SPECTRAL REPRESENTATIONS OF MATRIX ELEMENTS

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Submitted to JETP editor July 3, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **42**, 779-782 (March, 1962)

Spectral representations are obtained for the matrix elements of the product of n scalar Heisenberg operators.

THIS paper presents a generalization of the integral representation of Dyson, [3] based on the methods of Schwinger [1] and Gribov, [2] in which anomalous regions of integration do not arise.

1. Consider the mean in vacuum of the product of three scalar operators

$$F_{123}^{(-)}(x_{12}, x_{23}) = \langle 0 | \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3) | 0 \rangle, \qquad (1)$$

where $x_{ik} = x_i - x_k$. The function (1) contains only positive frequencies and consequently is analytic relative to time coordinates in the region

$$x_{12}^0 \to x_{12}^0 - i\varepsilon_1, \quad \varepsilon_1 > 0,$$
$$x_{23}^0 \to x_{23}^0 - i\varepsilon_2, \quad \varepsilon_2 > 0, \qquad (2)$$

where the ϵ are arbitrary positive constants, which we consider to be infinitesimally small.

According to (2), this function will have a spectral representation with a factor

$$\exp\{-i\alpha_1 x_{12}^2 - i\alpha_2 x_{13}^2 - i\alpha_3 x_{23}^2\}, \quad x^2 = x_0^2 - \mathbf{x}^2$$

in the integrand if

$$\alpha_1 x_{12}^0 + \alpha_2 x_{13}^0 > 0, \quad \alpha_2 x_{13}^0 + \alpha_3 x_{23}^0 > 0, \qquad (3)$$

i.e.,

$$F_{123}^{(-)}(x_{12}, x_{23}) = \int \exp(-i\alpha_1 x_{12}^2 - i\alpha_2 x_{13}^2 - i\alpha_3 x_{23}^2)\,\theta(\alpha_1 x_{12}^0 + \alpha_2 x_{13}^0)\,\theta(\alpha_2 x_{13}^0 + \alpha_3 x_{23}^0)\,\varepsilon(\alpha_1, \alpha_2, \alpha_4)\,\psi_{123}(\alpha_1, \alpha_2, \alpha_3) \times d\alpha_1\, d\alpha_2\, d\alpha_3. \qquad (4)$$

The spectral representation of the T-product can be obtained by assuming $x_{12}^0 > 0$ and $x_{23}^0 > 0$ in (4). In this case, it follows from (4) and the symmetry properties of the T-product that

$$F_{123}^{(c)}(x_{12}, x_{23}) = \int \exp(-i\alpha_1 x_{12}^2 - i\alpha_2 x_{13}^2 - i\alpha_3 x_{23}^2)\,\theta(\alpha_1)\,\theta(\alpha_2)\,\theta(\alpha_3)\,\psi_{123}(\alpha_1, \alpha_2, \alpha_3)\, d\alpha_1\, d\alpha_2\, d\alpha_3.$$

Introducing the Fourier transform of the function $\psi_{123}(1/4\alpha_1, 1/4\alpha_2, 1/4\alpha_3)$,

$$\psi_{123}(\alpha_1, \alpha_2, \alpha_3) = (2\pi i)^3 \int \exp\left\{-i\frac{\varkappa_{12}^2}{4\alpha_1} - i\frac{\varkappa_{13}^2}{4\alpha_2} - i\frac{\varkappa_{23}^2}{4\alpha_3}\right\} \times I_{123}(\varkappa_{12}^2, \varkappa_{13}^2, \varkappa_{23}^2)\, d\varkappa_{12}^2\, d\varkappa_{13}^2\, d\varkappa_{23}^2,$$

we obtain

$$F_{123}^{(c)}(x_{12}, x_{23}) = (2\pi i)^9 \int_0^\infty D^{(c)}(x_{12}, \varkappa_{12})\, D^{(c)}(x_{13}, \varkappa_{13})\, D^{(c)}(x_{23}, \varkappa_{23}) \times I_{123}(\varkappa_{12}^2, \varkappa_{13}^2, \varkappa_{23}^2)\, d\varkappa_{12}^2\, d\varkappa_{13}^2\, d\varkappa_{23}^2;$$

$$D^{(c)}(x, m) = \frac{1}{(2\pi)^4} \int e^{ikx} \frac{1}{m^2 - k^2 - i\varepsilon}\, dk; \qquad (5)$$

the parameters κ^2 take on only positive values, since they characterize the mass spectra.

From considerations of relativistic invariance, it follows that the conditions (3) in (4) can be replaced by the requirement

$$\alpha_1 x_{12}^0 > 0, \quad \alpha_2 x_{13}^0 > 0, \quad \alpha_3 x_{23}^0 > 0 \qquad (6)$$

and one can write

$$F_{123}^{(-)}(x_{12}, x_{23}) = (2\pi i)^9 \int_0^\infty D^{(-)}(x_{12}, \varkappa_{12}) D^{(-)}(x_{13}, \varkappa_{13}) D^{(-)}(x_{23}, \varkappa_{23}) \times I_{123}(\varkappa_{12}^2, \varkappa_{13}^2, \varkappa_{23}^2)\, d\varkappa_{12}^2\, d\varkappa_{13}^2\, d\varkappa_{23}^2;$$

$$D^{(-)}(x, m) = \frac{i}{(2\pi)^3} \int e^{ikx}\theta(-k^0)\,\delta(k^2 - m^2)\, dk. \qquad (7)$$

2. Let us turn to a consideration of the matrix element of the product of three operators

$$F_{123}(x_{12}, x_{23}) = \langle P | \varphi_1(x_1 - \bar{x})\, \varphi_2(x_2 - \bar{x})\varphi_3(x_3 - x) | Q \rangle', \qquad (8)$$

where $\bar{x} = (x_1 + x_2 + x_3)/3$, the prime indicates calculation only of connected diagrams, and P and Q are the total momenta of the arbitrary states $|P\rangle$ and $|Q\rangle$. From the spectral condition it follows that

$$F_{123}(x_{12}, x_{23}) = \int e^{-ix_{12}p_1 - ix_{23}p_2}\theta\left(\frac{2P+Q}{3} + p_1\right)\theta\left(\frac{P+2Q}{3} + p_2\right) \times \widetilde{F}_{123}(p_1, p_2)\, dp_1\, dp_2, \qquad (9)$$

where

$$\widetilde{F}_{123}(p_1, p_2) \neq 0$$

for

$$\left(\tfrac{1}{3}(2P+Q)+p_1\right)^2 \geqslant (m_{123}^{12})^2, \quad \tfrac{1}{3}(2P+Q)+p_1 \in L^+,$$

$$\left(\tfrac{1}{3}(P+2Q)+p_2\right)^2 \geqslant (m_{123}^{23})^2, \quad \tfrac{1}{3}(P+2Q)+p_2 \in L^+;$$

here the m are the minimum masses of the intermediate states, and $\theta(p) = \theta(p^0)\,\theta(p^2)$ is an invariant discontinuous function.

Function (8) can be written in the form

$$F_{123}(x_{12}, x_{23}) = \int \exp\{-ix_{12}k_{12} - ix_{13}k_{13} - ix_{23}k_{23} - ix_{12}u_1 - ix_{23}u_2\}\,\mathcal{F}_{123}(k_{12}, k_{13}, k_{23}, u_1, u_2)\,dk_{12}, dk_{13}, dk_{23}\,du_1\,du_2,$$

from which we obtain

$$\widetilde{F}_{123}(p_1, p_2) = \int \delta(p_1 - k_{12} - k_{13} - u_1)\delta(p_2 - k_{13} - k_{23} - u_2) \times \mathcal{F}_{123}(k_{12}, k_{13}, k_{23}, u_1, u_2)\,dk_{12}\,dk_{13}\,dk_{23}\,du_1\,du_2. \quad (10)$$

The occurrence of spectral representations of type (7) means that we can set

$$\mathcal{F}_{123}(k_{12}, k_{13}, k_{23}, u_1, u_2) = \theta(k_{12})\,\theta(k_{13})\,\theta(k_{23})\,\theta\left(\tfrac{1}{3}(2P+Q)+u_1\right) \times \theta\left(\tfrac{1}{3}(P+2Q)+u_2\right) f_{123}(k_{12}^2, k_{13}^2, k_{23}^2, u_1, u_2) = \theta(k_{12})\,\theta(k_{13})\,\theta(k_{23})\,\widetilde{I}_{123}(k_{12}^2, k_{13}^2, k_{23}^2, u_1, u_2). \quad (11)$$

Using (11), we obtain

$$F_{123}(x_{12}, x_{23}) = (2\pi i)^9 \int_0^\infty D^{(-)}(x_{12}, \varkappa_{12})\,D^{(-)}(x_{13}, \varkappa_{13}) \times D^{(-)}(x_{23}, \varkappa_{23})\,I_{123}(\varkappa_{12}^2, \varkappa_{13}^2, \varkappa_{23}^2, x_{12}, x_{23})\,d\varkappa_{12}^2\,d\varkappa_{13}^2\,d\varkappa_{23}^2, \quad (12)$$

where I_{123} is the Fourier transform of $\widetilde{I}_{123}$ in the variables u_1 and u_3. From the causality condition it follows that the functions

$$I_{123}(\varkappa_{12}^2, \varkappa_{13}^2, \varkappa_{23}^2, x_{12}, x_{23}),$$

$$\delta(p_1 + p_2 + p_3)\,\widetilde{I}_{123}(\varkappa_{12}^2, \varkappa_{13}^2, \varkappa_{23}^2, p_1, p_1 + p_2)$$

are symmetrical relative to the indices (1, 2, 3), ($\kappa_{ik}^2 = \kappa_{ki}^2$).

From Eqs. (9)—(11) and the symmetry properties of the function I_{123} it follows that

$$\widetilde{I}_{123}(\varkappa_{12}^2, \varkappa_{13}^2, \varkappa_{23}^2, u_1, u_2) \neq 0 \quad (13)$$

for

$$\varkappa_{12} + \varkappa_{13} \geqslant \max\left\{0, \max(m_{123}^{12}, m_{132}^{13}) - \sqrt{\left[\tfrac{1}{3}(2P+Q)+u_1\right]^2}, \max(m_{231}^{31}, m_{321}^{21}) - \sqrt{\left[\tfrac{1}{3}(P+2Q)-u_1\right]^2}\right\},$$

$$\tfrac{1}{3}(2P+Q)+u_1 \in L^+, \quad \tfrac{1}{3}(P+2Q)-u_1 \in L^+;$$

$$\varkappa_{12} + \varkappa_{23} \geqslant \max\left\{0, \max(m_{213}^{21}, m_{231}^{23}) - \sqrt{\left[\tfrac{1}{3}(2P+Q)-u_1+u_2\right]^2}, \max(m_{132}^{32}, m_{312}^{12}) - \sqrt{\left[\tfrac{1}{3}(P+2Q)+u_1-u_2\right]^2}\right\},$$

$$\tfrac{1}{3}(2P+Q)-u_1+u_2 \in L^+, \quad \tfrac{1}{3}(P+2Q)+u_1-u_2 \in L^+;$$

$$\varkappa_{13} + \varkappa_{23} \geqslant \max\left\{0, \max(m_{312}^{31}, m_{321}^{32}) - \sqrt{\left[\tfrac{1}{3}(2P+Q)-u_2\right]^2}, \max(m_{123}^{23}, m_{213}^{13}) - \sqrt{\left[\tfrac{1}{3}(P+2Q)+u_2\right]^2}\right\},$$

$$\tfrac{1}{3}(2P+Q)-u_2 \in L^+, \quad \tfrac{1}{3}(P+2Q)+u_2 \in L^+.$$

3. A matrix element of general form

$$F_{12\ldots n}(x_{12}, x_{23}, \ldots, x_{n-1,n}) = \langle P \,|\, \varphi_1(x_1 - \bar{x})\,\varphi_2(x_2 - \bar{x}) \ldots \varphi_n(x_n - \bar{x}) \,|\, Q\rangle',$$

$$\bar{x} = (x_1 + x_2 + \ldots + x_n)/n, \quad (14)$$

can be written in the form

$$F_{12\ldots n}(x_{12}, x_{23}, \ldots, x_{n-1,n}) = (2\pi i)^{3n(n-1)/2} \int_0^\infty D^{(-)}(x_{12}, \varkappa_{12}) \times D^{(-)}(x_{13}, \varkappa_{13}) \ldots D^{(-)}(x_{n-1,n}, \varkappa_{n-1,n}) \times I_{12\ldots n}(\varkappa_{12}^2, \varkappa_{13}^2, \ldots, \varkappa_{n-1,n}^2, x_{12}, x_{23}, \ldots, x_{n-1,n}) \times d\varkappa_{12}^2\,d\varkappa_{13}^2 \ldots d\varkappa_{n-1,n}^2, \quad (15)$$

where the function $I_{12\ldots n}$ is symmetrical with respect to the indices (1, 2, ..., n) [the number of its arguments κ^2 equals $n(n-1)/2$] and has a Fourier transform

$$\widetilde{I}_{12\ldots n}(\varkappa_{12}^2, \varkappa_{13}^2, \ldots, \varkappa_{n-1,n}^2, u_1, u_2, \ldots, u_{n-1}) \neq 0 \quad (16)$$

for

$$\varkappa_{12} + \varkappa_{13} + \ldots + \varkappa_{1n} \geqslant \max\{0, m_{12\ldots n}^{12} - \sqrt{([(n-1)P+Q]/n+u_1)^2}\},$$

$$\varkappa_{\cdot\cdot} + \varkappa_{14} + \ldots + \varkappa_{1n} + \varkappa_{23} + \varkappa_{24} + \ldots + \varkappa_{2n} \geqslant \max\{0, m_{12\ldots n}^{23} - \sqrt{([(n-2)P+2Q]/n+u_2)^2}\},$$

. . . .

$$\varkappa_{1n} + \varkappa_{2n} + \ldots + \varkappa_{n-1,n} \geqslant \max\{0, m_{12\ldots n}^{n-1,n} - \sqrt{([P+(n-1)Q]/n+u_{n-1})^2}\},$$

$$[(n-1)P+Q]/n+u_1, \quad [(n-2)P+2Q]/n + u_2, \ldots, [P+(n-1)Q]/n+u_{n-1} \in L^+.$$

Consideration of the symmetry properties of $I_{12\ldots n}$ brings in additional limitations on the variables κ. In particular, when P and Q correspond to vacuum states, then the functions $I_{12\ldots n}$ and $\widetilde{I}_{12\ldots n}$ become identical with each other and depend only on the variables κ^2.

[1] T. Schwinger, Proc. of the Seventh Annual Rochester Conference on High Energy Nuclear Physics, 1957, Session IV, p. 1. (Interscience, N.Y.).

[2] V. N. Gribov, JETP **34**, 1310 (1958), Soviet Phys. JETP **9**, 903 (1958).

[3] F. J. Dyson, Phys. Rev. **110**, 1460 (1958).

Translated by L. M. Matarrese
125