CHOICE OF INVARIANT VARIABLES FOR THE "MANY-POINT" FUNCTIONS

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Relations are found which connect the invariant variables of an amplitude for a Feynman diagram with n free lines ("n-point" function) and which allow one to express all possible invariant variables in terms of the $3n - 10$ independent ones. It is shown that these relations reduce to a linear system of kinematic conditions, arising from the energy-momentum conservation law, and (for $n \geq 6$) to the additional system of geometrical conditions connected with the four-dimensionality of space. It is shown what the choice of independent variables should be for the n-point function for arbitrary n; in particular a choice of independent variables (with adjacent subscripts) for which the geometrical conditions (generally speaking, a system of equations of fifth degree) reduce to a set of quadratic equations with respect to any of the variables is analyzed in detail. A graphical method for obtaining the necessary relations among the invariants is described.

THE study of Lorentz-invariant amplitudes, corresponding to Feynman diagrams with n free external lines ("n-point" functions) begins with the choice of independent invariant variables. These may be chosen to be the so-called kinematic invariants: double— $s_{ik} = (p_i + p_k)^2$, triple— $s_{ikl} = (p_i + p_k + p_l)^2$, etc., which are scalar quadratic combinations of the 4-momenta p_i corresponding to the free lines of the diagram ($p_i^2 = m_i^2$). As is well known, for the n-point function the number of such independent invariants is equal to $r = 3n - 10$.

In the usually studied simplest case of the 4-point function[1] ($r = 2$) the choice of the independent invariants is accomplished trivially: any pair of the three double invariants s_{12}, s_{23}, s_{13} (connected by a single linear relation) can be chosen as the independent variables; it is obvious that the three remaining invariants s_{34}, s_{14}, s_{24} (out of the total number $C_4^2 = 6$ of double invariants) coincide respectively with s_{12}, s_{23}, s_{13}, as a consequence of the conservation law $p_1 + p_2 + p_3 + p_4 = 0$. In the case of the 5-point function[2] ($r = 5$) the choice of the five independent invariants from among the ten ($C_5^2 = 10$) double invariants (triple invariants are equivalent to double invariants as a consequence of the conservation law $p_1 + p_2 + p_3 + p_4 + p_5 = 0$) is also largely arbitrary, provided only that in fixing the five independent invariants one does not consider those quadruple combinations of three double invariants that constitute a triple invariant, simultaneously with the double invariant identical with the indicated triple invariant; for example s_{12}, s_{23}, s_{13} (s_{123}) and s_{45} (see Appendix).

However, beginning with the 6-point function (for $n \geq 6$) the freedom in the choice of independent variables disappears and the picture becomes drastically more complex as a consequence of the four-dimensionality of space.

It is the aim of this article to show how the choice of independent variables should be accomplished for the n-point function for arbitrary n, as well as to establish the form of the conditions (kinematic and geometric) which connect the invariants to each other and permit one to obtain their range of variation.

1. CHOICE OF VARIABLES FOR THE n-POINT FUNCTION ($n \geq 6$)

Along with the kinematic invariants s_{ik}, s_{ikl}, s_{iklm}, etc. it is convenient to also consider the corresponding scalar products of the 4-momenta

$$(ik) \equiv 2(p_i p_k) = s_{ik} - m_i^2 - m_k^2 \qquad (1)$$

and their linear combinations

$$(ikl) \equiv 2(p_i p_k) + 2(p_i p_l) + 2(p_k p_l) = s_{ikl} - m_i^2 - m_k^2 - m_l^2,$$

$$(iklm) \equiv 2(p_i p_k) + 2(p_i p_l) + 2(p_i p_m) + 2(p_k p_l) + 2(p_k p_m) + 2(p_l p_m) = s_{iklm} - m_i^2 - m_k^2 - m_l^2 - m_m^2$$

etc., which are single-valuedly related to the kinematic invariants of corresponding multiplicity and which, as is easy to see, transform very simply into each other (with a lowering in the multiplicity):

$$(ikl) = (ik) + (il) + (kl),$$

$$(iklm) = (ik) + (il) + (im) + (kl) + (km) + (lm) = (ikl) + (im) + (km) + (lm) = (ikl) + (ikm) + (lm) - (ik)$$

etc.; this makes it possible to use in establishing relations among the invariants a simplifying graphical technique (see Appendix).

Since any invariant of higher multiplicity (triple, quadruple, etc.) can be reduced by elementary means to an aggregate of double invariants it is meaningful to start the analysis of invariant variables for the n-point function with the double invariants, whose number is given by $C_n^2 = n(n-1)/2$

A connection between the double invariants of the n-point function arises as a consequence of the conservation law

$$\sum_{i=1}^{n} p_i = 0, \qquad (2)$$

which can be transformed by means of scalar multiplication by $2p_k$ into a system of n linear equations for the double invariants (kinematic conditions):

$$\sum_{i=1}^{n} (ik) = 0, \qquad k = 1, 2 \ldots n, \qquad (3)$$

where $(ii) = 2m_i^2$. Consequently, the number of kinematically unrelated double invariants is equal for the n-point function to $p = C_n^2 - n = (n^2 - 3n)/2$. For the 4- and 5-point functions this number is equal to $r = 3n - 10$, the number of independent invariants $(p = r)$. Starting with the 6-point function, however, the number of double invariants unrelated by the kinematic conditions (3) becomes larger than the number of independent invariants by the quantity $q = p - r = (n-5)(n-4)/2$. This latter circumstance is due to the fact that in a four-dimensional space no more than four vectors can be linearly independent (already in the case of the 6-point function there is only one linear condition (2) to be imposed on the six vectors p_i, i.e., one would have five linearly independent vectors).

Without loss of generality we shall choose from among the n vectors of the n-point function as linearly independent the following four vectors (basis): p_1, p_2, p_3, p_4. Then on each quintuplet of vectors p_1, p_2, p_3, p_4, p_l $(l = 5, 6, \ldots n-1)$ which characterize the n-point function [the vector p_n is excluded by means of the conservation law (2)] there must be imposed in a 4-dimensional space the conditions of linear dependence (where $\alpha_i \neq 0$)

$$\sum_{i}^{1-4, l} \alpha_i p_i = 0, \qquad l = 5, 6 \ldots n-1, \qquad (4)$$

which, after scalar multiplication by $2p_k$ $(k = 1, 2, \ldots n-1)$, go over into a system of linear equations for the quantities α_i:

$$\sum_{i}^{1-4, l} (ik)\alpha_i = 0; \qquad k = 1, 2 \ldots n-1, \qquad l = 5, 6 \ldots n-1. \qquad (5)$$

The coefficients (ik) of this system of equations form a symmetric $(n-1)^2$ matrix of double invariants for the n-point function:

$$\left.\begin{array}{cccc|ccc}
2m_1^2 & (12) & (13) & (14) & (15) & \ldots & (1, n-1) \\
(12) & 2m_2^2 & (23) & (24) & (25) & \ldots & (2, n-1) \\
(13) & (23) & 2m_3^2 & (34) & (35) & \ldots & (3, n-1) \\
(14) & (24) & (34) & 2m_4^2 & (45) & \ldots & (4, n-1) \\
\hline
(15) & (25) & (35) & (45) & 2m_5^2 & \ldots & (5, n-1) \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
(1, n-1) & (2, n-1) & (3, n-1) & (4, n-1) & (5, n-1) & \ldots & 2m_{n-1}^2
\end{array}\right. ,$$

where the singled-out left upper corner consists of invariants made out of the four basis vectors p_1, p_2, p_3, p_4. For the determinant of this four-by-four matrix we have $\Delta_4(1, 2, 3, 4) \equiv |(ik)| \neq 0$ $(i, k \geq 4)$ (which is the condition that the vectors p_1, p_2, p_3, p_4 are linearly independent).

In order that the system of equations (5) have nontrivial solutions for the α_i one must require the vanishing of all determinants of fifth order (they are usually referred to as the Gramm determinants [3]), that can be constructed from the symmetric matrix of invariants by the addition of rows and columns to the fourth-order determinant $\Delta_4(1, 2, 3, 4)$. One obtains in this way $n-5$ conditions with symmetrical Gramm determinants of fifth order $(l = 5, 6, \ldots n-1)$

$$\Delta_5(1, 2, 3, 4, l) \equiv \begin{vmatrix}
2m_1^2 & (12) & (13) & (14) & (1l) \\
(12) & 2m_2^2 & (23) & (24) & (2l) \\
(13) & (23) & 2m_3^2 & (34) & (3l) \\
(14) & (24) & (34) & 2m_4^2 & (4l) \\
(1l) & (2l) & (3l) & (4l) & 2m_l^2
\end{vmatrix} = 0 \qquad (6)$$

and $(n-6)(n-5)/2$ conditions with "nonsymmetrical" Gramm determinants of fifth order $(l \neq k;$ $l, k = 5, 6, \ldots n-1)$

$$\Delta_5(1, 2, 3, 4, kl) \equiv \begin{vmatrix} 2m_1^2 & (12) & (13) & (14) & (1k) \\ (12) & 2m_2^2 & (23) & (24) & (2k) \\ (13) & (23) & 2m_3^2 & (34) & (3k) \\ (14) & (24) & (34) & 2m_4^2 & (4k) \\ (1l) & (2l) & (3l) & (4l) & (lk) \end{vmatrix} = 0. \quad (7)$$

[We remark that in the matrix and in the determinants one can include the invariants (in), i = 1, 2, ... n, which contain the vector p_n, and then accomplish their elimination with the help of Eq. (3); the fifth-order Gramm determinants corresponding to these invariants turn out to be linear combinations of the determinants (6) and (7).]

The aggregate of conditions (6) and (7) is equivalent to the condition (4) of linear dependence and represents the additional conditions connecting the double invariants, which in distinction to the kinematic conditions (3) arising from the conservation law (2) will be called geometrical conditions (as they are a consequence of our use of four-dimensional geometry). The number of geometrical conditions, as is easy to see from the symmetric matrix of invariants, is equal to $n-5+(n-6)(n-5)/2$, which coincides with the number $q = (n-5)(n-4)/2$.

In this way the n kinematic conditions (3) and the q geometrical conditions (6) and (7) leave from the total number $n(n-1)/2$ of double invariants only the required number $r = n(n-1)/2 - n - q = 3n - 10$ of independent invariant variables of the n-point function (for $n \geq 6$), whose choice however is not yet fully determined.

Before passing to the discussion of invariant variables for specific n-point functions let us remark that the nonsymmetric choice of p_1, p_2, p_3, p_4 as our basis vectors, as well as the exclusion of p_n from the relations (4) and (5), should not be thought of as restrictions on the formalism; by means of the symmetric with respect to all the vectors $p_1, p_2, \ldots p_n$ conservation laws (2) and linear relations (3) one may transform to any other basis p_α, p_β, p_γ, p_δ with $\Delta_4(\alpha, \beta, \gamma, \delta) \neq 0$ and exclude from Eqs. (4) and (5) any vector p_τ —the relations obtained in this manner turn out to be equivalent to those given above. In the following this will be illustrated in specific examples of n-point functions (n = 6, 7, and 8).

2. THE 6-POINT FUNCTION

In the case of the 6-point function (see Fig. 1) there are eight invariant variables (r = 8). The total number of double invariants, composed of the six vectors p_i, is equal to $(6 \cdot 5)/2 = 15$. On

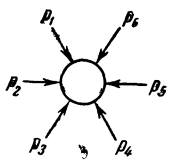

FIG. 1.

the 15 invariants we impose 6 kinematic conditions

$$\sum_{i=1}^{6} (ik) = 0, \qquad k = 1, 2, \ldots 6 \qquad (3')$$

and one geometrical condition

$$\Delta_5(1, 2, 3, 4, 5) = 0. \qquad (6')$$

It should be noted that as a consequence of the kinematic conditions (3′) all fifth-order Gramm determinants for the 6-point function are equal to each other:

$$\Delta_5(1, 2, 3, 4, 5) = -\Delta_5(1, 2, 3, 4; 56) = \Delta_5(1, 2, 3, 4, 6) = \ldots = \Delta_5(2, 3, 4, 5, 6),$$

where the transformation is accomplished by adding all rows (columns) to any one row (column) being transformed and replacing (with the help of the kinematic conditions) the sum of the elements of the determinant by the new element with the opposite sign. In this way the vanishing of the one determinant $\Delta_5(1, 2, 3, 4, 5)$ results in the vanishing of all the other determinants Δ_5 and, consequently, the choice of the basis as well as the exclusion of one linearly dependent vector from the geometrical condition are arbitrary in the case of the 6-point function.

The choice of independent invariant variables from among the total number of double invariants is best accomplished in two stages: one first picks 9 kinematically unrelated invariants and expresses the 6 remaining invariants in terms of the picked ones by means of the kinematic relations (3′); one then imposes on the 9 kinematically unrelated invariants the geometrical condition (6′), symmetric with respect to all invariants, thus reducing the number of independent variables to the required 8.

It is easy to show that not for any choice of the 9 kinematically unrelated invariants from among the 15 double invariants can the system of equations (3′) be uniquely solved for the 6 remaining invariants, because the determinant of the system may turn out to be zero (in that case the system of equations is undetermined and possesses an infinity of solutions). The system of equations (3′)

has a nonvanishing determinant only for a symmetric choice of the 9 kinematically unrelated double invariants, in particular for invariants with adjacent subscripts (s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}) and with subscripts differing by three units (s_{14}, s_{25}, s_{36}). In that case the solution of the system of equations (3′) results in linear expressions for the remaining 6 invariants (s_{13}, s_{24}, s_{35}, s_{46}, s_{51}, s_{62}), with each of the indicated invariants expressed simultaneously in terms of all the 9 picked invariants. As a consequence of this the geometrical condition (6′) turns out to be an algebraic equation of fifth order with respect to any one of the 9 kinematically unrelated invariants; its solution, necessary for the expression of one of these invariants in terms of the rest (to determine the range of variation), presents considerable difficulties. It is characteristic, however, that in the formulas under consideration there appear just such linear combinations of double invariants that correspond to triple invariants. For this reason it becomes convenient to consider in addition to double invariants also the triple invariants s_{ikl} when making the choice of the 9 kinematically unrelated invariants for the 6-point function.

In the case of the 6-point function the number of triple invariants is $C_6^3 = 20$; however, as a consequence of the conservation law $\sum_{i=1}^{6} p_i = 0$, it is sufficient to consider only 10 triple invariants since the remaining 10 are equal to the first 10 in pairs ($s_{ikl} = s_{mnp}$). Together with the 15 double invariants we thus have a total of 25 invariants from which to choose the 9 kinematically unrelated ones. The remaining 16 invariants are connected to the picked ones by means of 16 linear relations, 6 of which correspond to the Eqs. (6′) for the double invariants and the other 10 correspond to the transformation equations that express the triple invariants in terms of the picked double (and triple) invariants. In order to establish the necessary 16 relations it is convenient to make use of the graphical technique introduced in the Appendix.

The choice of the kinematically unrelated invariants may now be accomplished in the following symmetric manner: only double and triple invariants with adjacent subscripts are picked— s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}, $s_{123} \equiv s_{456}$, $s_{234} \equiv s_{561}$, $s_{345} \equiv s_{612}$. Then the relations expressing the remaining 16 invariants in terms of the kinematically unrelated ones have the form (see Appendix)

$$(13) = (123) - (12) - (23), \qquad (46) = (456) - (45) - (56),$$
$$(26) = (126) - (12) - (16), \qquad (35) = (345) - (34) - (45),$$
$$(15) = (156) - (16) - (56), \qquad (24) = (234) - (23) - (34)$$
$$(36) = s_{45} + s_{12} - s_{123} - s_{126},$$
$$(14) = s_{23} + s_{56} - s_{156} - s_{456},$$
$$(25) = s_{34} + s_{16} - s_{126} - s_{156},$$
$$s_{134} = s_{56} + s_{34} - s_{234} - (12), \quad s_{146} = s_{23} + s_{16} - s_{156} - (45),$$
$$s_{136} = s_{45} + s_{16} - s_{126} - (23), \quad s_{125} = s_{34} + s_{12} - s_{126} - (56),$$
$$s_{124} = s_{56} + s_{12} - s_{123} - (34), \quad s_{236} = s_{45} + s_{23} - s_{123} - (16). \tag{8}$$

The additional geometrical condition (6′) for the 9 kinematically unrelated invariants can now be written with the help of Eq. (8) in a form suitable for solution for any one of the picked invariants, for example s_{12}, namely

$$\left|\begin{array}{cc|ccc} 2m_1^2 & (12) & [13] & (14) & (15) \\ (12) & 2m_2^2 & 2m_2^2 + (23) & (24) & (25) \\ \hline [13] & 2m_2^2 + (23) & 2m_2^2 + 2m_3^2 + 2\,(23) & (24) + (34) & (25) + (35) \\ (14) & (24) & (24) + (34) & 2m_4^2 & (45) \\ (15) & (25) & (25) + (35) & (45) & 2m_5^2 \end{array}\right|$$
$$= a\,(12)^2 + b\,(12) + c = 0, \tag{9}$$

where $[13] = (13) + (12)$, and where the invariants (13), (14), (15), (24), (25), (35) must be expressed in terms of the picked invariants by means of Eq. (8). At that the determinant in (9) has been so transformed that the desired invariant s_{12} appears only in the left upper corner, so that we get a quadratic equation with respect to s_{12} (in an analogous fashion one obtains quadratic equations for any of the other picked invariants).

The solution of Eq. (9) may be simply found with the help of the following determinant identity[3,4]:

$$|(ik)| \cdot \underset{(i,\,k)}{|(ik)|} = \begin{vmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{vmatrix}, \tag{10}$$

where $|(ik)| = a\,(12)^2 + b(12) + c$ is the determinant appearing on the left side of Eq. (9), and A_{ik} is the cofactor of the i,k entry of the matrix (ik). It is easy to see that $a = -\underset{(i,k)>2}{|(ik)|}$ and $b = -2\,[A_{12} + (12)\underset{(i,k)>2}{|(ik)|}]$, whereas directly from the identity (10) we find for the discriminant of Eq. (9)

$$b^2 - 4ac = 4A_{11}A_{22}.$$

As a result we obtain for the picked invariant s_{12} under discussion the two geometrical conditions

$$(12)_{1,2} = [-b \pm 2\sqrt{A_{11}A_{22}}]/2a, \tag{11}$$

characterizing its two ranges of variation. (Analogous relations may be obtained for any of the other chosen invariants.)

3. THE 7-POINT FUNCTION

The 7-point function (see Fig. 2) depends on 11 independent invariant variables ($r = 11$). The total number of double invariants composed of the

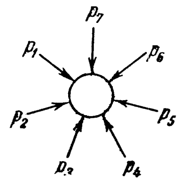

FIG. 2

seven vectors p_i is equal to $C_7^2 = 21$. They are connected by the seven kinematic conditions

$$\sum_{i=1}^{7} (ik) = 0, \qquad k = 1, 2 \ldots 7 \qquad (3'')$$

and the three geometrical conditions

$$\Delta_5(1, 2, 3, 4, 5) = 0, \quad \Delta_5(1, 2, 3, 4, 6) = 0, \quad \Delta_5(1, 2, 3, 4; 56) = 0. \qquad (6'')$$

The Eqs. (6″) arise from the two relations of linear dependence of the vectors

$$\sum_{l}^{1-4,\,5} \alpha_l p_l = 0, \qquad \sum_{k}^{1-4,\,6} \beta_k p_k = 0,$$

written for the basis p_1, p_2, p_3, p_4 with $\Delta_4(1, 2, 3, 4) \neq 0$. It is easy to show that any other pair of relations of linear dependence for the vectors formulated in terms of another basis can be reduced to the indicated pair of relations with the help of linear operations and the conditions (3″) and, consequently, the corresponding geometrical conditions are equivalent to the conditions (6″).

The choice of invariant variables proceeds in accordance with the principles formulated in the case of the 6-point function. The first stage of choosing 14 kinematically unrelated invariants must involve from the beginning invariants of higher multiplicity, namely triple invariants (quadruple invariants reduce in the case of the 7-point function to triple invariants because of the conservation law $\sum_{i=1}^{7} p_i = 0$), in order to insure that the geometrical conditions (6″)—as in the case of the 6-point function—be quadratic equations with respect to the picked invariants. The number of triple invariants is equal in the case of the 7-point function to $C_7^3 = 35$. From among the 56 double and triple invariants we choose 14 invariants with adjacent subscripts: 7 double

$$s_{12}, \quad s_{23}, \quad s_{34}, \quad s_{45}, \quad s_{56}, \quad s_{67}, \quad s_{71}$$

and 7 triple

$$s_{123}, \quad s_{234}, \quad s_{345}, \quad s_{456}, \quad s_{567}, \quad s_{671}, \quad s_{712}.$$

The problem of expressing the remaining 42 invariants in terms of the chosen 14, as well as the problem of subsequently solving the system of three quadratic equations (6″), is resolved by the same methods as were used in the case of the 6-point function (see also Appendix). Let us remark here that each of the three equations (6″) depends simultaneously on all kinematically unrelated invariants (the invariants are not distributed among the three different conditions), i.e., the geometrical conditions do not violate the symmetry in the choice of invariants.

4. THE 8-POINT FUNCTION

The 8-point function (see Fig. 3) depends on 14 independent invariant variables ($r = 14$). The total

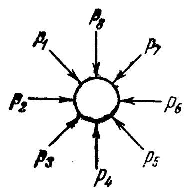

FIG. 3

number of double invariants composed of the eight vectors p_i is equal to $C_8^2 = 28$. They are connected by the eight kinematic conditions

$$\sum_{i=1}^{8} (ik) = 0 \qquad (k = 1, 2 \ldots 8), \qquad (3''')$$

and the six geometric conditions

$$\Delta_5(1, 2, 3, 4, 5) = 0, \qquad \Delta_5(1, 2, 3, 4, 6) = 0,$$
$$\Delta_5(1, 2, 3, 4, 7) = 0, \qquad \Delta_5(1, 2, 3, 4; 56) = 0,$$
$$\Delta_5(1, 2, 3, 4; 57) = 0, \qquad \Delta_5(1, 2, 3, 4; 67) = 0. \qquad (6''')$$

Following the procedure used in the case of the 7-point function one can show that the choice of basis p_1, p_2, p_3, p_4 with $\Delta_4(1, 2, 3, 4) \neq 0$ for the three equations of linear dependence

$$\sum_{i}^{1-4,5} \alpha_i p_i = 0, \qquad \sum_{k}^{1-4,\,6} \beta_k p_k = 0, \qquad \sum_{l}^{1-4,\,7} \gamma_l p_l = 0$$

imposes no restrictions in the case of the 8-point function, and an equivalent transition to any other basis can be performed.

Along with the double invariants one considers in the case of the 8-point function simultaneously 56 triple invariants and 35 quadruple invariants (half of the total number of 70 quadruple invariants is equal in pairs to the other half, $s_{iklm} = s_{pqrs}$, as a consequence of the conservation law $\sum_{i=1}^{8} p_i = 0$). In the first stage one chooses 20 kinematically unrelated invariants with adjacent subscripts: 8 double

$$s_{12},\quad s_{23},\quad s_{34},\quad s_{45},\quad s_{56},\quad s_{67},\quad s_{78},\quad s_{81},$$

8 triple

$$s_{123},\quad s_{234},\quad s_{345},\quad s_{456},\quad s_{567},\quad s_{678},\quad s_{781},\quad s_{812}$$

and 4 quadruple

$$s_{1234} \equiv s_{5678},\qquad s_{2345} \equiv s_{6781},\qquad s_{3456} \equiv s_{7812},\qquad s_{4567} \equiv s_{8123}.$$

The remaining 99 invariants are expressed by elementary means in terms of the 20 chosen ones with the help of the graphical methods (see Appendix). The geometrical conditions (6‴) reduce to quadratic equations with respect to any one of the chosen invariants in the same way as was discussed in the case of the 6-point function.

5. THE n-POINT FUNCTION

For the n-point function (see Fig. 4) the number of independent invariant variables is equal to $r = 3n - 10$, with the total number of double invariants equal to $C_n^2 = n(n-1)/2$. The double invariants are connected by the n kinematic conditions (3) and the $q = (n-5)(n-4)/2$ geometrical conditions (6) and (7), formulated in an arbitrary basis p_α, p_β, p_γ, p_δ from among the n vectors of the n-point function, in particular the basis p_1, p_2, p_3, p_4.

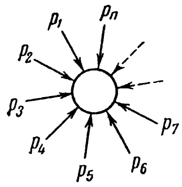

FIG. 4

The choice of $p = (n^2 - 3n)/2$ kinematically unrelated invariants proceeds according to the established principle of utilizing invariants with adjacent subscripts with invariants of higher multiplicity taken into account: triple, quadruple, etc., up to and including invariants of n/2 multiplicity (for even n) or $(n-1)/2$ multiplicity (for odd n). At that among the p chosen invariants there appear n triple invariants, n quadruple, etc., and finally n invariants of multiplicity n/2 or $(n-1)/2$ (let us note, however, that in the case of even n only half, namely n/2, of the invariants of n/2 multiplicity need be chosen, the remaining half being equivalent to the first half because of the relations $s_{ik\ldots m} = s_{pq\ldots r}$ which follow from the conservation law $\sum_{i=1}^{n} p_i = 0$). The remaining invariants are expressed in terms of the chosen p by elementary means based on the graphical method (see Appendix), and the geometrical conditions form —for the indicated choice of invariants—a system of q quadratic equations with respect to any one of the chosen invariants.

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APPENDIX

The construction of relations, connecting the kinematic invariants of the many-point function with each other and expressing unknown invariants in terms of a group of given invariants, proceeds on the basis of the conservation law $\sum_{i=1}^{n} p_i = 0$, which, as is easy to see, gives rise to a series of equalities of the form

$$\underbrace{s_{ik\ldots m}}_{n-\nu} = \underbrace{s_{pq\ldots r}}_{\nu},\qquad \nu = 0, 1, 2, \ldots n-1;$$

at that invariants of higher multiplicity $s_{ik\ldots m}$ should be decomposed in terms of invariants of lower multiplicity $s_i = m_i^2$, s_{ik}, s_{ikl}, etc. The choice of convenient equations for the unknown invariants and the appropriate decomposition of invariants of higher multiplicity in terms of invariants of lower multiplicity is easily accomplished in the following manner.

We shall represent the n-multiple invariant

$$\overbrace{(iklmp\ldots r)}^{n} = s_{iklmp\ldots r} - M^2_{iklmp\ldots r},$$

$$M^2_{iklmp\ldots r} = m_i^2 + m_k^2 + m_l^2 + m_m^2 + m_p^2 + \ldots + m_r^2$$

by a polygon with all diagonals (see Fig. 5), which can be arbitrarily decomposed into its geometrical elements (segments, triangles, quadrangles, etc.) subject to the condition that no line in the polygon should appear more than once in the decomposition elements and, at the same time, that

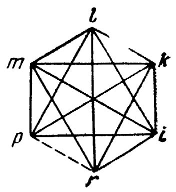


FIG. 5

no new lines should appear (or, if new lines do appear then they should be compensated). The various geometrical elements correspond to specific invariants: segments correspond to double invariants, triangles to triple invariants, etc., so that as a result we get all possible decompositions of the n-multiple invariant in terms of invariants of lower multiplicity.

To clarify the procedure we consider specific examples.

4-point function. From the conservation law $\sum_{i=1}^{4} p_i = 0$ follow three equations of the form $s_{ik} = s_{lm}$, establishing the equivalence between three pairs of the six double invariants, and four equations of the form $s_{klm} = m_i^2$ [or $(klm) = m_i^2 - M^2_{klm}$], which represent the sought for relations. If we take for the three nonidentical invariants s_{12}, s_{23}, and s_{13} [or correspondingly (12), (23), and (13)] whose subscripts are formed from the three numbers 1, 2, 3, then with the help of the graphical decomposition

$$(klm) \to {}_k\overset{l}{\triangle}_m = |^l_k + |^m_k + |^m_l \to (kl) + (km) + (lm) \quad \text{(A.1)}$$

we obtain just one relation connecting these invariants:

$${}_1\overset{2}{\triangle}_3 \to (12) + (23) + (13) = m_4^2 - M^2_{123}$$

or

$$s_{12} + s_{23} + s_{13} = M^2_{1234} \equiv \sum_{i=1}^{4} m_i^2.$$

5-point function. The conservation law $\sum_{i=1}^{5} p_i = 0$ leads to 10 equations of the form $s_{ikl} = s_{mn}$ [or $(ikl) = s_{mn} - M^2_{ikl}$], which reduce triple invariants to double in the case of the 5-point function and which at the same time establish with the help of Eq. (A.1) 10 symmetric relations among the double invariants:

$$(ik) + (il) + (kl) = s_{mn} - M^2_{ikl} \text{ or } s_{ik} + s_{il} + s_{kl} = s_{mn} + M^2_{ikl}.$$

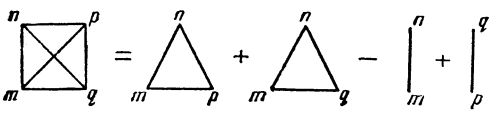
The choice of the 5 (from the total number 10) double invariants is arbitrary provided that one excludes from consideration those quadruple combinations of double invariants that appear in the above relations; at that for any five chosen invariants one can uniquely establish a system of five equations (from among the 10 indicated relations) with the help of which the remaining 5 invariants can be expressed in terms of the chosen ones. In particular, if in accordance with the general principle we choose as independent the invariants with adjacent subscripts s_{12}, s_{23}, s_{34}, s_{45}, s_{51}, or correspondingly (12), (23), (34), (45), (51), then the relations expressing the remaining invariants in terms of the independent ones are as follows (the desired quantities are underlined):

$${}_1\overset{2}{\triangle}_3 \to (12) + (23) + \underline{(13)}$$
$$= s_{45} - M^2_{123} \text{ or } s_{12} + s_{23} + \underline{s_{13}} = s_{45} + M^2_{123},$$
$${}_2\overset{3}{\triangle}_4 \to (23) + (34) + \underline{(24)}$$
$$= s_{51} - M^2_{234} \text{ or } s_{23} + s_{34} + \underline{s_{24}} = s_{51} + M^2_{234},$$
$${}_3\overset{4}{\triangle}_5 \to (34) + (45) + \underline{(35)}$$
$$= s_{12} - M^2_{345} \text{ or } s_{34} + s_{45} + \underline{s_{35}} = s_{12} + M^2_{345},$$
$${}_4\overset{5}{\triangle}_1 \to (45) + (51) + \underline{(14)}$$
$$= s_{23} - M^2_{451} \text{ or } s_{45} + s_{51} + \underline{s_{14}} = s_{23} + M^2_{451},$$
$${}_5\overset{1}{\triangle}_2 \to (51) + (12) + \underline{(25)}$$
$$= s_{34} - M^2_{512} \text{ or } s_{51} + s_{12} + \underline{s_{25}} = s_{34} + M^2_{512}.$$

6- and 7-point functions. For n equal to 6 and 7 one makes use of both triple and double invariants. For this reason the series of relations for the nonindependent double invariants is obtained by decomposition of the independent triple invariants in terms of double invariants with the help of Eq. (A.1). In particular we have 6 (for the 6-point function) and 7 (for the 7-point function) relations of the form

$${}_1\overset{2}{\triangle}_3 \to (123) = (12) + (23) + \underline{(13)},$$
$${}_2\overset{3}{\triangle}_4 \to (234) = (23) + (34) + \underline{(24)}$$

etc. The remaining relations for the nonindependent double invariants follow from the conservation law $\sum_i p_i = 0$: $s_{ik} = s_{mnpq}$ [or $(ik) = (mnpq) + M^2_{mnpq} - M^2_{ik}$] for the 6-point function, and $s_{ikl} = s_{mnpq}$ [or $(ikl) = (mnpq) + M^2_{mnpq} - M^2_{ikl}$] for the 7-point function. At that the right side gets decomposed according to the graphical relation

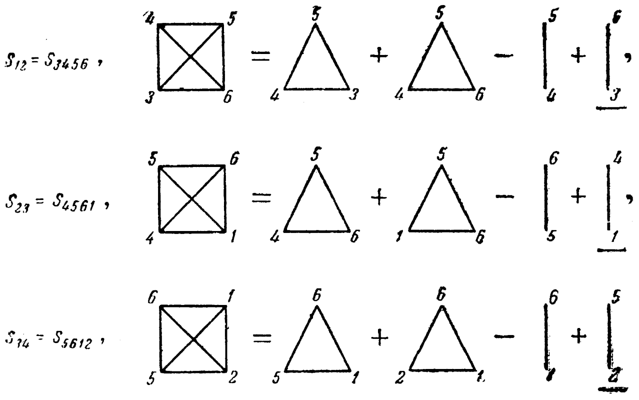


(A.2)

which corresponds to the decomposition of the quadruple invariant (mnpq) into invariants of lower multiplicity:

$$(mnpq) = (mnp) + (mnq) - (mn) + (qp)$$

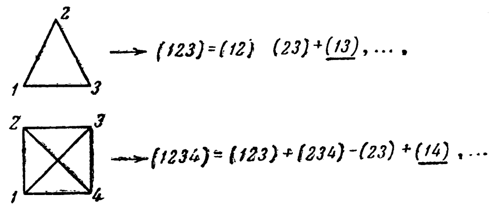
[the term (mn) with a minus sign comes about as a consequence of the need to compensate one of the segments $|^n_m$, which appears in two triangles on the right side of Eq. (A.2)]. In this way we obtain the three relations for the 6-point function:



(A.3)

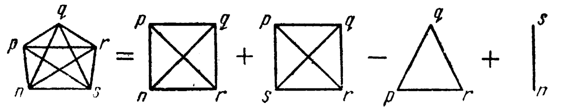
[see Eq. (8) for the invariants (36), (14), and (25)] and seven relations of the form $s_{123} = s_{4567}$, $s_{234} = s_{5671}$, etc., with analogous decompositions of the "squares" for the 7-point function. As a result all nonindependent invariants are expressed in terms of the independent ones. The nonindependent triple invariants can be expressed by means of Eq. (A.1) as a sum of nonindependent double invariants, whose relation to the independent invariants is already known [it was in this way that the expressions for the invariants s_{134}, s_{136}, s_{124}, s_{146}, s_{125}, s_{236} in Eq. (8) were obtained].

8- and 9-point functions. For n equal to 8 and 9 one makes use of double, triple, and quadruple invariants. Consequently we have directly from the decomposition of the independent triple and quadruple invariants, according to the Eqs. (A.1) and (A.2), 16 (for the 8-point function) and 18 (for the 9-point function) relations for the nonindependent double invariants of the form



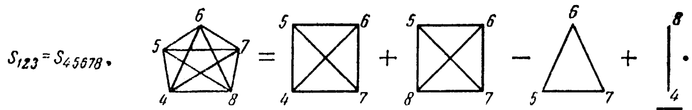
(A.4)

The remaining nonindependent double invariants are determined with the help of the relations $s_{ikl} = s_{npqrs}$ [or $(ikl) = (npqrs) + M^2_{npqrs} - M^2_{ikl}$] for the 8-point function, and $s_{iklm} = s_{npqrs}$ [or $(iklm) = (npqrs) + M^2_{npqrs} - M^2_{iklm}$] for the 9-point function, where on the left we have all possible independent invariants and on the right we have quintuple invariants with adjacent subscripts, which are decomposed according to the graphical relation



(A.5)

or (npqrs) = (npqr) + (pqrs) − (pqr) + (ns). For example, the invariant (48) is determined in the case of the 8-point function from the relation



(A.6)

The nonindependent triple and quadruple invariants are expressed in terms of independent invariants by decomposition in terms of known double (and triple) invariants using Eqs. (A.1) and (A.2).

n-point function. In accordance with the above established principle one determines first all nonindependent double invariants in terms of all the independent invariants by making use of the decompositions of the form (A.1), (A.2), (A.5), etc., of the independent invariants of higher multiplicity, and by making use of the equations arising from the conservation law $\sum_{i=1}^{n} p_i = 0$. Thereafter all nonindependent invariants of various multiplicities are decomposed in terms of independent invariants and already known nonindependent double invariants (see the cases of the 6-, 7-, 8-, and 9-point functions).

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86