_1 : \bar{\psi}(x) \frac{i}{2} [\gamma_l, \gamma_n] \psi(x) \partial T^n(x)/\partial x_l :.$$

Our model contains a large number of parameters, but it allows us to relate in a simple fashion a large number of physical processes. For example, we can connect the following processes: 1) the interaction of nucleons with the electromagnetic field (nucleon form factors) (Fig. 1), 2) pion-nucleon scattering (the contribution from the pion-pion interaction in the P state) (Fig. 2), 3) pion-pion scattering (Fig. 3), 4) the interaction of pions with the electromagnetic field (form factors of charged pions) (Fig. 4, the photon can be virtual),

*We use the rational units:

$$\hbar = 1, c = 1, m_\pi = 1; \ ab = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}, \ \{\gamma^n, \gamma^m\} = 2g^{mn},$$
$$-g^{00} = g^{11} = g^{22} = g^{33} = -1.$$

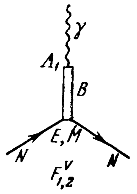

FIG. 1

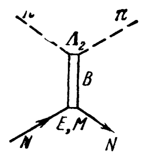

FIG. 2

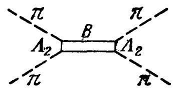

FIG. 3

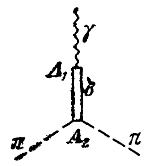

FIG. 4

5) the decay of the π^0 meson into two photons and into a photon, electron, and positron (form factor of the π^0 meson) (Fig. 5, one photon, or both, can be virtual), 6) the photoproduction of pions on pions (Fig. 6), 7) photoproduction of pions on nucleons (contribution from the pion-pion interaction) (Fig. 7), 8) nucleon-nucleon scattering (one-, two-, and three-pion exchange) (Fig. 8), 9) pion production in pion-nucleon scattering (πN interaction in the final state) (Fig. 9), and 10) pion production in nucleon-nucleon scattering (two-pion exchange) (Fig. 10).

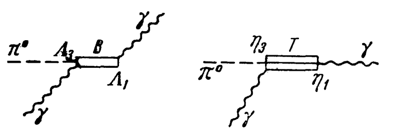

FIG. 5

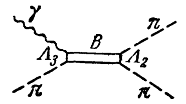

FIG. 6

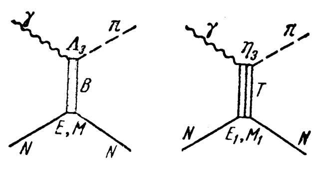

FIG. 7

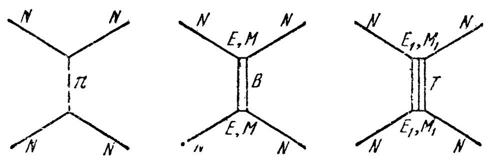

FIG. 8

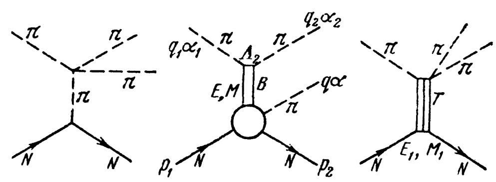

FIG. 9

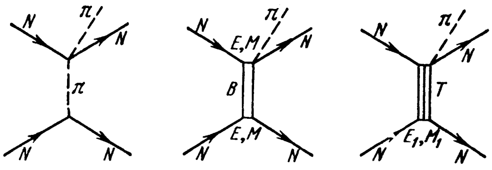

FIG. 10

In a similar fashion we can also consider the Compton effect on pions and nucleons, the anomalous magnetic moment of the deuteron, pion production in electron-electron scattering, etc.

This model has been used to some extent by a number of authors.[5] In the following we shall discuss some of the processes mentioned above.

2. NUCLEON FORM FACTORS

A precise experimental determination of the nucleon form factors would allow us to test our model (particularly with respect to the tripion) and to fix the large number of parameters, which, as we shall show in the following, govern a whole series of different processes. The nucleon form factors $F_{1,2}^{S,V}$ are given by

$$\langle p_2 \mid j^n(0) \mid p_1 \rangle$$
$$= \bar{u}(p_2)\{\hat{F}_e \gamma^n + \hat{F}_\mu \tfrac{1}{2}[\gamma(p_2 - p_1), \gamma^n]\}\, u(p_1), \qquad (1)$$

where

$$j^n(x) = i\,(\delta S/\delta A_n(x))\, S^+ \qquad (2)$$

is the electromagnetic current operator, p_1 and p_2 are the four-momenta of the nucleon before and after the collision, and

$$\hat{F}_e = \tfrac{1}{2} e (F_1^S + \tau_3 F_1^V), \qquad (3)$$

$$\hat{F}_\mu = \mu_S F_2^S + \tau_3 \mu_V F_2^V, \qquad (4)$$

$$\mu_S = \tfrac{1}{2}(\mu_p' + \mu_n) = -0.06\, e/2m,$$

$$\mu_V = \tfrac{1}{2}(\mu_p' - \mu_n) = 1.85\, e/2m. \qquad (5)$$

The bipion diagram gives a contribution only to $F_{1,2}^V$, the tripion diagram only to $F_{1,2}^S$. We include these diagrams (Fig. 1) and approximate the contribution from the other diagrams by constants which are determined by the normalization condition. We then obtain (k^2 is the photon "mass")

$$F_{1,2}^V(k^2) = 1 + a_{1,2}^V k^2/(m_B^2 - k^2), \qquad (6)$$

$$F_{1,2}^S(k^2) = 1 + a_{1,2}^S k^2/(m_T^2 - k^2), \qquad (7)$$

$$a_1^V = \frac{2\Lambda_1 E}{e m_B^2}, \quad a_2^V = \frac{\Lambda_1 M}{\mu_V m_B^2}, \quad a_1^S = \frac{2\eta_1 E_1}{e m_T^2}, \quad a_2^S = \frac{\eta_1 M_1}{\mu_S m_T^2}. \qquad (8)$$

In their recent experiments, Hofstadter et al[6] obtained form factors which are precisely of the form (6) and (7) with the parameters

$$a^V_{1,2} = 1.2, \quad a^S_1 = 0.56, \quad a^S_2 = -3, \quad m^2_B = 20, \quad m^2_T = 9.38. \tag{9}$$

It should be noted that these data are preliminary and may contain large errors, especially for F^S_2. For example if we assume that the proton and neutron form factors contain a 10% error, m^2_B may vary between 10 and 30 and m^2_T between 8 and 16.

The value $m^2_T = 9.38$ could imply either that there exists a bound state or that (in the absence of bound states and resonances in the three-pion system) the dispersion integral for F^S_1 is well approximated by the one-pole terms. It is therefore very important to get more accurate data on $F^S_{1,2}$ in order to check the tripion model.

For preliminary estimates as well as for methodological reasons, we shall in the following use the parameter values:

$$a^V_{1,2} = 1.2, \qquad m^2_B = 22.4, \tag{10}$$

which are in agreement with the experimental data on the elastic[7] and inelastic[1] pion-nucleon scattering.

In order to guarantee the approximate vanishing of the electric charge radius of the neutron, we must have

$$a^V_1/m^2_B \approx a^S_1/m^2_T. \tag{11}$$

The parameters a^S_1 and m^2_T are thus related by (10) and (11). We consider two possibilities:
a) we choose for a^S_1 the value given by (9) and obtain m^2_T from (10) and (11):

$$a^S_1 = 0.56, \qquad m^2_T = 10.5, \tag{12a}$$

which corresponds to a three-pion resonance; or
b) we take for m^2_T the value given by (9) and use (10) and (11) to find a^S_1:

$$m^2_T = 9.38, \qquad a^S_1 = 0.5; \tag{12b}$$

which would correspond to a three-pion bound state. Finally, we set, in accordance with (9),

$$a^S_2 = -3. \tag{13}$$

At present we can only say that these parameters are not in disagreement with the experimental data on the nucleon form factors. However, more accurate experiments may lead to a change in the chosen parameter values; this applies primarily to the parameters (12) and (13).

According to (8) and (10) to (13) we have

$$\Lambda_1 E = 1.2 m^2_B e/2, \qquad \Lambda_1 M = 1.2 m^2_B \mu_V, \quad m^2_B = 22.4, \tag{14}$$

$$\eta_1 E_1 = 0.56 m^2_T e/2, \quad \eta_1 M_1 = -3 m^2_T \mu_S, \quad m^2_T = 10.5, \tag{15a}$$

or

$$\eta_1 E_1 = 0.5 m^2_T e/2, \quad \eta_1 M_1 = -3 m^2_T \mu_S, \quad m^2_T = 9.38. \tag{15b}$$

3. PION-NUCLEON SCATTERING

The elastic pion-nucleon scattering with account of the two-pion resonance has been considered by Bowcock, Cottingham, and Lurié[7] in terms of dispersion theory. We shall show that the same result can be obtained on the basis of our model and shall connect the parameters of our model with the parameters used in [7]. The matrix element for πN scattering has the form

$$\langle p_2, q_2\alpha_2 | S - 1 | p_1, q_1\alpha_1 \rangle = i(2\pi)^4 \delta(p_2 + q_2 - p_1 - q_1)(4q^0_2 q^0_1)^{-1/2} \bar{u}(p_2) T u(p_1),$$

$$T = \delta_{\alpha_2\alpha_1} T^{(+)} + \tfrac{1}{2}[\tau_{\alpha_2}, \tau_{\alpha_1}] T^{(-)},$$

$$T^{(\pm)} = A^{(\pm)} + \tfrac{1}{2}\gamma(q_1 + q_2) B^{(\pm)}. \tag{16}$$

The bipion diagram (Fig. 2) gives the contribution (using the notation of [7])

$$A^{(-)} = -\frac{\Lambda_2 M(s - \bar{s})}{m^2_B - t}, \quad B^{(-)} = \frac{2\Lambda_2(E + 2mM)}{m^2_B - t}. \tag{17}$$

It is easily seen that the same expressions are obtained in [7] with

$$m^2_B = t_r, \quad \Lambda_2 E = -6\pi C_1, \quad \Lambda_2 M = -6\pi C_2, \tag{18}$$

where t_r, C_1, and C_2 are the parameters of [7].

By comparison with experiment it has been found in [7] that

$$t_r = 22.4, \quad C_1 = -1.0, \quad C_2 = -0.272. \tag{19}$$

Bowcock, Cottingham, and Lurié have also used the data on the form factors $F^V_{1,2}$ with $a^V_{1,2} = 1.2$. Besides the parameters (19) they estimated the width of the two-pion resonance (27) to be

$$\gamma = 0.376. \tag{20}$$

These are the parameters that we shall use in the following.

4. PION-PION SCATTERING

Our model is based on the assumption that there exists a $\pi\pi$ resonance in the P state. As applied to $\pi\pi$ scattering, it allows us to connect the constants of the model with the resonance parameters and to estimate the S waves within its framework. The matrix element for pion-pion scattering is equal to[9]

$$\langle q_2'\rho_2', q_1'\rho_1' | S - 1 | q_2\rho_2, q_1\rho_1\rangle$$
$$= -i(2\pi)^4 \delta(q_2' + q_1' - q_2 - q_1)(16 q_1^0 q_2^0 q_1'^0 q_2'^0)^{-1/2}$$
$$\times [\delta_{\rho_1\rho_2}\delta_{\rho_1'\rho_2'} A(s, \bar{s}, t) + \delta_{\rho_1'\rho_1}\delta_{\rho_2'\rho_2} B(s, s, t)$$
$$+ \delta_{\rho_1'\rho_2}\delta_{\rho_1\rho_2'} C(s, s, t)], \quad (21)$$

$$s = (q_1 + q_2)^2 = 4(1 + q^2),$$
$$\bar{s} = (q_1 - q_1')^2 = -2q^2(1 - \cos\theta),$$
$$t = (q_1 - q_2')^2 = -2q^2(1 + \cos\theta). \quad (22)$$

Including the bipion diagrams (Fig. 3) and approximating the contribution from the other diagrams by constants, we obtain

$$A = -\Lambda_2^2\left[\frac{s-t}{m_B^2 - \bar{s}} + \frac{s - \bar{s}}{m_B^2 - t}\right] + \Lambda,$$
$$B = -\Lambda_2^2\left[\frac{\bar{s}-t}{m_B^2 - s} + \frac{\bar{s} - s}{m_B^2 - t}\right] + \Lambda,$$
$$C = -\Lambda_2^2\left[\frac{t-\bar{s}}{m_B^2 - s} + \frac{t - s}{m_B^2 - \bar{s}}\right] + \Lambda \quad (23)$$

(the constants are equal owing to crossing symmetry) or, for the amplitudes of transitions with definite isotopic spin,

$$A^0 = 3A + B + C = -2\Lambda_2^2\left[\frac{s-t}{m_B^2 - \bar{s}} + \frac{s-\bar{s}}{m_B^2 - t}\right] + 5\Lambda,$$
$$A^1 = B - C = -\Lambda_2^2\left[2\frac{\bar{s}-t}{m_B^2 - s} + \frac{\bar{s}-s}{m_B^2 - t} + \frac{s-t}{m_B^2 - \bar{s}}\right],$$
$$A^2 = B + C = -\Lambda_2^2\left[\frac{\bar{s}-s}{m_B^2 - t} + \frac{t-s}{m_B^2 - \bar{s}}\right] + 2\Lambda. \quad (24)$$

Finally, the unitarity condition leads to the following relation between these amplitudes and the scattering phase shifts:

$$\frac{1}{2}\int_{-1}^{1} d\cos\theta A^l(q^2, \cos\theta) P_l(\cos\theta)$$
$$= -16\pi\frac{q^0}{q}\exp(i\delta_l^l)\sin\delta_l^l. \quad (25)$$

These expressions correspond in the language of dispersion theory to the resonance approximation in the Cini-Fubini representation.[10] The parameter Λ_2^2 is connected with the width of the resonance. Indeed, it follows from (24) and (25) that near the resonance

$$\frac{\exp(i\delta_1^1)\sin\delta_1^1}{q^3} = \frac{\Lambda_2^2}{3\pi m_B}\frac{1}{m_B^2 - s}. \quad (26)$$

Comparing this expression with the formula of Bowcock, Cottingham, and Lurié,[7]

$$\exp(i\delta_1^1)\sin\delta_1^1/q^3 = \gamma/(m_B^2 - s - i\gamma q^3), \quad (27)$$

we find

$$\Lambda_2^2 = 3\pi\gamma m_B. \quad (28)$$

We see that, regarding the bipion as a real particle and taking Λ_2 as real, we immediately obtain a resonance in the P state in which the phase shift goes through +90°.

With the parameter values of [7] we find

$$\Lambda_2^2 = 16.8. \quad (29)$$

Then the scattering lengths ($\lambda = -\Lambda/16\pi$)

$$a_0^0 = 5\lambda + \Lambda_2^2/\pi m_B^2, \quad a_0^2 = 2\lambda - \Lambda_2^2/2\pi m_B^2,$$
$$a_1^1 = \frac{\Lambda_2^2}{12\pi}\frac{3m_B^4 - 16}{m_B^4(m_B^2 - 4)} \quad (30)$$

are equal to

$$a_0^0 = 5\lambda + 0.24, \quad a_0^2 = 2\lambda - 0.12, \quad a_1^1 = 0.072. \quad (31)$$

5. THE FORM FACTORS OF CHARGED PIONS

Using the parameter values obtained above we can calculate the electromagnetic form factors of the pions. The electromagnetic form factor of the pion is given by

$$\langle q_2 | j^n(0) | q_1\rangle = \frac{q_2^n + q_1^n}{\sqrt{4q_2^0 q_1^0}} F((q_2 - q_1)^2), \quad (32)$$

where j^n is the electromagnetic current operator (2), and q_1 and q_2 are the four-momenta of the pion before and after the collision. The bipion diagram (Fig. 4) gives the following contribution to $F(k^2)$ for the $\pi^\pm$ meson:

$$\pm\Lambda_1\Lambda_2/(m_B^2 - k^2). \quad (33)$$

Assuming that the other diagrams give contributions which depend weakly on k^2 for small k^2, so that they can be approximated by a constant which is determined by the condition $F(0) = \pm e$, we obtain the following expression for the form factor of the charged pion:

$$F(k^2) = \pm e\left(1 + \frac{\Lambda_1\Lambda_2}{em_B^2}\frac{k^2}{m_B^2 - k^2}\right). \quad (34)$$

In particular, the mean square radius of the pion is equal to

$$\langle r^{(2)}\rangle^2 = 6\Lambda_1\Lambda_2/em_B^4. \quad (35)$$

Using the above-mentioned values of the constants (29), (14), (18), and (19), we obtain

$$\Lambda_1\Lambda_2 = \Lambda_2^2\Lambda_1 E/\Lambda_2 E = 12e. \quad (36)$$

This yields for the pion radius

$$\langle r^{(2)}\rangle = 0.38 \ (0.53\ \mathrm{F}). \quad (37)$$

6. NEUTRAL PION DECAY

The structure of the π^0 meson reveals itself in its decay. The matrix element for the decay of a

π^0 meson with four-momentum q into two photons with four-momenta k_1 and k_2 and polarization vectors ϵ_1 and ϵ_2 is of the form

$$\langle k_1k_2 | S | q\rangle = \frac{i(2\pi)^4 \delta(q-k_1-k_2)}{(8q^0 k_1^0 k_2^0)^{1/2}} \varepsilon_{lmns} \varepsilon_2^l \varepsilon_1^m k_2^n k_1^s F(0). \quad (38)$$

The lifetime of the π^0 meson at rest is

$$\tau = 64\pi / |F(0)|^2. \quad (39)$$

Here F denotes the form factor for π^0 decay, which in general determines the decay of a π^0 meson into a photon and an electron-positron pair:

$$\langle p_2p_1k | S | q\rangle = i(2\pi)^4 \delta(q-k-p_1-p_2)(4q^0k^0)^{-1/2} \times \varepsilon_{lmns} e u(p_2)\gamma^l v(p_1)(p_2+p_1)^{-2} \varepsilon^m q^n k^s F((p_2+p_1)^2). \quad (40)$$

Including the bipion and tripion intermediate states (Fig. 5), we obtain for the π^0 meson form factor

$$F(k^2) = -\Lambda_1\Lambda_3\left(\frac{1}{m_B^2-k^2}+\frac{1}{m_B^2}\right) - \eta_1\eta_3\left(\frac{1}{m_T^2-k^2}+\frac{1}{m_T^2}\right), \quad (41)$$

and hence

$$F(0) = -2\Lambda_1\Lambda_3/m_B^2 - 2\eta_1\eta_3/m_T^2, \quad (42)$$

$$F(k^2) = F(0)\left[1 - \frac{k^2}{F(0)}\left(\frac{\Lambda_1\Lambda_3/m_B^2}{m_B^2-k^2} + \frac{\eta_1\eta_3/m_T^2}{m_T^2-k^2}\right)\right]. \quad (43)$$

For small k^2 we have $F(k^2) = F(0)(1+\alpha k^2)$.

The quantities $|F(0)|$ and α are known experimentally.[3] If the "masses" of the bipion and tripion are known, we can then determine the constants $\Lambda_1\Lambda_3$ and $\eta_1\eta_3$:

$$\Lambda_1\Lambda_3 = m_B^4\left(\tfrac{1}{2} - \alpha m_T^2\right)(m_B^2 - m_T^2)^{-1} F(0),$$

$$\eta_1\eta_3 = m_T^4\left(\tfrac{1}{2} - \alpha m_B^2\right)(m_T^2 - m_B^2)^{-1} F(0). \quad (44)$$

At present the parameter α and the lifetime τ of the π^0 meson are known only with large errors:

$\tau = (2.3 \pm 0.8)\, 10^{-16}$ sec [11] *, $\alpha = -0.24 \pm 0.16$ [3].

If we set $\tau = 2 \times 10^{-16}$ sec, $\alpha = -0.2$, we find from (14), (15), (39), and (44)

$$\Lambda_1\Lambda_3 = 0.24, \quad \eta_1\eta_3 = -0.10 \quad (45a)$$

or

$$\Lambda_1\Lambda_3 = 0.2, \quad \eta_1\eta_3 = -0.073 \quad (45b)$$

(apart from a common phase factor).

In this estimate we included only the bipion and tripion diagrams. If we also took into account the contribution from the other diagrams (in the form of a constant), we would obtain formula (43) and one relation for the two constants:

$$\Lambda_1\Lambda_3 m_B^{-4} + \eta_1\eta_3 m_T^{-4} = -\alpha F(0). \quad (46)$$

However, it is easily seen that the bipion (isotopic spin T = 1) and tripion (T = 0) diagrams exhaust the set of possible diagrams with two or three pions in the intermediate state (indeed, the photon-three-pion vertex must be an isotopic scalar). The omitted diagrams therefore contain four or more intermediate pions, and it is reasonable to assume that their contribution can be neglected. We note that it is impossible to obtain the observed[3] negative value of α if only the bipion diagram is taken into account.

7. PHOTOPRODUCTION OF PIONS ON PIONS

The results of Sec. 6 can be used to estimate the parameter which determines the photoproduction of pions on pions. The matrix element for the photoproduction of pions on pions (Fig. 6) is equal to (the tripion diagram does not contribute)

$$\langle q_3\rho_3,\ q_2\rho_2 | S | k\lambda,\ q_1\rho_1\rangle = -(2\pi)^4\delta(k+q_1-q_2-q_3) \times (4k^0q_1^0q_2^0q_3^0)^{-1/2}\varepsilon_{\rho_1\rho_2\rho_3}\varepsilon_{lmns}\varepsilon_\lambda^l q_1^m q_2^n q_3^s \times \Lambda_2\Lambda_3\{[m_B^2-(k+q_1)^2]^{-1} + [m_B^2-(k-q_2)^2]^{-1} + [m_B^2-(k-q_3)^2]^{-1}\}. \quad (47)$$

Using the parameter values (45a), (45b), (18), (19), and (14), we obtain, up to a phase factor,

$$\Lambda_2\Lambda_3 = \Lambda_1\Lambda_3\Lambda_2E/\Lambda_1E = 37e \quad \text{or} \quad 38.4\,e. \quad (48)$$

8. INELASTIC PION-NUCLEON SCATTERING

All the formulas obtained above, although written down in terms of our model, are essentially a consequence of the resonance approximation in dispersion theory. However, we may attempt to apply our model to more complicated processes.

Let us consider, for example, the inelastic scattering of pions by nucleons

$$\pi + N \to \pi + \pi + N.$$

It has been shown that the one-pion diagram (Fig. 9) gives an important and characteristic contribution to the cross section of this process.[12] This distinguished role of the one-pion diagram is explained by the fact that it has a pole in the nucleon momentum transfer close to the physical region, and by the resonance character of the pion-pion cross section. It is clear, however, that the one-pion diagram represents only the very first approximation.[1] It does not take into account the pion-nucleon resonance interaction in the final state.

The simplest diagram taking into account this interaction is the diagram in which the pion and

*Preliminary data.

the nucleon exchange two pions (Fig. 9). These pions must have isotopic spin 1 or 2, in order that the pion and nucleon in the final state have isotopic spin $^3\!/_2$. Since the state with isotopic spin 1 is a resonance state, we may assume that it is sufficient to take this state alone into account. This can be done easily in the bipion approximation. Indeed, the bipion has the same transformation properties and the same "interaction" with the pion and the nucleon as the photon (more precisely, its iso-vector part). The lower part of the bipion diagram (Fig. 9) therefore is identical with the amplitude for virtual photoproduction, if in the latter we make the substitution

$$e\tau_3 \to 2E\tau_\alpha, \quad \mu_V\tau_3 \to M\tau_\alpha. \qquad (49)$$

The matrix element for inelastic πN scattering corresponding to the bipion diagram of Fig. 9 is equal to

$$\langle q_2\alpha_2,\ q\alpha,\ p_2 | S | q_1\alpha_1,\ p_1\rangle = M_2(q_2\alpha_2,\ q\alpha) + M_2(q\alpha,\ q_2\alpha_2), \qquad (50)$$

$$M_2(q_2\alpha_2,\ q\alpha) = -\frac{(2\pi)^4\,\delta(p_2+q-p-k)\,\Lambda_2}{(8q^0q_1^0q_2^0)^{1/2}}\,\frac{q_1^n+q_2^n}{m_B^2-k^2}\,\varepsilon_{\alpha_1\alpha_2\gamma}\,T_n^{(\gamma)}(q\alpha), \qquad (51)$$

where $k = q_1 - q_2$, and $T_n^{(\gamma)}(q\alpha)$ is the iso-vector part of the amplitude for virtual photoproduction [13,14] with the substitution (49). If in the latter we only include the magnetic dipole transition in the 33 state, we find in a coordinate system where $\mathbf{p}_2 + \mathbf{q} = 0$

$$T^{(\gamma)}(q\alpha) = \left(\delta_{\alpha\gamma} - \frac{1}{3}\tau_\alpha\tau_\gamma\right)\{i(\sigma\mathbf{k})\,\mathbf{q} - i(\mathbf{kq})\,\sigma - 2\,[\mathbf{qk}]\}$$
$$\times\frac{4\pi(2M+E/m)}{2f}\,\frac{w}{m}\exp(i\delta_{33})\,\frac{\sin\delta_{33}}{q^3}. \qquad (52)^*$$

Here $w = p_2^0 + q^0$ is the total energy in the system under consideration, m is the mass of the nucleon, and f is the pseudovector coupling constant ($f^2/4\pi \approx 0.08$). This expression is valid with an accuracy up to terms of order $[(w-m)/m]^2$.

Let us further consider the reaction

$$\pi^- + p \to \pi^- + \pi^+ + n.$$

In this case the second term in (50) does not contribute. The bipion matrix element $M_2(q_2\pi^-,\ q\pi^+)$ is obtained from (51) and (52) by the substitution

$$\varepsilon_{\alpha_1\alpha_2\gamma}(\delta_{\alpha\gamma} - \tau_\alpha\tau_\gamma/3) = -i\sqrt{2/3}$$

and gives the following contribution to the cross section for the process $\pi^- + p \to \pi^- + \pi^+ + n$:

$$\frac{\partial^2\sigma_2}{\partial w^2\partial(-k^2)} = \frac{\pi^2}{(2\pi)^5 q_{1L}^2}\,\frac{4q_1^2q_2^2\sin^2\theta_{12}}{(m_B^2-k^2)^2}\,\frac{q^3}{9w}\left|\frac{4\pi(2\Lambda_2M+\Lambda_2E/m)}{2f}\,\frac{w}{m}\,\frac{\sin\delta_{33}}{q^3}\right|^2. \qquad (53)$$

Here q_{1L} is the momentum of the incident pion in the laboratory system, $k^2 = (q_1 - q_2)^2$ is the square of the four-momentum transferred by the π^- meson, and $w^2 = (p_2 + q)^2$ is the square of the total energy of the π^+ meson and the neutron in their center-of-mass system; the other quantities are all defined in this same system: q is the momentum of the π^+ meson, q_1 and q_2 are the momenta of the π^- meson before and after the reaction, and θ_{12} is the angle between them, so that

$$4q_1^2q_2^2\sin^2\theta_{12} = w^{-2}\{-k^4W^2 - k^2\,[(W^2-1)^2 - (W^2+1)(w^2+m^2) + w^2m^2] - (w^2-m^2)^2\}, \qquad (54)$$

where W is the total energy of this process in the center of mass system $[W^2 = (p_1 + q_1)^2]$.

Figure 11 shows the contribution of the bipion diagram to the total cross section:

$$\sigma_2(T) = \int_{(m+1)^2}^{(W-1)^2} dw^2 \int_{y_1}^{y_2} dy\,\frac{\partial^2\sigma_2}{\partial w^2\,\partial y}, \qquad (55)$$

$$y_{2,1} = 2\left(\omega_1\omega_2 - 1 \pm \sqrt{(\omega_1^2-1)(\omega_2^2-1)}\right),$$
$$\omega_1 = (W^2 - m^2 + 1)/2W, \quad \omega_2 = (W^2 - w^2 + 1)/2W, \qquad (56)$$

where T is the kinetic energy of the incident pion in the laboratory system. For the constants $\Lambda_2 M$ and $\Lambda_2 E$ the values (18) and (19) were used. The same figure also shows the contribution σ_1 from the one-pion diagram (Fig. 9) to the total cross section, where the interaction between the pions is taken into account only in the P resonance state [formula (27)] with the parameters taken from [7].

We see that the contribution of the P state $\pi\pi$ interaction from the one-pion diagram (Fig. 9), as well as of the πN interaction in the final state

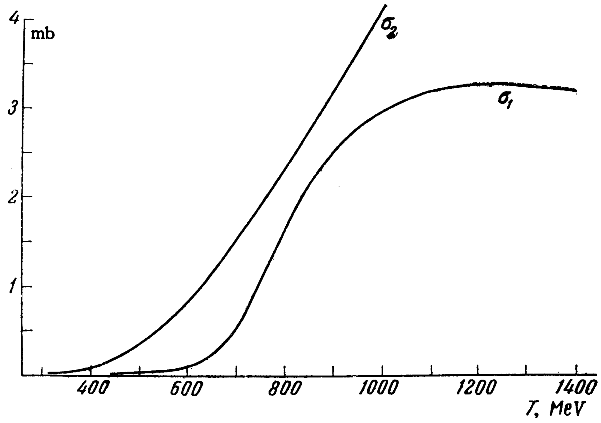

FIG. 11

*$(\sigma\mathbf{k}) = \sigma\cdot\mathbf{k}$; $[\mathbf{qk}] = \mathbf{q}\times\mathbf{k}$.

(bipion diagram in Fig. 9), is small for energies $T \lesssim 600$ MeV. It suffices to say that the experimental cross section at T = 427 MeV is equal to 3.5 mb. At these energies the main contribution to the cross section must therefore come from the S waves. (We note that the tripion diagram of Fig. 9 does not contribute to this process.)

For T > 600 MeV both resonance contributions increase. The contribution of the $\pi\pi$ resonance does not exceed 3.3 mb, which points out the importance of interactions in other states (the experimental cross section is ~ 10 mb at 1 BeV). The bipion contribution increases faster (it reaches 3.9 mb at 1 BeV) and tends to infinity at large energies. This is, of course, a consequence of our using a model with vector coupling. Since it is impossible to give a rigorous criterion for the energies up to which our model can be applied to inelastic processes, we cannot draw any convincing quantitative conclusions about the role of the bipion diagram at high energies. It is clear that there remains the possibility of introducing a cut-off parameter and attempting to achieve agreement with the experimental data with its help.

9. CONCLUDING REMARKS

It has been shown how a model with a bipion and a tripion (which is equivalent to the resonance approximation in dispersion theory) can be used to relate a large number of pion-nucleon experiments at low energies. The photoproduction of pions on nucleons can be considered in an analogous fashion (Fig. 7). The bipion contribution (to the iso-scalar part of the photoproduction amplitude) is determined by the parameters

$$\Lambda_3 E = \Lambda_2 \Lambda_3 \Lambda_2 E/\Lambda_2^2, \quad \Lambda_3 M = \Lambda_2 \Lambda_3 \Lambda_2 M/\Lambda_2^2, \qquad (57)$$

which can be estimated with the help of (18), (19), (29), and (48):

$$\Lambda_3 M = (2\mu_V/e)\, \Lambda_3 E, \qquad |\Lambda_3 E| = 22{,}4\, e. \qquad (58)$$

For the parameters determining the tripion contribution (to the iso-vector part of the amplitude) it is possible only to determine the ratio

$$\eta_3 M_1/\eta_3 E_1 = \eta_1 M_1/\eta_1 E_1 = -5{,}35\, (2\mu_S/e) \qquad (59)$$

[the numerical estimate follows from (15a)].

The bipion contribution to the nucleon-nucleon scattering (Fig. 8) is determined by the parameters

$$E^2 = (\Lambda_2 E)^2/\Lambda_2^2,$$

$$EM = \Lambda_2 E \Lambda_2 M/\Lambda_2^2, \quad M^2 = (\Lambda_2 M)/\Lambda_2^2, \qquad (60)$$

which, according to (14), (18), (19), and (29), are equal to

$$M^2 = (2\mu_V/e)^2 E^2, \quad EM = (2\mu_V/e)\, E^2, \quad E^2 = 21{,}2. \qquad (61)$$

Analogously, we have from (15) for the parameters determining the tripion contribution (Fig. 8)

$$M_1^2 = (2\mu_S/e)^2 E_1^2, \; E_1 M_1 = (2\mu_S/e)\, E_1^2, \qquad (62)$$

where E_1^2 is a free parameter.

The inelastic nucleon-nucleon scattering (Fig. 10) can be treated in exactly the same manner as the pion-nucleon scattering discussed in Sec. 7. The bipion contribution to this process is determined by the parameters (60) and (61). The tripion contribution can be neglected, since it does not lead to a 33 resonance.

The usefulness of our model is the greater, the higher the accuracy of the experimental data. It is highly desirable to obtain more accurate experimental data on the iso-scalar form factors of the nucleons, on the non-resonance πN phase shifts, on the inelastic πN scattering, on the $\pi\pi$ interaction, and on the photoproduction and the decay of the π^0 meson.

The authors are sincerely grateful to L. L. Nemenov and V. A. Meshcheryakov for useful discussions and to B. F. Feoktistov and Shen Chung-hua for numerical calculations.

[1] Pickup, Ayer, and Salant, Phys. Rev. Lett. **5**, 161 (1960). J. G. Rushbrooke and D. Radojičić, Phys. Rev. Lett. **5**, 567 (1960). I. Derado, Nuovo cimento **15**, 853 (1960). Bonsignori, Bortolani, and Stanghellini, Nuovo cimento **18**, 1237 (1960). Anderson, Bang, Burke, Carmony, and Schmitz, Phys. Rev. Lett. **6**, 365 (1961). Erwin, March, Walker, and West, Preprint.

[2] Y. Nambu, Phys. Rev. **106**, 1366 (1957). G. F. Chew, Phys. Rev. Lett. **4**, 142 (1960).

[3] N. P. Samios, Phys. Rev. **121**, 275 (1961).

[4] How-sen Wong, Phys. Rev. **121**, 289 (1961).

[5] Itabashi, Kato, Nakagawa, and Takeda, Progr. Theoret. Phys. **24**, 529 (1960). A. Stanghellini, Nuovo cimento **18**, 1258 (1960). Gourdin, Lurié, and Martin, Nuovo cimento **18**, 933 (1960). N. Dombey, Phys. Rev. Lett. **6**, 66 (1961). Y. Fujii, Progr. Theoret. Phys. **25**, 441 (1961).

[6] Hofstadter, de Vries, and Herman, Phys. Rev. Lett. **6**, 290 (1961). R. Hofstadter and R. Herman, Phys. Rev. Lett. **6**, 293 (1961).

[7] Bowcock, Cottingham, and Lurié, Nuovo cimento **16**, 918 (1960) and **19**, 142 (1961).

[8] Chew, Goldberger, Low, and Nambu, Phys. Rev. **106**, 1337 (1957).

[9] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[10] M. Cini and S. Fubini, Ann. of Physics **3**, 352 (1960).

[11] Glasser, Seeman, and Stiller, Proc. of the 1960 Annual Int. Conf. on High Energy Phys. at Rochester, Univ. of Rochester (1960), p. 30.

[12] C. Goebel, Phys. Rev. Lett. **1**, 337 (1958). G. F. Chew and F. E. Low, Phys. Rev. **113**, 1640 (1959). F. Salzman and G. Salzman, Phys. Rev. **120**, 599 (1960). F. Bonsignori and F. Selleri, Nuovo cimento **15**, 465 (1960).

[13] A. A. Logunov and L. D. Solov'ev, Nauch. Dokl. Vyssheĭ Shkoly **4**, 217 (1958. Kukin, Solov'ev and Frenkin, Nauchnye Doklady Vyssheĭ Shkoly **3**, 169 (1958). L. D. Solov'ev and G. N. Tentyukova, Preprint, Joint Institute of Nuclear Research, R-384.

[14] Fubini, Nambu, and Wataghin, Phys. Rev. **111**, 329 (1958).

Translated by R. Lipperheide
82