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HIGH-ENERGY BEHAVIOR OF THE ELASTIC PION-PION SCATTERING AMPLITUDE

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The imaginary part of the elastic scattering amplitude for neutral pseudoscalar particles is calculated in the high-energy limit for small momentum transfers. The resulting expression gives a constant cross section at large energies. The distribution of momentum transfers has a diffraction maximum in the forward direction. The shape of the maximum is independent of the energy.

1. INTRODUCTION

THE experimental data show that at high energies the elastic scattering of strongly interacting particles is essentially diffraction scattering. [1] This sort of behavior of the amplitude, however, cannot be obtained by the usual methods of field theory (for example, perturbation theory, Tamm-Dancoff method). In our opinion this is due to the fact that in all of these methods only a finite number of possible channels in the intermediate states are taken into account.

In this paper we propose a method for determining the asymptotic behavior of the imaginary part of the elastic scattering amplitude which assumes an unlimited number of inelastic channels in the intermediate states. At the same time we confine ourselves to the consideration of the exchange of a minimum number of particles between the colliding particles; our theory describes collisions with small momentum transfers. [2]† We shall make our calculations for neutral pseudoscalar particles of unit mass. It may be supposed, however, that our model can serve for the description of actual pions, since the isotopic spin evidently is not of much importance at high energies. [1]

From the very method of the calculations it follows that our results are accurate up to logarithmic factors (we shall obtain the solution of the equations in the form of an asymptotic power series), so that we can make no assertion about a logarithmic decrease of the total cross section (cf. [5]).

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†Compare papers by Chew and Frautschi [3,4] which contain similar ideas.

2. THE INTEGRAL EQUATIONS FOR THE SPECTRAL FUNCTIONS

Following Mandelstam [6] we assume that the Feynman amplitude for the process with four external lines is described by a single analytic function in all channels.

We write the unitarity relation in the "elastic approximation" for channel 3 (here and hereafter we use Mandelstam's notations [6])

$$A_3(z_1, t) = \frac{1}{32\pi^2}\sqrt{\frac{t-4}{t}}\iint \frac{dz_2 dz_3}{\sqrt{-k(z_1, z_2, z_3)}} A^*(z_2, t) A(z_3, t) \tag{1}$$

(the square of the energy in the c.m.s. is t). In the spectral region (1, 3) the imaginary part of A_3 (which is by hypothesis also the imaginary part of A_1) is given by the expression

$$\operatorname{Im} A_1(z_1, t) = \frac{-1}{4\pi^2}\sqrt{\frac{t-4}{t}}\iint_{z_0}^{\infty} \frac{dz_2 dz_3}{\sqrt{k(z_1, z_2, z_3)}}$$
$$\times \vartheta\left(z_1 - z_2 z_3 - \sqrt{(z_2^2-1)(z_3^2-1)}\right) A_1^*(z_2, t) A_1(z_3, t), \tag{2}$$

where $z_0 = 1 + 8(t-4)^{-1}$ (we have used here the symmetry property of the amplitude).

We shall regard Eq. (2) as an integral equation for A_1. It must be noted that the form of the equation does not depend on the asymptotic behavior of A_1 (this has already been pointed out by Gribov [5]), so that we can solve it without any supplementary assumptions (subtractions and so on).

3. THE APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION

Let us consider the asymptotic solution of the integral equation (2) for large s and small t. For this purpose we apply the Mellin integral transformation[5] to the equation. With the definition*

$$\tilde{A}_1(u, t) = \int_{z_0}^{\infty} \frac{dz}{z} z^u A_1(z, t), \qquad (3)$$

we find for real values of u

$$\mathrm{Im}\,\tilde{A}_1(u, t) = \frac{-1}{4\pi^2}\sqrt{\frac{t-4}{t}} \int\int_{z_0}^{\infty} \frac{dz_2\,dz_3}{[(z_2^2-1)(z_3^2-1)]^{(1-u)/2}}$$

$$\times c(\alpha, u) A_1^*(z_2, t) A_1(z_3, t), \qquad (2')$$

where

$$\alpha = \frac{z_2 z_3}{\sqrt{(z_2^2-1)(z_3^2-1)}}, \qquad c(\alpha, u) = \int_1^{\infty} \frac{dx}{\sqrt{x^2-1}} (x+\alpha)^{u-1}. \qquad (4)$$

Let us expand $c(\alpha, u)$ and the quantity in square brackets in asymptotic power series in z_2^{-1} and z_3^{-1}. Setting $c(1, u) \equiv c(u)$, after an elementary calculation we find

$$\mathrm{Im}\,\tilde{A}_1(u, t) = \frac{-1}{4\pi^2}\sqrt{\frac{t-4}{t}} \{c(u)\,|\tilde{A}_1(u, t)|^2$$

$$+ (1-u)(c(u) - c(u-1))$$

$$\times \mathrm{Re}\,[\tilde{A}_1^*(u-2, t)\,\tilde{A}_1(u, t)] + \ldots\}. \qquad (5)$$

If we represent the solution of this equation in the form of a series

$$\tilde{A}_1 = \tilde{A}_1^{(1)} + \tilde{A}_1^{(2)} + \ldots,$$

it is found that the main term in the asymptotic expansion of $\mathrm{Im}\,A_1(s, t)$ is given by the first term, which is the solution of the equation

$$\mathrm{Im}\,\tilde{A}_1^{(1)} = \frac{-1}{4\pi^2}\sqrt{\frac{t-4}{t}} c(u)\,|\tilde{A}_1^{(1)}|^2. \qquad (5')$$

Thus setting $\tilde{A}^{(1)} = [h_1(u, t) + ih_2(u, t)]^{-1} \equiv h^{-1}$, where h_1 and h_2 are real, we get

$$h_2(u, t) = \frac{1}{4\pi^2}\sqrt{\frac{t-4}{t}} c(u). \qquad (6)$$

We can determine $h_1(u, t)$ by using the analytical properties of h. Neglecting the left-hand "dynamical" cut (which gives a vanishing contribution to the asymptotic behavior), we find that $\tilde{A}_1^{(1)}(u, t)$ has a "dynamical" cut along the real axis of the t plane for $t > 4$. Besides this, A_1 can have "kinematical" singularities, caused by the fact that we have made the Mellin transformation with respect to z, and not to s. The function $h(u, t)$ has the same cuts as $\tilde{A}_1$, but poles at the points where $\tilde{A}_1$ is zero, and conversely.

*Since $A_1(z,t) = 0$ for $z < z_0$ and by hypothesis A_1 has at most a pole at infinity, the Mellin transformation exists in the complex plane of u for $\mathrm{Re}\,u < u_1$, where u_1 is a constant.

With the usual definition of the branch of the fractional power the t plane in the general case is divided into two parts by two cuts which go to the right and left along the real axis from the point $t = 4$. Thus we must use separate definitions of $h(u, t)$ in the upper and lower half-planes. If, on the other hand, u is a real integer, then there is a single function $h(u, t)$ analytic in the t plane with a cut going to the right along the real axis from $t = 4$. It has a spectral representation of the form

$$h(u, t) = \frac{t - t_0}{\pi} \int_4^{\infty} \frac{dt' h_2(u, t')}{(t'-t_0)(t'-t)} + \chi(u) + R(u, t), \qquad (7)$$

where h_2 is given by Eq. (6), $\chi(u)$ is an arbitrary function of u, and $R(u, t)$ is the sum of the pole terms corresponding to the roots of the function $\tilde{A}_1$.* In what follows we shall use the simpler formula (7) instead of the general representation; it can be shown that this simplification does not affect the final results.

Thus using Eq. (6) we have $(t > 4)$

$$h(u, t) = \frac{c(u)}{4\pi^2}\left\{\frac{1}{\pi}\sqrt{\frac{t-4}{t}} \ln\frac{\sqrt{t}+\sqrt{t-4}}{\sqrt{t}-\sqrt{t-4}} + i\sqrt{\frac{t-4}{t}}\right\}$$

$$+ \eta(u) \qquad (8)$$

(where $\eta(u)$ is again an arbitrary function of u), and finally

$$\tilde{A}_{13}(u, t) = \frac{c(u)}{4\pi^2}\sqrt{\frac{t-4}{t}}\left\{\left[\frac{c(u)}{4\pi^3}\sqrt{\frac{t-4}{t}} \ln\frac{\sqrt{t}+\sqrt{t-4}}{\sqrt{t}-\sqrt{t-4}}\right.\right.$$

$$\left.\left. + \eta(u)\right]^2 + \left(\frac{c(u)}{4\pi^2}\right)^2 \frac{t-4}{t}\right\}^{-1}. \qquad (9)$$

The function $c(u)$ can be expressed in terms of known functions:

$$c(u) = \pi^{1/2} 2^{u-1}\,\Gamma(1-u)/\Gamma(3/2 - u). \qquad (10)$$

In order to get $A_{13}(s, t)$ it is necessary to invert the Mellin transformation. This is impossible, however, as long as we know nothing about the function $\eta(u)$. We can get additional information if we consider Eq. (9) for $0 < [(t-4)/t]^{1/2} \ll 1$. Then

$$\tilde{A}_{13}(u, t) \approx \frac{c(u)}{8\pi^2(\eta(u))^2}\sqrt{\frac{t-4}{t}} \qquad (9')$$

and

$$A_{13}(s, t) = \frac{1}{2\pi i}\int_{u_0 - i\infty}^{u_0 + i\infty} du \left(1 + \frac{2s}{t-4}\right)^{-u} \tilde{A}_{13}(u, t)$$

$$\sim \frac{1}{2\pi i}\int_{u_0 - i\infty}^{u_0 + i\infty} du \left(\frac{2s}{t-4}\right)^{-u} \tilde{A}_{13}(u, t) \quad (u_0 < u_1). \qquad (11)$$

*The function $R(u, t)$ corresponds to ambiguities of the type indicated by Castillejo, Dalitz, and Dyson.[7] Hereafter we set $R(u, t) \equiv 0$.

For small values of t we use Eq. (9′) instead of Eq. (9).

It is known that in order to get the correct behavior of the amplitude near the threshold of channel 3,* near the boundary of the spectral region, A_{13} must behave like $[t-t_0(s)]^{-1/2}$, where $t_0(s) = 4s/(s-16)$.†

It can be seen from Eqs. (10), (9′), and (11) that this behavior is realized if $[\eta(u)]^2$ has a simple zero at the point $u = -1$. Thus

$$(\eta(u))^2 = (1+u)\,\eta_1(u) \qquad (\eta_1(-1) \equiv \eta_1 \neq 0)$$

and for $0 < [(t-4)/t]^{1/2} \ll 1$ we find

$$A_{13}(s, t) \approx \frac{1}{6\pi^2\eta_1}\frac{s}{\sqrt{t(t-4)}} + o(s). \qquad (12)$$

One cannot get from Eq. (9) an expression for $A_{13}(s,t)$ in closed form. In order to get some qualitative information about the behavior of $A_1(s,t)$ we assume that A_{13} is given by Eq. (12) in the entire range $4 \le t < \infty$. Then we find

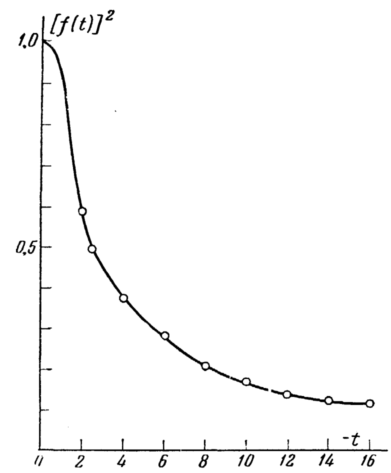
$$A_1(s, t) \sim sf(t)/12\pi^3\eta_1, \qquad (13)$$

where

$$f(t) = \frac{2}{\sqrt{t(t-4)}} \ln \frac{\sqrt{t-4}-\sqrt{t}}{\sqrt{t-4}+\sqrt{t}} \qquad (t<0), \qquad (14)$$

and by using the optical theorem we get the total cross section

$$\sigma \sim (3\pi\eta_1)^{-1} = \text{const}. \qquad (15)$$



Differential momentum-transfer distribution [Eq. (14)]

4. DISCUSSION OF THE RESULTS

As can be seen from the formulas (13)—(15), the theory given here describes the two main qualitative results that are known from experiments on the scattering of high-energy particles: the constant total cross section and the diffraction character of the elastic scattering, the shape of which does not depend on the energy (see figure).

For large values of $-t$ the function $f(t)$ falls off too slowly [as $(-t)^{-1}\ln(-t)$], but strictly speaking our results are not applicable in this region.

There are still some remarks to be made about the relation of our results to the work of Gribov.[5] In Gribov's paper it was shown that a pure power-law behavior of the amplitude at infinity is in contradiction with the generalized unitarity relation‡ [our Eq. (2)]. At the same time Gribov makes no assertion about the power-law index in the asymptotic expression for the amplitude. As can be seen from the present paper this index is fixed by the properties of the amplitude near threshold in the crossed channel, and (as was noted in Sec. 1) the absence of the logarithmic factor is a consequence of the approximations we have made.

The difficulty in principle with our theory is the presence of the unknown functions η and R. Setting $R \equiv 0$, we have been able to determine the position of the first zero of $\eta(u)$, but we do not know how to determine this function completely. The question as to whether oscillating solutions of the equations for the spectral function exist is still an open one. It is obvious that with the method of solution given in this paper it is impossible to settle the question of a logarithmic decrease or an oscillation of the amplitude. All of these problems require further study, and evidently call for the development of an essentially new method of solution of the nonlinear equation (2).

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*That is, in order for the phases to satisfy $\delta_l \approx a_l k^{2l+1}$ for infinitely small kinetic energy. This is physically equivalent to requiring that at small energies the interaction have a finite radius of action.

†One can convince oneself of this either by direct calculation or by reference to the results obtained from perturbation theory (cf. Mandelstam[8]).

‡But see[4]. Gribov's results are incorrect if the spectral function oscillates at infinity.

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