GROWTH OF FLUCTUATIONS IN A PLASMA WITH AN UNSTABLE DISTRIBUTION FUNCTION

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The growth of fluctuations is investigated in plasma characterized by an unstable distribution function (primarily systems consisting of a plasma and a beam of charged particles). The expectation values for the amplitudes of the growing oscillations excited by the beam passing through the plasma are determined, as are the correlation functions for various physical quantities (electric field, charge density of the plasma system, density and velocity of the beam). These correlation functions and the squares of the oscillation amplitudes contain terms that grow exponentially with time; the exponential is proportional to $N_B^{1/2}$ and is multiplied by a factor that is also proportional to $N_B^{1/2}$ (N_B is the beam density). If the beam velocity is high compared with the mean thermal velocity of the plasma electrons a resonance arises between the plasma oscillations excited by the beam and the Langmuir oscillations of the plasma. In this case the correlation functions grow with a factor proportional to $N_B^{1/3}$ but the exponential is multiplied by a factor that is independent of beam density.

1. It is well known that a plasma through which a stream of charged particles moves is unstable: in contrast with the fluctuations in a stable system, in such a system the fluctuations can increase with time. The frequencies of these "driven" oscillations and their growth rates have been determined by A. Akhiezer and Faĭnberg[1] and by Bohm and Gross.[2]

However, in addition to knowing the frequencies of these oscillations it is important to know the most probable values of the oscillation amplitudes and certain more general quantities—the average (over the fluctuations) products of various physical quantities measured at times that are not necessarily coincident. These averaged products can be called correlation functions. In particular, the correlation function for a given physical quantity at coincident times is the mean square of the oscillation amplitude of this quantity.

The present work is devoted to an analysis of the correlation functions in a plasma characterized by an unstable distribution function, primarily a plasma and a compensated beam of charged particles.

The instability in such a system can be analyzed only for a bounded time interval; hence, to describe this system we must assign the initial conditions that characterize the various physical properties. These initial values are obviously random quantities and some averaging process must be used to obtain their correlation functions. An extremely important feature in this connection is the fact that the averaged product of the fluctuations of the plasma particle distribution function can be given as a sum of a singular term, which describes the "autocorrelation" of a particle, and a smoothly varying term [cf. (11)], whose actual form is unimportant in certain cases. In particular, when the plasma is traversed by a beam of sufficiently low density the contribution of the smooth term in the correlation function is proportional to the beam density to a higher power than the contribution of the singular term. Hence, one can obtain the correlation function in a beam-plasma system without assuming that the fluctuations at the initial time are of equilibrium nature.

2. We determine the distribution function F_a for particles of the a-th species, characterized by mass m_a, charge ez_a and mean velocity $\mathbf{u}^a$, using the kinetic equation, neglecting the collision integral

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right) F_a(\mathbf{v}, \mathbf{r}, t) + \frac{ez_a}{m_a}\frac{dF_a^0(\mathbf{v}-\mathbf{u}^a)}{d\mathbf{v}}\mathbf{E}(\mathbf{r}, t) = 0 \tag{1}$$

with the initial conditions

$$F_a(\mathbf{v}, \mathbf{r}, 0) - F_a^0(\mathbf{v}-\mathbf{u}^a) = g_a(\mathbf{v}, \mathbf{r}),$$

where F_a^0 is the distribution function, averaged over the fluctuations, and $\mathbf{E}$ is the self consistent field, related to the functions F_a by the equations

$$\operatorname{div} \mathbf{E} = 4\pi e \sum_a z_a \int d\mathbf{v} F_a(\mathbf{v}), \qquad \operatorname{rot} \mathbf{E} = 0 \qquad (2)^*$$

(the particle velocities are assumed to be non-relativistic).

By taking the Laplace transform (in time) we can express the particle distribution functions and the various physical quantities determined by these functions (density and mean velocity of particles of the a-th species, charge density, electric field) at time t in terms of the fluctuations of the distribution functions g. For a given Fourier spatial component of the charge density we find

$$\rho(\mathbf{k}, t) = \int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\varepsilon(\mathbf{k}, \omega)} ie \sum_a z_a \int d\mathbf{v} \frac{g_a(\mathbf{v}, \mathbf{k})}{\omega - \mathbf{kv}}, \qquad (3)$$

where ϵ is the dielectric permittivity of the plasma

$$\varepsilon(\mathbf{k}, \omega) = 1 + 4\pi \sum_a \varkappa_a(k, \omega - \mathbf{k}\mathbf{u}^a) \qquad (4)$$

and κ_a is the electrical susceptibility of the a-th component

$$\varkappa_a(k, \omega) = \frac{e^2 z_a^2}{k^2 m_a} \int \frac{\mathbf{k}}{\omega - \mathbf{kv} + i0} \frac{dF_a^0(v)}{dv} d\mathbf{v}. \qquad (5)$$

The density fluctuations of particles of the a-th species are given by a similar expression:

$$\delta\rho_a(\mathbf{k}, t) \equiv e z_a \delta N_a(\mathbf{k}, t)$$
$$= \int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\varepsilon(\mathbf{k}, \omega)} \sum_{a'} \mu_{aa'}(\mathbf{k}, \omega)\, iez_{a'} \int d\mathbf{v} \frac{g_{a'}(\mathbf{v}, \mathbf{k})}{\omega - \mathbf{kv}}, \qquad (6)$$

where

$$\mu_{aa'}(\mathbf{k}, \omega) = \delta_{aa'} \varepsilon(\mathbf{k}, \omega) - 4\pi\varkappa_a(k, \omega - \mathbf{k}\mathbf{u}^a).$$

The integration over ω in (3) and (6) is carried out along the line $\operatorname{Im} \omega = \gamma$, which passes above all poles of the function ϵ^{-1}.

Averaging over all possible initial values of the particle distribution functions we write the correlation function of the charge density in the form

$$C(\mathbf{k}, t, t') \equiv \int d\mathbf{r} e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')} \langle \rho(\mathbf{r}, t)\, \rho(\mathbf{r}', t') \rangle$$
$$= \iint_{-\infty+i\gamma}^{\infty+i\gamma} \frac{d\omega\, d\omega'}{(2\pi)^2} \frac{\exp(-i\omega t - i\omega' t')}{\varepsilon(\mathbf{k}, \omega)\, \varepsilon(-\mathbf{k}, \omega')} A(\mathbf{k}, \omega, \omega'), \qquad (7)$$

where

$$A = \sum_{a,a'} A_{aa'},$$

$$A_{aa'}(\mathbf{k}, \omega, \omega') = -e^2 z_a z_{a'} \int d\mathbf{v}\, (\omega - \mathbf{kv} + i0)^{-1} \int d\mathbf{v}'$$
$$\times (\omega' + \mathbf{kv}' + i0)^{-1} \int d\mathbf{r} e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')}$$
$$\times \langle g_a(\mathbf{v}, \mathbf{r})\, g_{a'}(\mathbf{v}', \mathbf{r}') \rangle \qquad (8)$$

(the symbol $\langle \ldots \rangle$ denotes an average over the fluctuations).

A knowledge of the correlation functions allows us to determine the expectation values for the amplitudes of the oscillation of the physical quantities. For example, the mean square amplitude for the charge density oscillation characterized by the wave vector **k** is given by the relation

$$\langle |\rho(\mathbf{k}, t)|^2 \rangle = C(\mathbf{k}, t, t), \qquad (9)$$

where $\rho(\mathbf{k}, t)$ is the spatial Fourier component of the charge density.

In accordance with (2) the correlation function for the electric field is related to the correlation function for the charge density as follows:

$$\langle E_i(\mathbf{r}, t)\, E_j(\mathbf{r}', t') \rangle = \frac{2}{\pi} \int d\mathbf{k} e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} \frac{k_i k_j}{k^4} C(\mathbf{k}, t, t'). \qquad (10)$$

It follows from (7) that the behavior of the correlation function C is determined by the position of the poles of the functions ϵ^{-1} and A. It is evident from (8) that the poles of A lie in the lower half plane of ω; hence, contributions from the poles in A yield terms in the correlation function that diminish in time. The effect of the poles of ϵ^{-1}, however, is different; in a stable system these poles also lie in the lower half plane and C diminishes with increasing t and t′. On the other hand, if the system is unstable ϵ^{-1} has poles in the upper half plane so that the correlation function contains terms that grow exponentially with increasing t and t′ as well as terms that grow when either one of these quantities increases.

It should be noted that in deriving (7) we have neglected random forces acting in the time intervals $(0, t)$ and $(0, t')$, whose contribution is proportional to t/τ or t'/τ (τ is the mean time between collisions) and vanishes as $\tau \to \infty$.

3. The expressions in (7)–(10) relate the correlation functions with the fluctuations in the particle distribution functions at the initial time (the time at which the system is "switched on"). The Fourier component of the averaged product of the initial values of the distribution function can be written in the following general form:

$$\int d\mathbf{r} e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')} \langle g_a(\mathbf{v}, \mathbf{r})\, g_{a'}(\mathbf{v}', \mathbf{r}') \rangle$$
$$= \delta_{aa'}\, \delta(\mathbf{v} - \mathbf{v}')\, F_a^0(\mathbf{v} - \mathbf{u}^a) + Y_{aa'}(\mathbf{v}, \mathbf{v}', \mathbf{k}), \qquad (11)$$

where the first term corresponds to an ideal gas (cf. Kadomtsev [3]) while the second is due to the electromagnetic interaction between the particles. It is important that the second term is a smooth function of velocity whereas the first contains the delta function $\delta(\mathbf{v} - \mathbf{v}')$. In a plasma in thermody-

*rot = curl

namic equilibrium the function Y is

$$Y_{aa'}(\mathbf{v}, \mathbf{v}', \mathbf{k}) = -(4\pi/T)e^2 z_a z_{a'} F_a^0(v) F_{a'}^0(v')(R_0^{-2}+k^2)^{-1},$$

where R_0 is the plasma Debye radius.

Substituting (11) in (8) and making use of (5) we have

$$A(\mathbf{k}, \omega, \omega') = \frac{-k^2}{\omega\omega'}\sum_a T_a \frac{\omega\varkappa_a(k, \omega - \mathbf{ku}^a) + \omega'\varkappa_a(k, \omega' + \mathbf{ku}^a)}{\omega + \omega'} + \delta A(\mathbf{k}, \omega, \omega'), \quad (12)$$

where δA denotes that part of A which does not have a pole at $\omega + \omega' = 0$ [the contribution due to Y and part of the contribution of the δ-function term in (11)]. For simplicity we assume that the distribution functions F_a^0 are Maxwellian and characterized by temperature T_a.)

In a plasma characterized by a stable distribution function the basic contribution to the correlation function comes from the first of the two terms in (12), which has a pole at $\omega + \omega' = 0$. This then is the term that makes the basic contribution in a plasma traversed by a beam of low density (this is the case because the characteristic frequencies in such a system satisfy the inequality $|\eta_{\mathbf{k}} + \eta_{-\mathbf{k}}| \ll |\eta_{\mathbf{k}}|$.

We now consider A for a plasma with a non-equilibrium but stable distribution function:*

$$A(k, \omega, \omega') = e^2\pi^2\delta(\omega + \omega') \times \sum_a z_a^2 \int d\mathbf{v} F_a^0(v)\,\delta(\omega - \mathbf{kv}) + \delta A(k, \omega, \omega'). \quad (13)$$

Substituting (13) in (7) and taking $t, t' \gg |\mathrm{Im}\,\omega_{\mathbf{k}}^{-1}|$, where $\omega_{\mathbf{k}}$ is a root of the equation $\epsilon(k, \omega_{\mathbf{k}}) = 0$, we have

$$C(k, t - t') = e^2 \int_{-\infty}^{\infty} \frac{d\omega e^{-i\omega(t-t')}}{|\varepsilon(k, \omega)|^2} \sum_a z_a^2 \int d\mathbf{v}\, F_a^0(v)\,\delta(\omega - \mathbf{kv}). \quad (14)$$

A similar formula has been given earlier by Rostoker.[5]

4. We now consider in greater detail the case in which the plasma is traversed by a "cold" beam ($T_B \approx 0$) whose density is smaller than the plasma density. In this case the dielectric permittivity is given by

$$\varepsilon(\mathbf{k}, \omega) = \varepsilon^0(\mathbf{k}, \omega) - \Omega_B^2(\omega - \mathbf{ku})^{-2}, \quad (15)$$

where ϵ^0 is the dielectric permittivity of the free plasma and $\Omega_B^2 = 4\pi e^2 m^{-1} N_B$ (N_B is the electron density in the beam). The quantity ϵ differs appreciably from ϵ^0 only near the characteristic frequencies

$$\eta_{\mathbf{k}} = \mathbf{ku}\left\{1 - \frac{\Omega_B}{|\mathbf{ku}|}[\varepsilon^0(k, \mathbf{ku})]^{-1/2}\right\},$$
$$\tilde{\eta}_{\mathbf{k}} = \mathbf{ku}\left\{1 + \frac{\Omega_B}{|\mathbf{ku}|}[\varepsilon^0(k, \mathbf{ku})]^{-1/2}\right\}, \quad (16)$$

where the frequency $\eta_{\mathbf{k}}$ corresponds to a growing oscillation while $\tilde{\eta}_{\mathbf{k}}$ corresponds to a damped oscillation.

Substituting (15) in (7) we can write the charge density correlation function in the form $C = C^{(1)} + C^{(2)} + C^{(3)}$; the term $C^{(1)}$ grows when both times t and t', increase; the term $C^{(2)}$ grows if either t or t' increases, and the term $C^{(3)}$ diminishes with increasing t, t'. The growth terms are given by

$$C^{(1)}(\mathbf{k}, t, t') = \frac{-k^2T}{16\pi}\frac{\Omega_B}{|\mathbf{ku}|}\frac{\exp(-i\eta_{\mathbf{k}}t - i\eta_{-\mathbf{k}}t')}{|\varepsilon^0(k, \mathbf{ku})|^2}\frac{\mathrm{Im}\,\varepsilon^0(k, \mathbf{ku})}{\mathrm{Im}\sqrt{\varepsilon^0(k, \mathbf{ku})}},$$

$$C^{(2)}(\mathbf{k}, t, t') = \frac{k^2T}{16\pi}\frac{\Omega_B}{|\mathbf{ku}|}\frac{\mathrm{Im}\,\varepsilon^0(k, \mathbf{ku})}{|\varepsilon^0(k, \mathbf{ku})|^2} \times \left\{\frac{\exp\{-i\eta_{\mathbf{k}}(t-t')\} + \exp\{i\eta_{-\mathbf{k}}(t-t')\}}{\mathrm{Im}\sqrt{\varepsilon^0(k, \mathbf{ku})}} + i\,\frac{\exp\{-i\eta_{\mathbf{k}}t - i\tilde{\eta}_{-\mathbf{k}}t'\} - \exp\{-i\tilde{\eta}_{\mathbf{k}}t - i\eta_{-\mathbf{k}}t'\}}{\mathrm{Re}\sqrt{\varepsilon^0(k, \mathbf{ku})}}\right\}$$
$$+ \frac{k^2T}{16\pi}\Omega_B\left\{\frac{\exp\{-i\eta_{\mathbf{k}}t\}}{\sqrt{\varepsilon^0(k, \mathbf{ku})}} I(-\mathbf{k}, t') + \frac{\exp\{-i\eta_{-\mathbf{k}}t'\}}{\sqrt{\varepsilon^0(k, -\mathbf{ku})}} I(\mathbf{k}, t)\right\}, \quad (17)$$

where

$$I(\mathbf{k}, t) = \frac{i}{\pi}\int_{-\infty+i0}^{\infty+i0}\frac{d\omega}{\omega}\frac{e^{-i\omega t}}{\omega - \mathbf{ku}}\frac{1}{\varepsilon^0(k, \omega)}.$$

The expression for I is simplified considerably if $t \gg \Omega^{-1}$ (Ω is the plasma Langmuir frequency). In this case $I(\mathbf{k}t) = (\omega_0 - \mathbf{k}\cdot\mathbf{u})^{-1} \times \exp(-i\omega_0 t)$ where ω_0 is the frequency (complex) of the electron plasma oscillations ($\mathrm{Re}\,\omega_0 \approx \Omega$).

Using (7) it is a simple matter to find the damped part of the charge density correlation function $C^{(3)}$. The quantity $C^{(3)}$ differs from the charge density correlation function in the absence of a beam by virtue of the terms proportional to $\sqrt{N_B}$; these terms depend on the times t and t' in the combination t − t' and t and t' separately.

We may note that the growing part of the charge density correlation function [and, in accordance with (10) the growing part of the electric field correlation function] is proportional to $\sqrt{N_B}$ and has a growth rate that is also proportional to $\sqrt{N_B}$.

5. We now determine the charge density correlation function for the beam. In this case we use the relation

*The conditions that must be satisfied by such a stable function F_a^0 have been given by A. Akhiezer, Lyubarskii, and Polovin.[4]

$$C_{aa'}(\mathbf{k}, t, t') \equiv \int d\mathbf{r} e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')} \langle \delta\rho_a(\mathbf{r}, t)\, \delta\rho_{a'}(\mathbf{r}', t')\rangle$$

$$= \iint\limits_{-\infty+i\gamma}^{\infty+i\gamma} \frac{d\omega d\omega'}{(2\pi)^2} \frac{\exp(-i\omega t - i\omega' t')}{\varepsilon(\mathbf{k}, \omega)\, \varepsilon(-\mathbf{k}, \omega')}$$

$$\times \sum_{c,c'} \mu_{ac}(\mathbf{k}, \omega)\, \mu_{a'c'}(-\mathbf{k}, \omega')\, A_{cc'}(\mathbf{k}, \omega, \omega'), \quad (18)$$

which follows from (6) and (8); we assume that both particle species a and a′ are present in the beam and that $T_B = 0$ so that

$$\sum_{c,c'} \mu_{ac}(\mathbf{k}, \omega) \mu_{a'c'}(-\mathbf{k}, \omega') A_{cc'}(\mathbf{k}, \omega, \omega')$$

$$= (4\pi)^2 \varkappa_a(k, \omega - \mathbf{ku})\, \varkappa_{a'}(k, \omega' + \mathbf{ku}) A(\mathbf{k}, \omega, \omega').$$

We obtain the following expressions for the growing parts of the charge density correlation functions in the beam $C_B^{(1)} + C_B^{(2)}$:

$$C_B^{(1)}(\mathbf{k}, t, t') = |\, \varepsilon^0(k, \mathbf{ku})\,|^2 C^{(1)}(\mathbf{k}, t, t'),$$

$$C_B^{(2)}(\mathbf{k}, t, t') = \frac{k^2 T}{16\pi} \frac{\Omega_B}{|\mathbf{ku}|} \operatorname{Im} \varepsilon^0(k, \mathbf{ku})$$

$$\times \left\{ \frac{\exp\{-i\eta_{\mathbf{k}}(t-t')\} + \exp\{i\eta_{-\mathbf{k}}(t-t')\}}{\operatorname{Im} \sqrt{\varepsilon^0(k, \mathbf{ku})}} \right.$$

$$\left. + i \frac{\exp\{-i\eta_{\mathbf{k}} t - i\tilde{\eta}_{-\mathbf{k}} t'\} - \exp\{-i\tilde{\eta}_{\mathbf{k}} t - i\eta_{-\mathbf{k}} t'\}}{\operatorname{Re} \sqrt{\varepsilon^0(k, \mathbf{ku})}} \right\}. \quad (19)$$

Thus, in general the growing fluctuations in the beam are of the same order of magnitude as the growing fluctuations in the plasma and are therefore relatively large quantities.

The damped part of the beam charge density correlation function is

$$C_B^{(3)}(\mathbf{k}, t, t') = \frac{k^2 T}{16\pi} \frac{\Omega_B}{|\mathbf{ku}|} \exp\{-i\tilde{\eta}_{\mathbf{k}} t - i\tilde{\eta}_{-\mathbf{k}} t'\} \frac{\operatorname{Im} \varepsilon^0(k, \mathbf{ku})}{\operatorname{Im} \sqrt{\varepsilon^0(k, \mathbf{ku})}}. \quad (20)$$

We note that when $T_B = 0$ the correlation function for the velocity fluctuations of the beam components

$$V_{ij}^{aa'}(\mathbf{k}, t, t') \equiv \int d\mathbf{r} e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')} \langle \delta u_i^a(\mathbf{r}, t)\, \delta u_j^{a'}(\mathbf{r}', t')\rangle,$$

$$\left(\delta u_i^a(\mathbf{r}, t) = N_a^{-1} \int d\mathbf{v}\, v_i F_a(\mathbf{v}, \mathbf{r}, t) - u_i^a\right),$$

can be expressed in terms of the correlation function for the density fluctuations for these components:

$$V_{ij}^{aa'}(\mathbf{k}, t, t') = \left\{u_i + \frac{k_i}{k^2}\left(i \frac{\partial}{\partial t} - \mathbf{ku}\right)\right\}$$

$$\times \left\{u_j - \frac{k_j}{k^2}\left(i \frac{\partial}{\partial t'} + \mathbf{ku}\right)\right\} \frac{C_{aa'}(\mathbf{k}, t, t')}{e^2 z_a z_{a'} N_a N_{a'}}.$$

Hence, the correlation function for the electron velocity fluctuations in the beam is

$$V_{ij}^B(\mathbf{k}, t, t') = (u_i u_j / e^2 N_B^2)\, C_B(\mathbf{k}, t, t'). \quad (21)$$

6. If the beam velocity is appreciably greater than the mean thermal velocity of the plasma electrons the growth rate for the fluctuations $\operatorname{Im} \eta_{\mathbf{k}}$ has a sharp maximum for wave vectors satisfying the relation $|\mathbf{k} \cdot \mathbf{u}| = \Omega$. This maximum corresponds to a resonance between the plasma oscillations excited by the beam and the plasma Langmuir oscillations.[1]

If the following condition is satisfied

$$(\Omega_B/\Omega)^{2/3} \ll \operatorname{Im} \varepsilon^0(k, \Omega),$$

where

$$\operatorname{Im} \varepsilon^0(k, \Omega) = 2\sqrt{\pi}\, (\Omega/ks)^3 \exp\{-(\Omega/ks)^2\}, \quad s^2 = 2T/m,$$

the correlation functions near the resonance values of the wave vector $|\mathbf{k} \cdot \mathbf{u}| \approx \Omega$ are given by the expressions in Secs. 4 and 5.

In the other limiting case $[(\Omega_B/\Omega)^{2/3} \gg \operatorname{Im} \epsilon^0(\mathbf{k}, \Omega)]$ the characteristic frequencies of a beam-plasma system with $|\mathbf{k} \cdot \mathbf{u}| = \Omega$ are not characterized by (16), but by the relations

$$\eta_{\mathbf{k}} = \Omega \operatorname{sign} \mathbf{ku}\, [1 - (\Omega_B/4\Omega)^{2/3}] + i\sqrt{3}\, \Omega\, (\Omega_B/4\Omega)^{2/3},$$

$$\tilde{\eta}_{\mathbf{k}} = \Omega \operatorname{sign} \mathbf{ku}\, [1 + 2(\Omega_B/4\Omega)^{2/3}] - \tfrac{1}{6} i\Omega \operatorname{Im} \varepsilon^0(k, \Omega),$$

$$\tilde{\eta}'_{\mathbf{k}} = \eta_{\mathbf{k}}^*, \quad (22)$$

where $\eta_{\mathbf{k}}$ corresponds to a growing oscillation and $\tilde{\eta}_{\mathbf{k}}$, $\tilde{\eta}'_{\mathbf{k}}$ to damped oscillations.

We obtain an expression for the charge density correlation function near resonance by substituting (12) and (22) in (7) and neglecting terms proportional to $N_B^{1/3}$:

$$C(\mathbf{k}, t, t') = C^0(k, t - t') + \frac{k^2 T}{72\pi} \left(e^{-i\eta_{\mathbf{k}} t} + e^{-i\tilde{\eta}_{\mathbf{k}} t} + e^{-i\eta_{\mathbf{k}}^* t}\right)$$

$$\times \left(e^{-i\eta_{-\mathbf{k}} t'} + e^{-i\tilde{\eta}_{-\mathbf{k}} t'} + e^{-i\eta_{-\mathbf{k}}^* t'}\right), \quad (23)$$

where C^0 is the known value of the charge density correlation function in the plasma in the absence of the beam:

$$C^0(k, t - t') = \frac{k^2 T}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \frac{d\omega}{\omega} e^{-i\omega(t-t')} \operatorname{Im} \frac{-1}{\varepsilon^0(k, \omega)}.$$

In accordance with (18) the charge density correlation function in the beam is of the form

$$C_B(\mathbf{k}, t, t') = \frac{k^2 T}{72\pi} \left(\frac{2\Omega_B}{\Omega}\right)^{4/3} \left(e^{-i\eta_{\mathbf{k}} t + i\alpha} + e^{-i\tilde{\eta}_{\mathbf{k}} t} + e^{-i\eta_{\mathbf{k}}^* t - i\alpha}\right)$$

$$\times \left(e^{-i\eta_{-\mathbf{k}} t' - i\alpha} + e^{-i\tilde{\eta}_{-\mathbf{k}} t'} + e^{-i\eta_{-\mathbf{k}}^* t' + i\alpha}\right), \quad (24)$$

where $\alpha = (2\pi/3) \operatorname{sign} \mathbf{k} \cdot \mathbf{u}$.

As before, the correlation function for the electron velocity fluctuations in the beam is related to C_B by (21).

If t, and $t' \gg \Omega^{-1}(\Omega/\Omega_B)^{2/3}$, then when $|\mathbf{k}\cdot\mathbf{u}| = \Omega$ we have

$$C(\mathbf{k}, t, t') = \frac{k^2T}{72\pi}\exp\left\{\Omega(t+t')\sqrt{3}\left(\frac{\Omega_B}{4\Omega}\right)^{2/3}\right\}$$
$$\times\exp\{-i\Omega(t-t')\,\mathrm{sign}\,\mathbf{ku}\,[1-(\Omega_B/4\Omega)^{2/3}]\};$$

$$C_B(\mathbf{k}, t, t') = (2\Omega_B/\Omega)^{4/3}C(\mathbf{k}, t, t'). \qquad (25)$$

Thus, at resonance the charge density correlation function has a growth rate proportional to $N_B^{1/3}$ and the exponential is multiplied by a factor that is independent of beam density (it is assumed that the condition $(\Omega_B/\Omega)^{2/3} \gg \mathrm{Im}\,\epsilon^0(\mathbf{k}, \Omega)$ is satisfied). Near resonance the quantity C_B is proportional to $N_B^{2/3}$ (not $N_B^{1/2}$ as in the nonresonance case) and is characterized by a high growth rate. We assume, in accordance with (9), that $t = t'$ in (23)—(25) so that we can find the mean square amplitude of the resonance plasma oscillations. In particular, the oscillation amplitude for the electric field at $t \gg \Omega^{-1}(\Omega/\Omega_B)^{2/3}$ is

$$\{\langle|\mathbf{E}(\mathbf{k}, t)|^2\rangle\}^{1/2} = \frac{1}{3}\sqrt{2\pi T}\exp\{\Omega t\sqrt{3}\,(\Omega_B/4\Omega)^{2/3}\}. \qquad (25')$$

We now estimate the width of the oscillation resonances excited by the beam in a plasma. We may note that the growth rate near resonance is of the form

$$\mathrm{Im}\,\eta_{\mathbf{k}} = \sqrt{3}\,\Omega\left(\frac{\Omega_B}{4\Omega}\right)^{2/3}\left\{1-\frac{4(\Omega-|\mathbf{ku}|)^2}{9\Omega_B^2}\left(\frac{\Omega_B}{4\Omega}\right)^{2/3}\right\},$$

so that the most rapidly growing oscillations are those whose wave vectors satisfy the relation $||\mathbf{k}\cdot\mathbf{u}| - \Omega| < \Omega_B^{1/3}\Omega^{1/6}t^{-1/2}$. The quantity multiplying the exponential in the expressions for the expectation values of the square of the oscillation amplitudes in the plasma falls off by a factor $(N/N_B)^{1/2}$ when $|\mathbf{k}\cdot\mathbf{u}|$ changes from Ω to $\Omega[1 \pm (\Omega_B/\Omega)^{2/3}]$. Thus, immediately after introduction of the beam into the plasma the oscillations with greatest amplitude are those with frequencies in the range $(\Omega - \Delta\omega,\ \Omega + \Delta\omega)$ where $\Delta\omega \sim \Omega(\Omega_B/\Omega)^{2/3}$. Starting at $t \sim \Omega^{-1}(\Omega/\Omega_B)^{2/3}$ it is necessary to take account of the reduction of the resonance width with time: $\Delta\omega \sim \Omega_B^{1/3}\Omega^{1/6}t^{-1/2}$. However, a nonlinear theory is needed to describe the behavior of the plasma accurately when $t \gg \Omega^{-1}(\Omega/\Omega_B)^{2/3}$.

7. In order to investigate the instability of the beam-plasma system it was necessary to assume some initial time, at which the system is switched on. At this instant of time the dielectric permittivity changes sharply from that of the free plasma ϵ^0 to that of the beam-plasma system ϵ. However, it is not necessary to assume that the fluctuations at the initial time are of equilibrium nature. It must be assumed, however, that these fluctuations do not increase greatly during the time in which the leading edge of the beam passes through the plasma (while the system is still inhomogeneous): for this reason the condition $\mathrm{Im}\,\eta_{\mathbf{k}} \ll u/L$ must be satisfied, where L represents the dimension of the system in the direction of motion of the beam.

On the other hand, to analyze a beam plasma system as an infinite medium it must be assumed that the wavelength of the oscillations is small compared with the dimensions of the system. Hence, the results given apply when the inequality $|\mathbf{k}\cdot\mathbf{u}| \gg u/L \gg \mathrm{Im}\,\eta_{\mathbf{k}}$ is satisfied, where, in accordance with (16) and (22), $\mathrm{Im}\,\eta_{\mathbf{k}} \sim \Omega_B$ in the nonresonance case and $\mathrm{Im}\,\eta_{\mathbf{k}} \sim \Omega(\Omega_B/\Omega)^{2/3}$ in the resonance case $(\mathbf{k}\cdot\mathbf{u} \approx \Omega)$. It is evident that this inequality is satisfied for small values of Ω_B and that it imposes certain limitations on L.

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