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SOME FEATURES OF COLLECTIVE ENERGY LOSSES OF FAST ELECTRONS MOVING IN AN ANISOTROPIC PLASMA

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Plasma oscillations in a quasiparticle gas with an arbitrary dispersion law (anisotropic plasma) are considered. In the anisotropic case the function $\omega_p(\mathbf{k})$ which expresses the dispersion law of the plasma oscillations depends (for a wave vector $\mathbf{k} \to 0$) on the direction of propagation of the plasmon; this leads to a peculiar angular dependence of the characteristic energy losses at small scattering angles. Surface plasma waves in an anisotropic plasma are considered. The effect of retardation of the interaction on the spectrum of the characteristic losses is taken into account. The calculations are carried out for uniaxial crystals.

As is well known,[1,2] a beam of fast electrons (the electron energies employed usually range from several keV to several times ten keV) experiences on passing through metallic films (d ~ 100—1000 A) so-called characteristic energy losses, which are multiples of a certain minimum value. These losses are attributed to the excitation of plasma waves in the metal.*

The electron plasma in a crystal is anisotropic, and this leads to several essential features compared with the isotropic model. The frequency of the plasma oscillations $\omega_p = \omega_p(\mathbf{k})$ depends not only on the absolute value of the wave vector $\mathbf{k}$, but also on its direction, and when $\mathbf{k} \to 0$ the ordinary dispersion (i.e., the dependence of ω_p on $|\mathbf{k}|$) is insignificant, whereas the dependence of ω_p on the direction of $\mathbf{k}$ continues to hold also when $\mathbf{k} \to 0$. Consequently, the discrete characteristic energy losses of electrons in monocrystalline films should exhibit considerable peculiarities. The present paper is devoted to an investigation of these questions.

We shall first derive (in Sec. 1) the dispersion law for plasma oscillations in the anistropic case, using the classical kinetic equation, although this question has been considered earlier ([3-5] and others). In the following section we shall investigate the characteristic energy losses of fast electrons in crystals.

*In addition, it is well known that transitions between bands are possible and also lead to discrete energy losses. When fast electrons are scattered through small angles, however, the plasma losses always predominate, since the probability of plasma emission is $w \approx k^{-2}$, where k is the momentum transfer (see, for example, [1]), whereas the probability of an interband transition depends on k in a more or less monotonic fashion (there is nothing to favor small k).

1. COLLECTIVE OSCILLATIONS IN A QUASI-PARTICLE GAS WITH ARBITRARY DISPERSION LAW

Modern theory of metals[6,7] is based on the assumption that the electron gas in the metal is an aggregate of Fermi quasiparticles with arbitrary dispersion $\epsilon(\mathbf{p})$. Interaction between the quasiparticles (Fermi-liquid effects) also permits the existence of a Bose branch of the elementary excitation spectrum (zero-sound oscillations). The theory of collective oscillations in a system of charged Fermi quasiparticles with Coulomb interaction can be constructed under the assumption that the function $\epsilon(\mathbf{p})$ has an arbitrary form (for long waves). This is explained by the presence of a Coulomb singularity in the interaction potential, due to the long-range nature of the Coulomb forces. The result is expressed only in terms of the function $\epsilon(\mathbf{p})$ —the dispersion law of the quasiparticles—so that many conclusions can be drawn regarding the form of this function.

Following Landau[8] (see also the paper by Klimontovich and Silin[9,10]) we write down the energy for the aggregate of quasiparticles in the form

$$E = E_0 + \sum_s \int d\mathbf{r}\, d\tau_p\, \varepsilon(\mathbf{p})\, \delta n_s(\mathbf{r}, \mathbf{p}, t) + \frac{1}{2}\sum_{ss'}\int d\mathbf{r}\, d\mathbf{r}'\, d\tau_p\, d\tau_{p'} \times \Phi_{ss'}(\mathbf{p}, \mathbf{p}'; \mathbf{r}, \mathbf{r}')\, \delta n_s(\mathbf{r}, \mathbf{p}, t)\, \delta n_{s'}(\mathbf{r}', \mathbf{p}', t), \quad (1)$$

where $d\tau_p = V d\mathbf{p}/(2\pi)^3$ and δn_s is the deviation of the distribution function from the equilibrium value n_0 (s is the spin index).*

*Formula (1) hold if the damping of the quasiparticles is neglected. In this approximation n_0 coincides with the Fermi distribution function (compare with[11]).

The last term in (1) describes the correlation between the particles. As shown by Silin and Klimontovich,[10,12] the correlation function $\Phi_{ss'}$ for the electrons in the metal can be represented in the form

$$\Phi_{ss'}(\mathbf{p}, \mathbf{p}'; \mathbf{r}, \mathbf{r}') = v(|\mathbf{r} - \mathbf{r}'|) + f_{ss'}(\mathbf{p}, \mathbf{p}'; \mathbf{r}, \mathbf{r}'), \quad (2)$$

where v is the Coulomb interaction potential, and $f_{ss'}$ takes into account exchange effects and other correlations between particles. In this case $f_{ss'}$ differs appreciably from zero only at the distances on the order of $|\mathbf{r}-\mathbf{r}'| \lesssim a$, where a is the lattice constant. Consequently, the Fourier component $f_{ss'}(\mathbf{k}; \mathbf{p}, \mathbf{p}')$ has no divergences when $\mathbf{k} \to 0$, since the Fourier component of the Coulomb potential $v(k) = 4\pi e^2/k^2$ has a singularity at $k \to 0$. Consequently, for long waves $(\mathbf{k} \to 0)$ the Coulomb interaction potential is the most important one.

If f is neglected, the kinetic equation obtained from (1) coincides with the ordinary kinetic equation in the self-consistent field approximation, used by Vlasov[13] and Landau[14] to investigate plasma oscillations and by Gol'dman[15] for the degenerate Fermi gas. It has the following form (without the collision integral)

$$(\omega - \mathbf{kv})\, \delta n_s(\mathbf{k}, \mathbf{p}) + (\mathbf{kv}) \frac{\partial n_0}{\partial \varepsilon} \frac{1}{V} v(k) \sum_{s'} \int d\tau_{p'}\, \delta n_{s'}(\mathbf{k}, \mathbf{p}') = 0, \quad (3)$$

from which we get the following dispersion equation, which determines the dependence of the frequency on the wave vector, i.e., the plasmon dispersion law $\omega = \omega_p(\mathbf{k})$:

$$1 + 2v(k) \frac{1}{V} \int d\tau_p \frac{\mathbf{kv}}{\omega - \mathbf{kv}} \frac{\partial n_0}{\partial \varepsilon} = 0, \quad (4)$$

where $\mathbf{v} = \partial\epsilon/\partial\mathbf{p}$ —velocity of the quasiparticles. We neglect here the reaction of the plasmon on the electrons [i.e., on the function $\epsilon(\mathbf{p})$]. This holds true for long waves, when the density changes appreciably over distances that are long compared with the interatomic distance.

The plasma oscillations correspond to finite ω as $\mathbf{k} \to 0$. Expanding $1/(\omega - \mathbf{k}\cdot\mathbf{v})$ in powers of $\mathbf{k}\cdot\mathbf{v}/\omega$, we obtain for the frequency of the plasma oscillation ω_p, in the zeroth approximation,

$$\omega_p^2 = -2v(k) \frac{1}{V} \int d\tau_p (\mathbf{kv})^2 \frac{\partial n_0}{\partial \varepsilon}. \quad (5)$$

Since $\partial n_0/\partial\epsilon = -\delta(\epsilon - \epsilon_F)$, where ϵ_F is the end-point Fermi energy, this expression yields for $\mathbf{k} \to 0$:

$$\omega_p^2(\mathbf{n}) = \frac{e^2}{\pi^2\hbar} \oint (\mathbf{v}_F\mathbf{n})^2 \frac{ds}{v_F}, \quad (6)$$

where the integral is taken over the Fermi surface $\epsilon(\mathbf{p}) = \epsilon_F$; $\mathbf{n}$ is a unit vector in the $\mathbf{k}$ direction; $\mathbf{v}_F$ is the electron velocity on the Fermi boundary.*

The most important consequence of (6) is the dependence of the frequency ω_p at zero propagation vector, $\mathbf{k} \to 0$, on the direction of $\mathbf{k}$. The frequency ω_p is thus a non-analytic function of $\mathbf{k}$.†

In the higher-order approximations in $\mathbf{k}$, ω becomes dependent on the absolute value of $\mathbf{k}$, i.e., ordinary dispersion is obtained. The corresponding expansions can be readily obtained from (4). It turns out here, however, that it is necessary to take into account both f in formula (2) and the quantum corrections. In addition, at large values of k the damping of the plasma waves becomes significant. In a well reasoned approach based on the model used in the present paper (in which near-order correlations are disregarded), the ordinary plasmon dispersion can be neglected. The theory developed here holds true for sufficiently small k satisfying the condition

$$k \ll k_c,$$

where the "cutoff" momentum k_c has a value[17] ‡

$$k_c = \omega_p/v_F. \quad (7)$$

Returning to Formula (6) we rewrite it in the form

$$\omega_p^2(\mathbf{n}) = \Omega_{ik}^2 n_i n_k, \quad (8)$$

$$\Omega_{ik}^2 = \frac{e^2}{\pi^2\hbar} \oint \frac{v_F^i v_F^k}{v_F} dS. \quad (9)$$

$\omega^2(\mathbf{n})$ is a positive definite form in n_i. After reduction to the principal axes, Eq. (8) assumes the form

$$\omega_p^2(\mathbf{n}) = \Omega_1^2 n_1^2 + \Omega_2^2 n_2^2 + \Omega_3^2 n_3^2. \quad (10)$$

We see therefore that the surface $f(\mathbf{n}) = 1/\omega_p(\mathbf{n})$

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*Plasma oscillations in crystals were also considered by other authors[3,4]. Bonch-Bruevich[3] used the quantum field theory and obtained (in the notation used here) the following expression for the frequency of oscillation of an electron plasma in a solid:

$$\omega_p^2 = \frac{e^2}{\pi^2\hbar} \int \frac{\partial^2\varepsilon}{\partial p_i \partial p_k} n_i n_k d\mathbf{p},$$

where, unlike (6), the integration is over the entire region of p-space inside the Fermi surface. Transforming the volume integral into a surface integral we see that this formula coincides with (6).

†A similar situation obtains for sound waves and excitons in crystals[16].

‡We can estimate the plasma frequency from the expression $\omega_p = (4\pi ne^2/m)^{1/2}$, which holds for the isotropic case (n – electron density, m – effective mass).

is an ellipsoid with semi-axes Ω_1^{-1}, Ω_2^{-1}, and Ω_3^{-1}, which we shall henceforth call the plasma-frequency ellipsoid. Its form depends on the symmetry of the crystal. For crystals of cubic symmetry all three principal values coincide, $\Omega_1 = \Omega_2 = \Omega_3$, and the ellipsoid degenerates into a sphere, i.e., in cubic crystals the electron plasma is isotropic (in the zeroth approximation). For uniaxial crystals (i.e., crystals of hexagonal, tetragonal, and rhombohedral systems), the plasma-frequency surface is an ellipsoid of revolution. Finally, for biaxial crystals we have an ellipsoid in general form. We shall consider later on several specific physical phenomena for the case of greatest importance, that of uniaxial crystals.

2. COLLECTIVE ENERGY LOSSES IN CRYSTALS

The above-noted singularities of the function $\omega_p(\mathbf{k})$ in an isotropic plasma manifest themselves in investigations of the inelastic scattering of fast electrons, which is connected with the emission of quanta of plasma oscillations, at very small angles. Let us consider for simplicity a uniaxial crystal with two principal values of the tensor Ω_{ik}^2 [see (10)], $\Omega_{\parallel}$ (parallel to the principal axis of the crystal) and $\Omega_{\perp}$ (perpendicular to the axis), and let us carry out the calculation when the surface of the metal is perpendicular to the principal axis and when the surface of the metal is parallel to the principal axis. The electron beam is normally incident on the surface of the metal. We consider scattering through small angles θ, when ordinary dispersion of the plasma frequency can be neglected.

A. Principle Axis Perpendicular to the Surface. Let $\mathbf{p}_0$ be the momentum of the electron prior to scattering, $\mathbf{p}_1$ the momentum after scattering, and $\mathbf{k}$ the momentum of the plasmon, with components $k_{\parallel}$ and $k_{\perp}$ parallel and perpendicular to $\mathbf{p}_0$, respectively (Fig. 1). The plasma frequency, according to (10), is

$$\omega_p^2 = \Omega_{\parallel}^2 k_{\parallel}^2/(k_{\parallel}^2 + k_{\perp}^2) + \Omega_{\perp}^2 k_{\perp}^2/(k_{\parallel}^2 + k_{\perp}^2).$$

The energy and momentum conservation laws yield

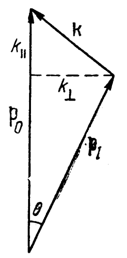

FIG. 1

$$v_0^2 k_{\parallel}^2 = \omega_p^2 = \Omega_{\parallel}^2 k_{\parallel}^2/(k_{\parallel}^2 + k_{\perp}^2) + \Omega_{\perp}^2 k_{\perp}^2/(k_{\parallel}^2 + k_{\perp}^2),$$
$$k_{\perp} = p_0\theta \tag{11}$$

[θ —scattering angle (Fig. 1); v_0 —velocity of scattered electron]. From (11) we can determine the dependence of the energy loss,* which is equal to ω_p, on the scattering angle θ. This dependence has the form

$$\omega_p^2(\theta) = \tfrac{1}{2}\{\Omega_{\parallel}^2 - v_0^2 p_0^2\theta^2 + [(\Omega_{\parallel}^2 - v_0^2 p_0^2\theta^2)^2 + 4\Omega_{\perp}^2 v_0^2 p_0^2\theta^2]^{1/2}\}. \tag{12}$$

This formula is valid when $k_{\parallel} \ll k_c$ and $k_{\perp} \ll k_c$, for otherwise we cannot neglect the ordinary dispersion of the plasmon. In addition, when $k \gtrsim k_c$ the plasma-loss spectrum is smeared out by the finite lifetime of the plasmon, connected with the possibility of plasmon decay into an electron-hole pair.† The first of these inequalities is always satisfied, since

$$k_{\parallel} = \omega_p/v_0 = k_c v_F/v_0 \ll k_c. \tag{13}$$

The second condition leads to the inequality

$$\theta \ll \omega_p/v_F p_0. \tag{14}$$

For example, for a plasma loss of 10 eV and for a scattered-electron loss of 10 keV, we obtain (putting $v_F \sim 10^8$ cm/sec)

$$\theta \ll 0.03.$$

In this case the greatest change in energy loss occurs in the angle interval $\theta \sim \theta_0 = \omega_p/v_0 p_0$. The investigation should thus be carried out at very small scattering angles. Another experimental difficulty lies in the need for preparing single-crystal metal films that are transparent to fast electrons.

An investigation of the dependence of the plasma losses on the scattering angle θ at small θ enables us to determine $\Omega_{\perp}$ and $\Omega_{\parallel}$ [see (12)].

B. Principal Axis Parallel to the Surface. In this case the plasma energy losses will depend also on the azimuthal angle φ between the direction of scattering in the plane of the plate and the principal axis (i.e., between the plane in which the normal to the surface and the scattered wave vector of the electron lie, and the plane drawn through the normal and the principal axis).

*We are dealing with a discrete loss spectrum. The smallest energy loss is equal to ω_p (we assume that the energy and the frequency, and also the momentum and the wave vector, are measured in the same units: $\varepsilon = \omega$ and $p = k$).

†We do not consider here the question of plasma-loss intensity, which increases with decreasing scattering angle in both the isotropic and anisotropic case.

In this case

$$\omega_p^2 = \Omega_\parallel^2 \frac{k_\perp^2}{k_\perp^2 + k_\parallel^2} \cos^2 \varphi + \Omega_\perp^2 \left(\frac{k_\perp^2}{k_\perp^2 + k_\parallel^2} \sin^2 \varphi + \frac{k_\parallel^2}{k_\perp^2 + k_\parallel^2} \right).$$

Elementary manipulations similar to those used above lead to the following value for the energy loss

$$\omega_p^2 (\theta, \varphi) = \tfrac{1}{2} \{\Omega_\perp^2 - v_0^2 p_0^2 \theta^2 + [(\Omega_\perp^2 - v_0^2 p_0^2 \theta^2)^2 + 4 v_0^2 p_0^2 \theta^2 (\Omega_\parallel^2 \cos^2 \varphi + \Omega_\perp^2 \sin^2 \varphi)]^{1/2}\}. \quad (15)$$

This formula, like (12), is valid if condition (14) is satisfied.

As can be seen from (12) and (15), for any relation between $\Omega_\parallel$ and $\Omega_\perp$ ($\Omega_\parallel > \Omega_\perp$ or $\Omega_\parallel < \Omega_\perp$), one of the experiments, subject to condition A or B (i.e., when the surface of the metal is perpendicular to the principal axis or parallel to it), should disclose a "negative dispersion," i.e., a decrease in plasma loss with increasing scattering angle (ordinary dispersion of the plasma frequency, without account of the anisotropy of ω_p, always leads to an increase in the loss with increasing scattering angle θ).

3. SURFACE WAVES IN AN ANISOTROPIC PLASMA

The spectrum of the characteristic energy losses of fast electrons passing through a thin metallic film or reflected from a metal surface exhibits, along with energy losses that are multiples of ω_p, several anomalous lines. Ritchie [18] first advanced the hypothesis that these anomalous losses are connected with the possible existence of surface plasma waves, which represent space-charge oscillations along the metal-vacuum surface.* In the anisotropic case the frequency of the surface plasma waves is [18,19]

$$\omega_s = \omega_p / \sqrt{2}. \quad (16)$$

Stern and Ferrell [19] considered surface plasma oscillations on metal-dielectric and metal-metal boundaries. In the present paper we discuss surface plasma oscillations on a metal-vacuum interface with account of the anisotropy of the plasma frequency $\omega_p = \omega_p(\mathbf{n})$.

As will be shown in the next section, [formula (26)], the anisotropic plasma can be described, neglecting spatial dispersion (i.e., as $\mathbf{k} \to 0$), with the aid of a dielectric-constant tensor

$$\varepsilon_{ik} (\omega) = \delta_{ik} - \Omega_{ik}^2 / \omega^2. \quad (17)$$

In the case of a uniaxial crystal we have two principal values: $\epsilon_\parallel = 1 - \Omega_\parallel^2/\omega^2$ (parallel to the axis) and $\epsilon_\perp = 1 - \Omega_\perp^2/\omega^2$ (perpendicular to the axis).

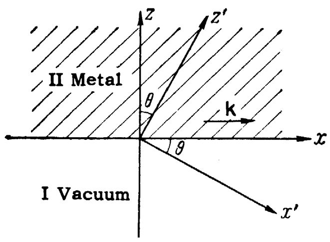

FIG. 2

Assuming the metal to occupy the half space $z > 0$ (see Fig. 2), we can write the following expression for the scalar potential φ in the region $z > 0$ (metal) and $z < 0$ (vacuum):

$$\varphi_{\mathrm{I}} = \varphi_0 e^{ikx} e^{kz}, \quad z < 0 \quad (k > 0);$$
$$\varphi_{\mathrm{II}} = \varphi_0 e^{ikx} e^{\varkappa z}, \quad z > 0 \quad (\mathrm{Re}\,\varkappa < 0). \quad (18)$$

In this case φ_{I} and φ_{II} satisfy the equations

$$\partial^2 \varphi_{\mathrm{I}} / \partial x_i^2 = 0, \quad z < 0;$$
$$\varepsilon_{ik} (\omega)\, \partial^2 \varphi_{\mathrm{II}} / \partial x_i\, \partial x_k = 0, \quad z > 0.$$

Expression (18) is a plane wave propagating along the x axis and damped exponentially on both sides of the surface. On the metal-vacuum interface the z-component of the induction vector must be continuous

$$D_{z\mathrm{I}} = D_{z\mathrm{II}}|_{z=0}, \qquad D_i = -\varepsilon_{ik} \partial \varphi / \partial x_k.$$

We shall call the plane in which the normal to the surface and the principal axis of the crystal lie the principal plane, and consider first the case when the propagation vector lies in the principal plane (Fig. 2). The angle between the principal axis and the normal to the surface will be denoted θ. We then have in the system of principal axes of the crystal (when $z > 0$)

$$\varepsilon_\perp \partial^2 \varphi_{\mathrm{II}} / \partial x'^2 + \varepsilon_\parallel \partial^2 \varphi_{\mathrm{II}} / \partial z'^2 = 0;$$
$$D_{z\mathrm{II}} = \varepsilon_\perp \sin \theta\, \partial \varphi_{\mathrm{II}} / \partial x' - \varepsilon_\parallel \cos \theta\, \partial \varphi_{\mathrm{II}} / \partial z'.$$

The first equation leads to the following value of κ in (18):

$$\varkappa = \frac{-k \sqrt{\varepsilon_\perp \varepsilon_\parallel} + ik (\varepsilon_\perp - \varepsilon_\parallel) \sin \theta \cos \theta}{\varepsilon_\perp \sin^2 \theta + \varepsilon_\parallel \cos^2 \theta}, \quad (19)$$

and from the condition of continuity of the z component of the induction vector on the surface of the metal we obtain the relation

$$\sqrt{\varepsilon_\perp (\omega)\, \varepsilon_\parallel (\omega)} = -1, \quad (20)$$

*The ratio of the intensity of the surface plasma energy loss to the intensity of the ordinary plasma loss will be of the order of unity in the case of sufficiently thin films.

which plays the role of a dispersion equation for the surface plasma waves as $\mathbf{k} \to 0$ ($\sqrt{\epsilon_\perp \epsilon_\parallel}$ is assumed to have the same sign as $\epsilon_\perp$ and $\epsilon_\parallel$, which are both negative).

Equation (20) leads in this case to the following value for the frequency of the surface plasma oscillations

$$\omega_s^2 = \Omega_\perp^2 \Omega_\parallel^2 / (\Omega_\perp^2 + \Omega_\parallel^2). \tag{21}$$

For an isotropic plasma with $\Omega_\perp = \Omega_\parallel = \omega_p$, formula (21) yields the well known expression $\omega_s = \omega_p/\sqrt{2}$ [see (16)].

With the aid of similar derivations we obtain when the propagation vector is normal to the principal plane

$$\omega_s^2 = \frac{\Omega_\perp^2 (\Omega_\perp^2 \sin^2\theta + \Omega_\parallel^2 \cos^2\theta)}{\Omega_\perp^2 + (\Omega_\perp^2 \sin^2\theta + \Omega_\parallel^2 \cos^2\theta)}. \tag{22}$$

In the general case we obtain finally the following expression for the frequency of the surface plasma waves in a uniaxial crystal

$$\omega_s^2(\theta, \varphi) = \frac{\Omega_\perp^2 [\Omega_\perp^2 \sin^2\theta \sin^2\varphi + \Omega_\parallel^2 (\cos^2\theta \sin^2\varphi + \cos^2\varphi)]}{\Omega_\perp^2 + [\Omega_\perp^2 \sin^2\theta \sin^2\varphi + \Omega_\parallel^2 (\cos^2\theta \sin^2\varphi + \cos^2\varphi)]}, \tag{23}$$

where θ is the angle between the normal to the surface of the metal and the principal axis, while φ is the angle between the propagation vector and the principal plane.

4. DISPERSION OF PLASMA WAVES WITH LIMITINGLY SMALL MOMENTA

The analysis of collective oscillations in metals, considered in Secs. 1—3, does not hold true when $k \to 0$. Indeed, when $k \to 0$ we get $\omega \to \omega_p = \text{const}$, and consequently the phase velocity of the plasmon is $v_{ph} = \omega_p/k \to \infty$. We cannot neglect the electrodynamic retardation here. The formulas given above are valid if the following double inequality is satisfied

$$\omega_p/c \ll k \ll \omega_p/v_F, \tag{24}$$

where c is the velocity of light (see Sec. 1 concerning the meaning of the upper limit).

We obtain here formulas that are also valid when $k \lesssim \omega_p/c$, which may be important for the investigation of the characteristic energy losses of relativistic electrons with velocities close to c (electrons with energy 25 keV[20] have $v_0 = 9 \times 10^9$ cm/sec).

We use the classical kinetic equation with self-consistent interaction. The adequacy of this approach (as $k \to 0$) was demonstrated in Sec. 1. Neglecting the collision integral (for $\omega\tau \gg 1$) we have

$$\frac{\partial \delta n}{\partial t} + \mathbf{v}\frac{\partial \delta n}{\partial \mathbf{r}} + e\mathbf{E}\mathbf{v}\frac{\partial n_0}{\partial \varepsilon} = 0; \qquad \delta n = n - n_0.$$

We consider plane waves, $\delta n \sim \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, we obtain for the non-equilibrium addition to the distribution function

$$\delta n = e\mathbf{E}\mathbf{v}/i(\omega - \mathbf{k}\mathbf{v}).$$

Calculating with the aid of δn the electron current $\mathbf{j}$, we transform the Maxwell equation $\operatorname{curl}\mathbf{H} = c^{-1}4\pi\mathbf{j} - c^{-1}i\omega\mathbf{E}$ to the form $\operatorname{curl}\mathbf{H} = -c^{-1}i\omega\mathbf{D}$, the induction vector $\mathbf{D}$ being

$$\mathbf{D} = \mathbf{E} - \frac{4\pi}{i\omega}\mathbf{j} = \mathbf{E} + \frac{8\pi e^2}{\omega}\frac{1}{V}\int d\tau_p \frac{(\mathbf{E}\mathbf{v})\mathbf{v}}{\omega - \mathbf{k}\mathbf{v}}\frac{\partial n_0}{\partial \varepsilon} = \hat{\varepsilon}\mathbf{E}, \tag{25}$$

and consequently the dielectric-constant tensor ϵ_{ik} has a value (as $\mathbf{k} \to 0$)

$$\varepsilon_{ik} = \delta_{ik} - \Omega_{ik}^2/\omega^2, \tag{26}$$

where Ω_{ik}^2 is the plasma-frequency tensor, determined in accord with (9).

We now obtain from Maxwell's relations a dispersion equation for $\omega(\mathbf{k})$. It has the form

$$\det\{(k_i k_j - \Omega_{ij}^2/c^2) - (k^2 - \omega^2/c^2)\,\delta_{ij}\} = 0. \tag{27}$$

In the case of a uniaxial crystal, (27) breaks up into the two equations

$$\begin{gathered}\omega^4 - \omega^2(\Omega_\perp^2 + \Omega_\parallel^2 + c^2k^2) \\ + (\Omega_\perp^2\Omega_\parallel^2 + \Omega_\perp^2 c^2 k_\perp^2 + \Omega_\parallel^2 c^2 k_\parallel^2) = 0, \\ \omega^2 - \Omega_\perp^2 - c^2k^2 = 0,\end{gathered} \tag{28}$$

from which we get

$$\begin{gathered}\omega_{1,2}^2 = \tfrac{1}{2}(\Omega_\perp^2 + \Omega_\parallel^2 + c^2k^2) \\ \pm \tfrac{1}{2}[(\Omega_\perp^2 - \Omega_\parallel^2)^2 - 2(\Omega_\perp^2 - \Omega_\parallel^2)c^2(k_\perp^2 - k_\parallel^2) + c^4k^4]^{1/2}, \\ \omega_3^2 = \Omega_\perp^2 + c^2k^2.\end{gathered} \tag{29}$$

The root $\omega = \omega_3$ corresponds to the transverse optical wave. For the first and second waves the vector $\mathbf{E}$ is in general neither longitudinal nor transverse with respect to the wave vector $\mathbf{k}$.

Let us consider the limiting cases of large and small k.

1) $k \gg \omega_p/c$. From (29) we get

$$\omega_1^2 = \Omega_\perp^2 n_\perp^2 + \Omega_\parallel^2 n_\parallel^2, \qquad \omega_2^2 = c^2k^2 + \Omega_\perp^2 n_\parallel^2 + \Omega_\parallel^2 n_\perp^2.$$

The solution $\omega = \omega_1$ corresponds in this case to a longitudinal plasma wave [see (10)].

2) $k \ll \omega_p/c$. Similarly, formula (29) yields

$$\omega_1^2 = \Omega_\parallel^2 + c^2k_\perp^2, \qquad \omega_2^2 = \Omega_\perp^2 + c^2k_\parallel^2.$$

Returning to case 1), we obtain the correction for the frequency of the plasma oscillation, needed to account for the retardation, in the form of an expansion in ω_p/ck:

$$\omega_p^2 = \Omega_\perp^2 n_\perp^2 + \Omega_\parallel^2 n_\parallel^2 - (\Omega_\perp^2 - \Omega_\parallel^2)^2 n_\perp^2 n_\parallel^2 / c^2 k^2 + \ldots \quad (30)$$

In the isotropic case there is no correction term, since the optical and plasma oscillations do not get "entangled."

We give also an expression for the induction in a plasma wave

$$D_\perp = \frac{(\Omega_\parallel^2 - \Omega_\perp^2)\, n_\parallel^2 n_\perp}{\Omega_\perp^2 n_\perp^2 + \Omega_\parallel^2 n_\parallel^2} E, \quad D_\parallel = \frac{(\Omega_\perp^2 - \Omega_\parallel^2)\, n_\perp^2 n_\parallel}{\Omega_\perp^2 n_\perp^2 + \Omega_\parallel^2 n_\parallel^2} E. \quad (31)$$

In this case the conditions $\mathbf{D}\cdot\mathbf{n} = 0$ and $\mathbf{E}\cdot\mathbf{n} = E$ are satisfied. Unlike the isotropic case, we have $\mathbf{D} \neq 0$ [in the isotropic plasma $\mathbf{D} \equiv 0$ by virtue of the condition $\epsilon(\omega_p) = 0$].

In conclusion we note that in the frequency interval between $\Omega_\perp$ and $\Omega_\parallel$ the components of the dielectric constant tensor $\epsilon_\perp = 1 - \Omega_\perp^2/\omega^2$ and $\epsilon_\parallel = 1 - \Omega_\parallel^2/\omega^2$ have different signs, and consequently the metal will have sharply anisotropic optical properties: it may be opaque in one direction (say along the principal optical axis) but transparent in a perpendicular direction (assuming that the ordinary absorption is small).

In conclusion, I am deeply grateful to I. M. Lifshitz for a discussion of this work.

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