SOME PECULIARITIES IN THE BEHAVIOR OF A RELATIVISTIC PLASMA WITH AN ANISOTROPIC DISTRIBUTION OF ELECTRON VELOCITIES

G. M. ZASLAVSKIĬ and S. S. MOISEEV

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The nature of cyclotron and aperiodic instabilities in a relativistic plasma is investigated. It is shown that the stability of a relativistic plasma against cyclotron resonance and aperiodic instability is greater than that of a nonrelativistic plasma.

THE problem of the behavior and stability of a nonrelativistic plasma with an anisotropic velocity distribution has been studied in a number of researches (see for example, [1-3]). Such features of the relativistic plasma as the increase of the radiation in a magnetic field, the dependence of the mass of the particles on the velocity, the sharp break in the "tail" of the velocity distribution ($v \sim c$), permit us to expect definite changes in the behavior of the plasma. For example, the cyclotron instability[2] depends on the shape of the "tail" of the distribution function. Some properties of the relativistic plasmas are considered below in the kinetic approximation for the case of anisotropy in the velocity distribution.

1. CYCLOTRON INSTABILITY OF A RELATIVISTIC PLASMA

Let us consider the problem of the cyclotron instability of a plasma with relativistic electrons in a constant magnetic field $\mathbf{H}_0$. We are interested in processes with characteristic frequencies

$$\mathrm{Re}\,\omega \gg 1/\tau_D, \tag{1}$$

where τ_D is the scattering time for collisions. In the scattering of relativistic electrons on nonrelativistic particles, it is easy to get

$$\tau_D = m^2 v^3 \gamma^2 / 8\pi (ee')^2 Ln', \tag{2}$$

from the results of Belyaev and Budker.[4] Here, e, m, v are the charge, mass, and velocity of the electrons; e′, n′ are the charge and density of the nonrelativistic particles; L is the Coulomb logarithm; $\gamma = (1 - v^2/c^2)^{-1/2}$.

We also note that systematic account of the bremsstrahlung would lead to the problem of the stability of the nonstationary states (here, in particular, it would be impossible to look for a correction to the distribution function with a time dependence merely of the form $e^{i\omega\tau}$). However, as will be seen from what follows, in cases of practical interest the cyclotron instability arises at frequencies for which

$$\mathrm{Re}\,\omega \gg 1/\tau_{\mathrm{rad}}, \tag{3}$$

where the bremsstrahlung time τ_{rad} for $\gamma^2 \gg 1$ is equal to

$$\tau_{\mathrm{rad}} = 3m^3c^7/2e^4 H_0^2 v_\perp^2 \gamma \qquad (\mathbf{v}_\perp \perp \mathbf{H}_0). \tag{4}$$

The inequality (3) allows us to investigate the conditions for the appearance of the instability without account of the radiation. The nature of the development of the instability will depend here on the relation between τ_{rad} and $1/\mathrm{Im}\,\omega$.

The relativistic kinetic equation for the electron distribution function $f(t, \mathbf{r}, \mathbf{p})$, under the assumptions (1) and (3), has the form

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + \left\{e\mathbf{E} + \frac{e}{c}[\mathbf{v}(\mathbf{H}_0 + \mathbf{H})]\right\}\frac{\partial f}{\partial \mathbf{p}} = 0\,. \tag{5}$$*

Here $\mathbf{E}$ and $\mathbf{H}$ are the fields of the excitation wave, equal to

$$\mathbf{E} = \mathbf{E}^0 e^{i(\mathbf{kr}-\omega t)}; \qquad \mathbf{H} = \mathbf{H}^0 e^{i(\mathbf{kr}-\omega t)}. \tag{5'}$$

We seek a solution of Eq. (5) in the form $f = f_0 + f_1$, where f_1 is a small correction, and†

$$f_0 = f_0\left(\sqrt{1 + u_\alpha^2},\ (u_\alpha H_{0\alpha})^2\right) \tag{6}$$

is the solution of

$$[\mathbf{v}\mathbf{H}_0]\,\partial f_0/\partial \mathbf{p} = 0, \qquad u_\alpha = v_\alpha \gamma / c. \tag{6'}$$

Taking into account the relativistic invariance of the distribution function,[4] it is not difficult to write f_0 in an arbitrary system of units:

$$f_0 = f_0\left(U_i u_i,\ (\varepsilon_{iklm} u_i U_k F_{lm})^2\right).$$

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*$[\mathbf{v}(\mathbf{H}_0 + \mathbf{H})] \to \mathbf{v} \times (\mathbf{H}_0 + \mathbf{H})$.

†Greek indices run over three values, Latin ones over four.

Here u_i and U_i are the 4-velocities of the particle and of the average motion ($U_\alpha = 0$ in the system of units considered), respectively, F_{lm} is the electromagnetic field tensor that is associated with $\mathbf{H}_0$; ϵ_{iklm} is an antisymmetric unit tensor of fourth rank.

By calculations similar to those of Trubnikov,[5] it is easy to obtain a generalization of the expression for the dielectric permittivity tensor $\epsilon_{\alpha\beta}$ in a relativistic plasma with an arbitrary momentum distribution of the form (6). For a wave with frequency ω propagated along $\mathbf{H}_0$ ($\mathbf{k} \parallel \mathbf{H}_0$), we have

$$\varepsilon_{xx} = 1 + \frac{\omega_0^2}{2\omega}\int d\mathbf{p}\cdot p_\perp^2 \sum_{n=1,\,-1}\left(-i\pi\delta_+ + \frac{1}{\omega}F_2\right),$$

$$\varepsilon_{xy} = -\varepsilon_{yx} = -\frac{i\omega_0^2}{2\omega}\int d\mathbf{p}\cdot p_\perp^2 \sum_{n=1,\,-1} n\left(-i\pi F_1\delta_+ + \frac{1}{\omega}F_2\right),$$

$$\varepsilon_{zz} = 1 - \frac{2i\omega_0^2}{\omega}\int d\mathbf{p}\cdot p_\parallel^2 \frac{\partial f_0}{\partial p_\parallel^2}\delta_+(\gamma(\omega - kv_\parallel)),$$

$$\varepsilon_{yy} = \varepsilon_{xx}, \qquad \varepsilon_{xz} = \varepsilon_{zx} = \varepsilon_{yz} = \varepsilon_{zy} = 0.$$

Here

$$F_1 = \frac{\partial f_0}{\partial p_\parallel^2} + \frac{n\Omega}{\omega\gamma}\left(\frac{\partial f_0}{\partial p_\perp^2} - \frac{\partial f_0}{\partial p_\parallel^2}\right), \qquad F_2 = \frac{\partial f_0}{\partial p_\perp^2} - \frac{\partial f_0}{\partial p_\parallel^2},$$

$$\delta_+ = \delta_+\left[(\omega - kv_\parallel)\gamma - n\Omega\right], \qquad \delta_+(x) = \frac{i}{\pi}\mathrm{P}\left(\frac{1}{x}\right) + \delta(x),$$

$$\Omega = |eH_0/mc|, \qquad \omega_0^2 = 4\pi n_0 e^2/m,$$

n_0 is the electron density.

In the nonrelativistic limit, the components for $\epsilon_{\alpha\beta}$ coincide with the corresponding expressions from the work of Sagdeev and Shafranov,[2] with $\mathbf{k} \parallel \mathbf{H}_0$. The dispersion equation for the extraordinary wave ($n = 1$) has the form

$$\left(\frac{kc}{\omega}\right)^2 = 1 + \frac{\omega_0^2}{\omega}\int d\mathbf{p}\cdot p_\perp^2\left\{-i\pi\delta_+[(\omega - kv_\parallel)\gamma - \Omega]\right.$$
$$\left.\times\left[\frac{\partial f_0}{\partial p_\parallel^2} + \frac{\Omega}{\omega\gamma}\left(\frac{\partial f_0}{\partial p_\perp^2} - \frac{\partial f_0}{\partial p_\parallel^2}\right)\right] + \frac{1}{\omega}\left(\frac{\partial f_0}{\partial p_\perp^2} - \frac{\partial f_0}{\partial p_\parallel^2}\right)\right\}. \qquad (7)$$

If

$$\left|\frac{\partial f_0}{\partial p_\perp^2}\right| > \left|\frac{\partial f_0}{\partial p_\parallel^2}\right|,$$

then, as is well known,[2] the cyclotron instability is connected with the build-up of the extraordinary wave.

It is convenient to choose for f_0, the function

$$f_0 = A\exp\{-\sigma(\sqrt{1 + (p/mc)^2} + \sqrt{1 + (\sigma_1 p_\parallel/\sigma mc)^2})\}, \qquad (8)$$

which satisfies (6′). Here A is a normalizing factor, σ and σ_1 are the parameters of the distribution. In the nonrelativistic limit, (8) transforms to a Maxwell distribution with two temperatures ($T_\perp$, $T_\parallel$).

For weak anisotropy ($\sigma_1^2\overline{u_z^2}/\sigma^2 \ll 1$) we have from (8)

$$A = \frac{\sigma}{4\pi(mc)^3 K_2(\sigma)}\left[1 + \frac{1}{2}\left(\frac{\sigma_1}{\sigma}\right)^2\frac{K_3(\sigma)}{K_2(\sigma)}\right], \qquad (9)$$

$$\left(\frac{\sigma_1}{\sigma}\right)^2 = \frac{K_2(\sigma)}{K_3(\sigma)}\frac{\Delta P}{P}, \qquad (10)$$

where $K_\mu(\sigma)$ is the MacDonald function of index μ, $P = P_\perp \approx P_\parallel$, $\Delta P = P_\perp - P_\parallel > 0$ ($P_\parallel$ is the pressure along $\mathbf{H}_0$, $P_\perp$ is the pressure in the direction perpendicular to $\mathbf{H}_0$).

Substitution of (8) in (7) and a somewhat involved but simple integration lead, for $v\bar{\gamma}/c \gtrsim 1$, to the following expressions for the real and imaginary parts of the square of the index of refraction, N^2:

$$\mathrm{Re}\,N^2 = 1 + \omega_0^2/\omega\Omega, \qquad (11)$$

$$\mathrm{Im}\,N^2 = -2\pi^2\frac{\omega_0^2\Omega}{\omega(kc)^2}A\frac{(mc)^3}{\sigma}\times\left[\frac{(\sigma_1/\sigma)^2}{\sqrt{1 + (\varkappa\sigma_1/\sigma)^2}}\frac{\Omega}{\omega} - 1\right]$$
$$\times\exp\{-\sigma(\varkappa + \sqrt{1 + (\varkappa\sigma_1/\sigma)^2})\}, \qquad (12)$$

where $\kappa = \Omega/kc$. To derive (11) and (12), we used the inequalities

$$\varkappa \gg \bar{\gamma}, \qquad \omega_0^2 \gg \omega\Omega.$$

We now consider perturbations of the type $e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$; therefore, the instability arises if the expression for $\mathrm{Im}\,N^2$ becomes negative.

By making use of (9), (10), and (12), we easily get, for weak anisotropy, the minimum build-up time and the condition of instability for ultrarelativistic electrons ($\sigma\bar{\gamma} \sim 1$, $\sigma \ll 1$). Thus, for $(\kappa\sigma_1/\sigma)^2 \ll 1$, the growth time is equal to

$$\tau_{gr} \sim \bar{\gamma}^{1/2}\frac{\omega_0}{\Omega^2}\left(\frac{\Delta P}{P}\right)^{-1/2}\exp\left\{\frac{\Omega}{\omega_0}\left(\bar{\gamma}\frac{\Delta P}{P}\right)^{-1/2}\right\} \qquad (13)$$

and the instability arises for the frequencies

$$\omega < \sigma\Omega\Delta P/P. \qquad (14)$$

Similarly, for the case $(\kappa\sigma_1/\sigma)^2 \gg 1$,

$$\tau_{gr} \sim \bar{\gamma}^{3/2}\,\Omega^{-1}\left(\frac{\Delta P}{P}\right)^{-1/2}\exp\left\{\frac{\Omega^2}{\omega_0^2}\left(\bar{\gamma}\frac{\Delta P}{P}\right)^{-1/2}\right\}, \qquad (15)$$

$$\omega < \sigma\frac{\omega_0^2}{\Omega}\frac{\Delta P}{P}. \qquad (16)$$

Inasmuch as in the nonrelativistic case the boundary of the instability[2] is determined by the inequality $\omega < \Omega\Delta P/P$, the region of instability contracts in the ultrarelativistic case, in accord with (14) and (16), in proportion to σ, and the boundary of the instability is displaced in the direction of the long waves. The latter circumstance can lead to increase in stability of a relativistic plasma of restricted dimensions.

We note that, for example, for $(\kappa\sigma_1/\sigma)^2 \gg 1$, condition (3) for frequencies with minimum value of the time of the build-up takes the form

$$\frac{H_0}{er_0 n_0}\bar{\gamma}^{-3}\left[1+\frac{\Omega^2}{\omega_0^2}\left(\bar{\gamma}\frac{\Delta P}{P}\right)^{-1/2}\right]\gg 1$$

and is satisfied in a sufficiently wide range (r_0 is the classical radius of the electron).

The presence of an exponential factor in expressions (13) and (15) leads to a strong change in the relation between τ_{gr} and τ_{rad}. Thus, for

$$n \sim 10^{12}\ \mathrm{cm}^{-3},\quad H_0 \sim 10^4\ \mathrm{Oe},$$

$$\bar{\gamma} \sim 10,\quad \Delta P/P \sim 10^{-2}$$

we get from (15) and (16),

$$\tau_{gr} \sim 10^4\ \mathrm{sec},\quad \tau_{rad} \sim 1\ \mathrm{sec},\quad \lambda \sim 1/k > 100\ \mathrm{cm},$$

and for $n \sim 10^{13}\ \mathrm{cm}^{-3}$ and the previous values of the other parameters, $\tau_{gr} \sim 10^{-6}$ sec, $\tau_{rad} \sim 1$ sec. Strictly speaking, Eqs. (13) and (15) are valid for $\tau_{rad} \gg \tau_{gr}$.

Let us make some remarks on the effect of radiation on the stability of a nonrelativistic two-temperature Maxwell distribution, where the physical picture is more illustrative. Setting the radiation force equal to

$$\mathbf{F}_{rad} = \frac{2}{3}\frac{e^4}{m^2c^5}[\mathbf{H}_0[\mathbf{H}_0\mathbf{v}]]$$

and choosing the initial distribution in the form

$$f_{t=0}(v_\perp, v_\parallel) = n_0(2\pi mT_\perp)^{-1}(2\pi mT_\parallel)^{-1/2}\exp\{-mv_\perp^2/2T_\perp - mv_\parallel^2/2T_\parallel\},$$

we obtain a solution of the kinetic equation with account of radiation for the transparent plasma

$$f(t, v_\perp, v_\parallel) = n_0(2\pi mT_\perp)^{-1}(2\pi mT_\parallel)^{-1/2} \times \exp\left\{Kt - \frac{mv_\parallel^2}{2T_\parallel} - \frac{mv_\perp^2}{2T_\perp}e^{Kt}\right\}. \quad (17)$$

Here

$$K = \frac{4}{3}\frac{e^4H_0^2}{m^3c^5} = 2/\tau_{rad}^{(nonrel)} \quad (18)$$

and nonrelativistic radiation time τ_{rad}^{nonrel} is introduced. We obtain the change in the anisotropy with time, as seen from (17), by introducing the new temperature

$$T'_\perp = T_\perp e^{-Kt}. \quad (19)$$

Any existing initial anisotropy $T_\perp > T_\parallel$ vanishes in accord with (17) and (19), after a time

$$\tau = \frac{1}{2}\tau_{rad}^{(nonrel)}\ln(T_\perp/T_\parallel).$$

For $t > \tau$ an anisotropy of opposite sign appears, which, however, does not lead to cyclotron instability.[2]

We now consider the problem of cyclotron instability for frequencies

$$1/\mathrm{Re}\,\omega \ll \min\{\tau, \tau_{rad}^{(nonrel)}\}.$$

Then it is possible to fix the distribution function at a certain instant of time

$$t_0 < \min\{\tau, \tau_{rad}^{(nonrel)}\}$$

and solve the problem of the instability for the initial distribution $f(t_0)$, taking t_0 as the parameter. An expression was obtained by Sagdeev and Shafranov[2] for the maximum increment in the build-up of the waves:

$$\mathrm{Im}\,\omega = \frac{\sqrt{\pi}}{4}\omega_0\left(\frac{T'_\perp - T_\parallel}{T'_\perp}\right)^{3/2}\frac{T'_\perp}{T_\parallel}\left(\frac{2T_\parallel}{mc^2}\right)^{1/2} \times \exp\left\{-\frac{H_0^2}{8\pi n_0 T_\parallel}\frac{T'_\perp}{T'_\perp - T_\parallel}\right\}. \quad (20)$$

With account of (19), Eq. (20) shows the time variation of the increment; the latter is decreased by the presence of the radiation. It is not difficult to see that the region of instability obtained in [2] is lessened, and the boundary of the instability is shifted in the direction of longer waves. It must be noted that if $\mathrm{Im}\,\omega \ll 1/\tau$, then the instability is generally unable to develop. Inasmuch as the blocking of the radiation in the plasma falls off with increase in temperature,[6] the discussions given above are qualitatively applicable only for sufficiently hot nonrelativistic electrons.

2. INSTABILITY IN THE ABSENCE OF EXTERNAL FIELDS

As seen from (7), the cyclotron instability vanishes when $H_0 = 0$. It will be shown below, however, that an instability of the aperiodic type exists in the absence of external fields in a plasma with anisotropic velocity distribution.

The initial linearized set of equations has the form

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\frac{\partial f_1}{\partial \mathbf{r}} + \left\{e\mathbf{E} + \frac{e}{c}[\mathbf{vH}]\right\}\frac{\partial f_0}{\partial \mathbf{p}} = 0,$$

$$\mathrm{rot}\,\mathbf{H} = \frac{4\pi e}{c}\int \mathbf{v} f_1 d\mathbf{p} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},\qquad \mathrm{rot}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \quad (21)^*$$

where f_0 is the initial distribution in the form (8), f_1 is a small correction to f_0 due to the perturbation fields **E** and **H**, taken, as usual, in the form (5′). Choosing the z axis along **k**, the x axis along **E**, and substituting (5′) in (21), we get an expression for the correction f_1:

$$f_1 = ieH_y\frac{[\omega/kc - p_z/mc\gamma]\,\partial f_0/\partial p_x + (p_x/mc\gamma)\,\partial f_0/\partial p_z}{kv_z - \omega} \quad (22)$$

and finally the following dispersion equation

*rot = curl.

$$\frac{\omega^2}{c^2} - k^2 = \frac{\omega_0^2}{c}\int \frac{p_x dp}{p_z - m\gamma\omega / k}\left[\left(\frac{\omega}{kc} m - \frac{1}{\gamma c} p_z\right)\frac{\partial f_0}{\partial p_x} + \frac{p_x}{\gamma c}\frac{\partial f_0}{\partial p_z}\right]. \quad (23)$$

We shall be interested in the solution with small Re ω. In this case we find the boundary of the region of instability by setting $\omega = 0$ in (23). For the boundary value $k = k_0$, we have

$$k_0^2 = \frac{\omega_0^2 m\sigma}{c}\int \frac{p_x^2 dp}{\sqrt{p^2 + m^2c^2}}\left(\frac{\sigma_1}{\sigma mc}\right)^2 \frac{f_0}{\sqrt{1 + (\sigma_1 p_z / \sigma mc)^2}}. \quad (24)$$

As is seen from (24), an instability of the type considered ($k_0^2 > 0$) is possible for an anisotropic function. Carrying out the integration in (24) for weak anisotropy, we get

$$k_0^2 = \omega_0^2 c^{-2} (\sigma_1/\sigma)^2. \quad (25)$$

Making use of (10), we convert (25) to the form

$$k_0^2 = \omega_0^2 c^{-2} (\Delta T/T), \qquad \sigma \gg 1, \quad (26a)$$

$$k_0^2 = \sigma\omega_0^2 c^{-2} (\Delta P/P), \qquad \sigma \ll 1. \quad (26b)$$

Let us establish the values of k for which the instability arises upon transition through the boundary k_0. For this purpose, we solve Eq. (23) close to $|\omega|/kc \ll 1$. Limiting ourselves to the ultrarelativistic case and to a weak anisotropy, we get from (23)

$$k^2 = \pi A\sigma \frac{\omega_0^2}{c^2}\int_0^\infty dp\cdot p^2 \exp(-\sigma p/mc) \times \int_{-1}^{+1} d\xi\,(1 - \xi^2)\left[\frac{\omega}{kc} + \xi pmc\left(\frac{\sigma_1}{\sigma mc}\right)^2\right]\frac{1}{\xi - \omega / kc} \quad (27)$$

[Inasmuch as growing solutions are of interest to us, the contour of the integration in (23) is taken over the real axis]. Finally, we get

$$\frac{\omega}{kc} = -i\frac{4}{\pi}\frac{c^2}{\omega_0^2}\frac{k^2 - k_0^2}{\sigma + 3(\sigma_1/\sigma)^2}. \quad (28)$$

It is then evident that growing solutions are obtained for $k < k_0$. Inasmuch as the problem has been considered for an unbounded plasma, one can confirm that an instability arises if the characteristic dimensions $d \gg 1/k_0$. From (26b), we see that in the ultrarelativistic case the region of instability is decreased in proportion to $\sqrt{\sigma}$ and the boundary of the instability is shifted, as in the case of cyclotron instability, to the region of long waves.

Taking it into account that the minimum build-up time of the aperiodic instability is

$$\tau_a \sim \omega_0^{-1}\gamma^{1/2}(\Delta P/P)^{-3/2}, \quad (29)$$

we have for the ratio of times

$$\tau_D/\tau_a \sim \gamma^{-3/2}(\Delta P/P)^{3/2}/r_0^{3/2}\sqrt{n'}L, \quad (30)$$

where $n' \approx n_0$. Equation (30) shows that the neglect of collisions in the investigation of a given instability is valid in all cases of practical interest.

As is easy to see from (22) and (8), the mean perturbation current arising in the magnetic field as a result of the velocity anisotropy is directed against the electric field, which also leads to increase in the perturbation for $k < k_0$.

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