

## RESONANCE THEORY OF NUCLEAR REACTIONS INDUCED BY NEUTRONS WITH ENERGIES BELOW 5 MeV

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A method whereby reaction channels that are open at a given energy and compound-nucleus states are excluded is employed to deduce the Schrödinger equation with an infinitesimally small addition which automatically takes into account the diverging wave in the incident scattering channel, with the ordinary Hamiltonian replaced by an "effective" one. In the latter, the terms with smooth energy dependence are combined with dispersion terms averaged over the energy and related to processes occurring via the compound nucleus. An optical Hamiltonian is thus formed.

Knowledge of the formal solution of the equation is sufficient to find the diagonal transition matrix element of the exact collision problem and hence various cross sections for interaction between neutrons and nuclei. Optical penetration factors are separated in the neutron widths. The cross section for elastic scattering via the compound elastic nucleus and the reaction cross sections can be determined. A "large resonance structure" is singled out in the "force function" of the previous theories. The imaginary part of the optical potential is related to the true force function which is independent of the optical properties of the interaction. The theory is compared with experiment.

### 1. INTRODUCTION

THE presence of both narrow and broad resonances in the energy dependence of the total cross section for the interaction between neutrons and nuclei has called for a theoretical analysis not only of the compound nucleus model<sup>[1-3]</sup> but also of the optical model.<sup>[4]</sup> The optical model, however, unlike the compound-nucleus model, could yield only cross sections averaged over the energy. A need thus arose for a theory that describes from a unified point of view both the "fine" structure of the interaction cross sections, and the "large-scale" structure of the average cross sections.

The theory variant developed in the present paper is based on the mathematical apparatus of the formal theory of nuclear reactions.<sup>[5-7]</sup> In addition, we use Serdobol'skiĭ's idea<sup>[8a]</sup> of dividing the Hilbert space of the system Hamiltonian eigenfunctions within the configuration space of a system of  $A+1$  nucleons into two subspaces, the subspace of the wave functions of the open reaction channels, and the subspace of the wave functions of the compound nucleus and closed reaction channels. Unlike,<sup>[8]</sup> we separate in the subspace of the wave functions of the open reaction channels the subspace of wave functions of the input channel, on which, in final analysis, the initial Schrödinger

equation with the  $i\epsilon$ -additions are projected. The equation obtained enables us to change over to the terms of the Feshbach, Porter, and Weisskopf (FPV) optical model.<sup>[4]</sup> The "effective Hamiltonian" of the Schrödinger-equation projection in the input channel (see, for example,<sup>[8a]</sup>) contains, along with the optical terms that vary smoothly with the energy, also dispersion terms from the compound nucleus. The optical and fluctuation terms are separated by averaging over the energies.

Knowing the formal solution of this equation, we change over to a diagonal transition matrix element in the input channel, and from it to the diagonal element of the S-matrix in the chosen representation. The S-matrix is presented here as the sum of the optical  $S^0$  matrix and contributions from the compound nucleus. It is then easy to go over to the different cross sections: the total cross section, the cross section for elastic scattering on the optical potential (shape-elastic) and the cross section for elastic scattering via the compound nucleus (compound-elastic).

Breaking down the optical wave function into two factors, one dependent on the energy and the other not, we can separate the so-called optical penetration factor in the neutron width. By the same token we separate the true "force function"

in the neutron width. We also show how to relate the imaginary part of the optical potential  $W_c$  with the true "force function"  $\bar{b}_{\lambda c}^2/\bar{D}$ . At the same time, the customarily used force function

$$\frac{\bar{v}_{\lambda c}^2}{\bar{D}} = \frac{\Pi_c(E, R)}{p_c(E, R)} \frac{\bar{b}_{\lambda c}^2}{\bar{D}}$$

has a "large resonant structure," since it contains as a factor the optical penetration factor  $\Pi_c(E, R)$ , which is proportional to  $p_c(E, R)$ .

We calculated in the FPV complex rectangular well model the cross sections for the interaction between the neutrons and the nuclei Mn<sup>55</sup>, Th<sup>232</sup>, and U<sup>238</sup>. The results of the calculations are compared with experiment.

## 2. GENERAL THEORY

Following the terminology of Serdobol'skiĭ and of Agodi and Eberly,<sup>[8]</sup> we introduce the projection operators  $P$ . Let  $P_c$ ,  $P_r$ , and  $P_{\perp}$  be the projection operators in the subspace of the wave functions  $|\Phi_c\rangle$  of the input channel  $c$ , in the subspace of the wave functions  $|\Phi_r\rangle$  of the reaction channel  $r$ , and the wave functions of the compound nucleus states  $|\chi_{\lambda}\rangle$ . Using the same symbols introduced by Serdobol'skiĭ,<sup>[8a]</sup> we can obtain from the initial Schrödinger equations with  $i\epsilon$ -additions

$$(E + i\epsilon - H)|\psi^{(+)}\rangle = i\epsilon|\Phi_c\rangle \quad (2.1)$$

the equivalent equation ( $E$  —energy of relative motion of the system of colliding particle and the target-nucleus;  $H$  —Hamiltonian of the system;  $|\psi^{(+)}\rangle$  —wave function of the system, on which we impose at infinity in the input channel  $c$  the condition of incident plane wave plus outgoing wave (the latter is indicated in the wave function by the index  $(+)$ );  $|\Phi_c\rangle$  —wave function of free motion of the system in the input channel  $c$ , or incoming plane wave;  $i\epsilon$  —infinitesimal imaginary addition, which automatically separates the divergent wave at infinity. In the final formulas this addition is made to approach zero from the right, i.e.,  $\epsilon \rightarrow 0+$ ). That is to say, the equation equivalent to (2.1) for the projection of the wave function in the input channel of  $|\Phi_c\rangle$ , where the usual Hamiltonian  $H$  is replaced by the "effective Hamiltonian"  $H_{c\text{eff}}$ , will assume after slight transformations the form

$$(E + i\epsilon - H_{c\text{eff}})|P_c\psi^{(+)}\rangle = i\epsilon|\Phi_c\rangle. \quad (2.2)$$

Here

$$\begin{aligned} H_{c\text{eff}} = & H_c + \Gamma_{cr}^+ [E + i\epsilon - H_r - \Gamma_{r\perp}^+ (E - H_{\perp})^{-1} \Gamma_{\perp r}]^{-1} \\ & \times [\Gamma_{rc} + \Gamma_{r\perp}^+ (E - H_{\perp})^{-1} \Gamma_{\perp c}] + \Gamma_{c\perp}^+ [E - H_{\perp} - \Gamma_{\perp r} \\ & \times (E + i\epsilon - H_r)^{-1} \Gamma_{r\perp}]^{-1} \\ & \times [\Gamma_{\perp c} + \Gamma_{\perp r} (E + i\epsilon - H_r)^{-1} \Gamma_{rc}], \end{aligned} \quad (2.2a)$$

where we introduce the notation

$$H_c = P_c H P_c, \quad H_r = P_r H P_r, \quad H_{\perp} = P_{\perp} H P_{\perp}, \quad (2.2b)$$

$$\Gamma_{cr}^+ = P_c H P_r, \quad \Gamma_{rc} = P_r H P_c, \quad (2.2c)$$

$$\Gamma_{c\perp}^+ = P_c H P_{\perp}, \quad \Gamma_{\perp c} = P_{\perp} H P_c, \quad (2.2d)$$

$$\Gamma_{r\perp}^+ = P_r H P_{\perp}, \quad \Gamma_{\perp r} = P_{\perp} H P_r. \quad (2.2e)$$

As in the paper by Serdobol'skiĭ,<sup>[8a]</sup> we impose on the wave functions  $|\chi_{\lambda}\rangle$  the natural boundary conditions  $(d\chi_{\lambda}/dr)/\chi_{\lambda} \rightarrow 0$  as  $r \rightarrow \infty$  —an exponential decrease outside the region of nuclear forces. The addition  $i\epsilon$  is therefore left out of the Green's function  $(E + i\epsilon - H_{\perp})^{-1}$ .

The first term of the Hamiltonian (2.2a) obviously makes the main contribution to the real part of the optical potential, the second term corresponds to the direct transitions with redistribution of the particles, and the third term corresponds to a transition via the compound nucleus. The second member in the parentheses of the second term in (2.2a) is of second order of smallness compared with the first member in the parentheses (see the figure). The same can also be said concerning the second member in the parentheses of the third term of (2.2a). We therefore neglect the contributions from these terms.

Let us expand the Green's function  $(E + i\epsilon - H_r)^{-1}$  and  $(E - H_{\perp})^{-1}$  in terms of the eigenfunctions  $|\psi_r^{(+)}\rangle$  and  $|\chi_{\lambda}\rangle$  of the corresponding Hamiltonians  $H_r$  and  $H_{\perp}$ . After minor transformations, using Bethe's assumption<sup>[9]</sup> that the magnitudes and signs of the transition matrix elements containing the functions  $|\chi_{\lambda}\rangle$  are random, we obtain from (2.2a)

$$H_{c\text{eff}} = H_c + V_c^{(r)} + \sum_{\lambda} \frac{\Gamma_{c\perp}^+ |\chi_{\lambda}\rangle \langle \chi_{\lambda}| \Gamma_{\perp c}}{E - E_{\lambda} - \Delta_{\lambda r} + i\Gamma_{\lambda r}/2}. \quad (2.3)$$

We use here the notation

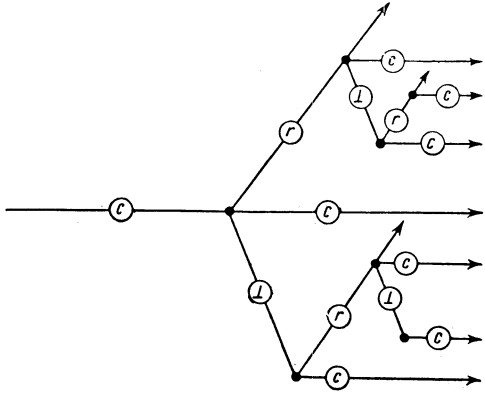
$$\begin{aligned} \text{Re} V_c^{(r)} = & \frac{1}{2\pi\hbar} \sum_r \int dE' \\ & \times \frac{\Gamma_{cr}^+ |u_r^{(+)}\rangle \langle u_r^{(+)}| \Gamma_{rc}}{E - E' - \sum_{\lambda} \langle u_r^{(+)} | \Gamma_{r\perp}^+ | \chi_{\lambda}\rangle \langle \chi_{\lambda}| \Gamma_{\perp r} | u_r^{(+)}\rangle / (E - E_{\lambda})}, \end{aligned}$$

$$\text{Im} V_c^{(r)} = -\frac{1}{2\hbar} \sum_r \Gamma_{cr}^+ |u_r^{(+)}\rangle \langle u_r^{(+)}| \Gamma_{rc},$$

$$\Delta_{\lambda r} = \frac{1}{2\pi\hbar} \sum_r \int dE' \frac{\langle \chi_{\lambda}| \Gamma_{\perp r} | u_r^{(+)}\rangle \langle u_r^{(+)} | \Gamma_{r\perp}^+ | \chi_{\lambda}\rangle}{E - E'},$$

$$\frac{\Gamma_{\lambda r}}{2} = \frac{1}{2\hbar} \sum_r \langle \chi_{\lambda}| \Gamma_{\perp r} | u_r^{(+)}\rangle \langle u_r^{(+)} | \Gamma_{r\perp}^+ | \chi_{\lambda}\rangle.$$

For the wave functions  $|\psi_r^{(+)}\rangle$  in the representation with quantum numbers  $\{a\nu l m J M\}$ , where  $a$  —quantum number of internal motion of the system comprising the particle and target nucleus,



s and  $\nu$  —channel spin and its projection,  $l$  and  $m$  —orbital momentum of the particle and its projection,  $J$  and  $M$  —total system momentum and its projection, we have in analogy with the corresponding expansions of Serdobol'skiĭ<sup>[8a]</sup> the asymptotic expansion

$$|\psi_r^{(+)}\rangle = i \left( \frac{\hbar}{2\pi m k} \right)^{1/2} \sum_{lmJM} Y_{lm}^* \left( \frac{\mathbf{k}}{k} \right) \langle s\nu lm | JM \rangle |u_r^{(+)}\rangle,$$

$$|u_r^{(+)}\rangle = \sum_{r'} \frac{I_{r'} \delta_{r'r} - S_{r'r}^J O_{r'}}{v_{r'}^{1/2}} |r'\rangle, \quad |r'\rangle = |a'E'l's'JM\rangle, \quad (2.4)$$

where  $|u_r^{(+)}\rangle$  —radial-spin-angle function of the system and the channel of the reaction  $r$ ,  $I_r$  and  $O_r$  —incoming and outgoing waves of particles of type  $r$  at infinity,  $v_r$  —relative velocity of particle of type  $r$  and residual nucleus, and  $|r\rangle$  —wave function of internal motion of the system in the state  $|a\rangle$  and spin-angle function  $|lsJM\rangle$ .\*

The wave functions  $|\psi_r^{(+)}\rangle$  are normalized to unit volume

$$\langle \psi_r^{(+)} | \psi_r^{(+)} \rangle = \delta_{r'r} \delta(\mathbf{k}' - \mathbf{k}), \quad (2.5)$$

and the radial-spin-angle functions  $|u_r^{(+)}\rangle$  are normalized to the velocity  $v_r$ :

$$\langle u_r^{(+)} | u_r^{(+)} \rangle = \frac{2\pi}{v_r} \delta_{r'r} \delta(\mathbf{k}' - \mathbf{k}) = 2\pi \hbar \delta(E' - E) \delta_{r'r},$$

$$\langle r' | r \rangle = \delta_{r'r}.$$

After averaging expression (2.3) over the energy, we get†

$$\bar{H}_{c \text{ eff}} \equiv H_c^0 = H_c + V_c^{(r)} - i \frac{\pi}{D_J} \overline{(\Gamma^+ | \chi_\lambda \rangle \langle \chi_\lambda | \Gamma)}. \quad (2.6)$$

Using (2.6), we obtain from (2.5)

$$H_{c \text{ eff}} = H_c^0 + \Delta H_{c \text{ eff}},$$

$$\Delta H_{c \text{ eff}} = \sum_\lambda \frac{\Gamma^+ | \chi_\lambda \rangle \langle \chi_\lambda | \Gamma}{E - E_\lambda - \Delta_{\lambda r} + i\Gamma_{\lambda r}/2} + i \frac{\pi}{D_J} \overline{(\Gamma^+ | \chi_\lambda \rangle \langle \chi_\lambda | \Gamma)}. \quad (2.7)$$

\*One must not confuse the letter  $r$  in the denominator of  $|u_r^{(+)}\rangle$ , which has the meaning of a coordinate, with  $|r\rangle$ , which has the meaning of a collective quantum number  $r = \{aElsJM\}$ .

†We henceforth omit the indices  $c$  and  $c$  in the operators  $\Gamma$ .

The formal solution of (2.2) has the form

$$|P_c \psi^{(+)}\rangle = |\psi_c^{(+)}\rangle + \frac{1}{E + i\varepsilon - H_c^0} \Delta H_{c \text{ eff}} |P_c \psi^{(+)}\rangle, \quad (2.8)$$

where the optical wave function is

$$|\psi_c^{(+)}\rangle = \frac{i\varepsilon}{E + i\varepsilon - H_c^0} |\Phi_c\rangle.$$

Separating from the Hamiltonians  $H_{c \text{ eff}}$  and  $H_c^0$  the interactions  $V_{c \text{ eff}}$  and  $V_c^0$ , and also the interaction  $V_c^{(c)}$  from  $H_c$ , we have respectively

$$H_{c \text{ eff}} = H_{0c} + V_{c \text{ eff}}, \quad H_c^0 = H_{0c} + V_c^0,$$

$$H_c = H_{0c} + V_c^{(c)}, \quad (2.9)$$

where  $H_{0c}$  is the Hamiltonian of free motion of the system in channel  $c$ .

It follows from a comparison of (2.6) with (2.9) that

$$\text{Re} V_c^0 = V_c^{(c)} + \text{Re} V_c^{(r)},$$

$$\text{Im} V_c^0 = V_c^{(r)} - \frac{\pi}{D_J} \overline{(\Gamma^+ | \chi_\lambda \rangle \langle \chi_\lambda | \Gamma)}. \quad (2.10)$$

The further procedure of going over to the matrix elements, and from then to the expression for the  $S^J$  matrix, is similar in many respects to the procedure used in [8a], and we shall write out only the final expression for the diagonal element of the  $S^J$  matrix:

$$S_{cc}^J = S_{cc}^{0J} - 2i \sum_\lambda \frac{\langle \tilde{u}_c^{0(-)} | \Gamma^+ | \chi_\lambda \rangle \langle \chi_\lambda | \Gamma | u_c^{(+)} \rangle}{-2\hbar(E - E_\lambda - \Delta_{\lambda r} + i\Gamma_{\lambda r}/2)}$$

$$- 2i \frac{i\pi}{D_J} \frac{1}{-2\hbar} \overline{\langle \tilde{u}_c^{0(-)} | \Gamma^+ | \chi_\lambda \rangle \langle \chi_\lambda | \Gamma | u_c^{(+)} \rangle}, \quad (2.11)$$

where  $\langle \tilde{u}_c^{0(-)} |$  is the radial-spin-angle function of the wave function  $\langle \tilde{\psi}_c^{0(-)} |$  of the reverse motion, described by the stationary Schrödinger equation with  $i\varepsilon$ -additions in the Hamiltonian  $O\tilde{H}_c^{0*}O^+$  (the operation  $O$  corresponds to a unitary transformation of the inversion of the particle spin direction and the symbol  $\sim$  corresponds to the reversal of the sign of the imaginary part of the optical potential  $V_c^0$ ), with

$$\langle \tilde{\psi}_c^{0(-)} | = \langle \Phi_c | \frac{-i\varepsilon}{E - i\varepsilon - O\tilde{H}_c^{0*}O} | = \langle \Phi_c | \frac{i\varepsilon}{E + i\varepsilon - H_c^0};$$

$|u_c^{(+)}\rangle$  is the radial-spin-angle function of the wave function of the exact problem  $|P_c^{(+)}\rangle$ .

It follows from (2.11), in particular,

$$\bar{S}_{cc}^J = S_{cc}^{0J}.$$

To determine the sought matrix element  $\langle \chi_\lambda | \Gamma | u_c^{(+)} \rangle$ , which enters into (2.11), we have the system of equations

$$\sum_\lambda \left[ \left( \delta_{\mu\lambda} - i \frac{\pi}{D_J} \mathcal{K}_{\mu\lambda} \right) \left( E - E_\lambda - \Delta_{\lambda r} + i \frac{\Gamma_{\lambda r}}{2} \right) - \mathcal{K}_{\mu\lambda} \right] Y_{\lambda c}$$

$$= \langle \chi_\mu | \Gamma | u_c^{(+)} \rangle, \quad (2.12)$$

where  $Y_{\lambda c}$  denotes the sought expression

$$Y_{\lambda c} = \frac{\langle \chi_{\lambda} | \Gamma | u_c^{(+)} \rangle}{E - E_{\lambda} - \Delta_{\lambda r} + i\Gamma_{\lambda r}/2}$$

and we have introduced the notation

$$\mathcal{X}_{\mu\lambda} = \langle \chi_{\mu} | \Gamma \frac{1}{E + i\varepsilon - H_c^0} \Gamma^+ | \chi_{\lambda} \rangle.$$

The "shift" and the width of the level  $\lambda$  in channel  $c$  are derived in this case from the formulas

$$\begin{aligned} \frac{\Gamma_{\lambda c}}{2} &= |\operatorname{Im}_{\varepsilon} \mathcal{X}_{\lambda\lambda}| = \pi |\langle \chi_{\lambda} | \Gamma \delta(E - H_c^0) \Gamma^+ | \chi_{\lambda} \rangle| \\ &= \frac{1}{2\hbar} |\langle \chi_{\lambda} | \Gamma | u_c^{(+)} \rangle \langle \tilde{u}_c^{(+)} | \Gamma^+ | \chi_{\lambda} \rangle|, \\ \Delta_{\lambda c} &= \frac{1}{2\pi\hbar} \int \frac{dE'}{E - E'} \frac{\langle \chi_{\lambda} | \Gamma | u_c^{0(+)} \rangle \langle u_c^{0(+)} | \Gamma^+ | \chi_{\lambda} \rangle}{E - E'} \quad (2.13) \end{aligned}$$

where we make use of the condition that the wave functions  $|u_c^{0(+)}\rangle$  be complete, a condition which has the form

$$\int dE |u_c^{0(+)}(E, \mathbf{r}')\rangle \langle \tilde{u}_c^{0(+)}(E, \mathbf{r})| = 2\pi\hbar\delta(\mathbf{r} - \mathbf{r}').$$

### 3. ANALYSIS OF THE STRUCTURE OF THE EXPRESSIONS CONTAINED IN THE S-MATRIX.

To separate the energy dependence of the optical transition-matrix elements we derive a relation connecting the optical wave function  $|u_c^{0(+)}\rangle$  with known quantities of the logarithmic-derivative type on the boundary of the nucleus. For this purpose we use the initial Schrödinger equations for the direct  $|u_c^{0(+)}\rangle$  and inverse  $\langle u_c^{0(-)}|$  solutions in the form

$$(E - H_c^0) |u_c^{0(+)}\rangle = 0, \quad (E' - O\tilde{H}_c^{0*}O^+) \langle \tilde{u}_c^{0(-)}| = 0. \quad (3.1)$$

Integrating over a volume with radius  $R$ , we can obtain from (3.1), using Green's theorem,

$$\langle \tilde{u}_c^{0(-)} | u_c^{0(+)} \rangle_R = 2\hbar p_c \Omega_c^2 \frac{\partial L_c^0}{\partial E} \frac{1}{(L_c - L_c^0)^2}, \quad (3.2)$$

where

$$\begin{aligned} L_c &= R \frac{O_c'}{O_c} \Big|_{r=R} = \Delta_c + ip_c, & \Omega_c^2 &= (I_c/O_c)_{r=R}, \\ L_c^0 &= R \frac{d(ru_c^{0(+)} / dr)}{ru_c^{0(+)}} \Big|_{r=R}. \end{aligned}$$

The energy-dependent part  $f_c(E, R)$  of the optical wave function  $|u_c^{0(+)}\rangle$  inside a nucleus of radius  $R$  can be separated and the function represented in the form\*

\*The independence of  $N_c(E, R)$  of  $E$  will hold true so long as  $k \ll K$ , where  $k$  and  $K$  are the wave numbers of the neutrons outside and inside the nucleus, respectively. This approximation holds true satisfactorily for incident-neutron energies not higher than 5–10 MeV, if  $\operatorname{Re}V_c^0 = 42$  MeV.

$$|u_c^{0(+)}\rangle = f_c(E, R) N_c(E, r) |c\rangle, \quad r \leq R. \quad (3.3)$$

By virtue of the condition of "generalized reality" of the optical wave function (for more details see, for example [8,10] and others) we have

$$\langle \tilde{u}_c^{0(-)} | = (-1)^{J-M} O | u_c^{0(+)} (-M) \rangle,$$

where  $M$  — projection of the spin  $J$ , and for the spin-angle function we have

$$\langle c | = (-1)^{J-M} O | c (-M) \rangle.$$

We thus have for  $r \leq R$

$$\langle \tilde{u}_c^{0(-)} | = f_c(E, R) N_c(E, r) \langle c |.$$

Substituting (3.3) in (3.2) we obtain

$$f_c(E, R) = i(2\hbar\Pi_c(E, R)/E_R)^{1/2} \Omega_c \Theta_c^0, \quad (3.4)$$

under the normalization condition

$$\langle N_c^* | N_c \rangle_R = 1.$$

Here

$$\begin{aligned} \Pi_c(E, R) &= P_c \frac{E_R}{\gamma_c^{02} |L_c - L_c^0|^2}, & \gamma_c^{02} &= - \left( \frac{\partial L_c^0}{\partial E} \right)^{-1}, \\ \Theta_c^{02} &= \frac{(L_c - L_c^0)^*}{L_c - L_c^0}, & E_R &= \frac{\hbar^2}{2MR^2}. \end{aligned}$$

Near optical resonance of the complex energy  $E_C^0 - iW_C$  (where  $E_C^0$  and  $W_C$  are the real and imaginary parts of the energy of single-particle resonance in a complex optical potential) we can use for the logarithmic derivative  $L_C^0$  the linear approximation

$$\begin{aligned} L_c^0 &= (E - E_c^0 + iW_c) \frac{\partial L_c^0}{\partial E} \Big|_{E=E_c^0, W_c=0} \\ &\approx (E - E_c^0 + iW_c) \frac{\partial \operatorname{Re} L_c^0}{\partial E} \Big|_{E=E_c^0, W_c=\text{const}}. \end{aligned} \quad (3.5)$$

The last equation holds true within the framework of the approximation (3.5) when  $W_C = \text{const}$  (i.e., independent of  $E$ ).

We then have from (3.4) the following expression for  $\gamma_C^{02}$

$$\gamma_c^{02} = - \left( \frac{\partial L_c^0}{\partial E} \right)^{-1}_{E=E_c^0, W_c=0} \approx - \left( \frac{\partial \operatorname{Re} L_c^0}{\partial E} \right)^{-1}_{E=E_c^0, W_c=\text{const}}. \quad (3.6)$$

Using the expressions (3.5) and (3.6) for (3.4) we get

$$\Pi_c(E, R) = \frac{E_R \Gamma_c^0 / 2}{(E - E_c^0 - \Delta_c^0)^2 + (\Gamma_c^0 / 2 + W_c)^2}, \quad (3.7)$$

where  $\Gamma_C^0 = 2p_C \gamma_C^{02}$  is the single-particle width and  $\Delta_C^0 = -\Delta_C \gamma_C^{02}$  is the "shift" of the single-particle level.

From the asymptotic expression

$$|u_c^{0(+)}\rangle \approx \frac{I_c - S_{cc}^0 O_c}{v_c^{1/2} r} |c\rangle$$

and (3.2) we can obtain an expression for the  $S_{cc}^0$  matrix

$$S_{cc}^0 \equiv \Omega_c^{02} = \Omega_c^2 \frac{L_c^* - L_c^0}{L_c - L_c^0} = \Omega_c^2 \left( 1 - \frac{2ip_c}{L_c - L_c^0} \right). \quad (3.8)$$

Substituting (3.5) and (3.6) in (3.8) and using (3.7), we obtain

$$S_{cc}^0 = \Omega_c^2 \left( 1 - \frac{i\Gamma_c^0}{E - E_c^0 - \Delta_c^0 + i(\Gamma_c^0/2 + W_c)} \right).$$

From the equations for the direct and conjugate solutions, namely

$$(E - H_c^0) |u_c^{0(+)}\rangle = 0, \quad (E' - \tilde{H}_c^0) \langle \tilde{u}_c^{0(+)} | = 0, \quad (3.9)$$

we can obtain by using Green's theorem, another useful relationship:

$$\langle \tilde{u}_c^{0(+)} | u_c^{0(+)} \rangle_R = \frac{2\hbar}{E_R} \Pi_c(E, R) \frac{\Theta_c^{02} \Omega_c^2}{\Omega_c^{02}}. \quad (3.10)$$

In the derivation of (3.10) we have used the expression

$$\langle \tilde{u}_c^{0(+)} | = \frac{1}{v_c^{1/2} r} (O_c - \tilde{S}_{cc}^{0*} I_c) \langle c |,$$

where  $\tilde{S}_{cc}^{0*}$  has in accord with (3.8) the form

$$\tilde{S}_{cc}^{0*} = (S_{cc}^0)^{-1} = \Omega_c^{-2} \frac{L_c - L_c^0}{L_c^* - L_c^0}.$$

For  $\langle u_c^{0(+)} |$  inside the nucleus we have an expansion similar to (3.3), namely

$$\langle \tilde{u}_c^{0(+)} | = \tilde{f}_c^*(E, R) \tilde{N}_c^*(E, r) \langle c |, \quad (3.11)$$

which when substituted in (3.10) yields

$$\tilde{f}_c^*(E, R) = -i \left( \frac{2\hbar \Pi_c(E, R)}{E_R} \right)^{1/2} (\Omega_c^0)^{-1}$$

under the normalization condition

$$\langle \tilde{N}_c | N_c \rangle_R = \Theta_c^0 \Omega_c / \Omega_c^0.$$

Introducing the reduced half-widths in channel  $c$  by means of the formulas

$$b_{\lambda c} = E_R^{-1/2} \langle \chi_\lambda | \Gamma | c N_c \rangle, \quad b_{c\lambda}^+ = E_R^{-1/2} \langle c N_c^* | \Gamma^+ | \chi_\lambda \rangle \quad (3.12)$$

and using expansion (3.3), we can represent the transition matrix elements contained in the expression for the  $S$  matrix (2.11) and in (2.12) in the form

$$\begin{aligned} \langle \tilde{u}_c^{0(-)} | \Gamma^+ | \chi_\lambda \rangle &= i (2\hbar)^{1/2} b_{c\lambda}^+ \Pi_c^{1/2} \Omega_c \Theta_c^0, \\ \langle \chi_\lambda | \Gamma | u_c^{0(+)} \rangle &= i (2\hbar)^{1/2} b_{\lambda c} \Pi_c^{1/2} \Omega_c \Theta_c^0. \end{aligned} \quad (3.13)$$

From the condition of "generalized reality" of the wave functions of the compound nucleus<sup>[8]</sup>

$$\langle \chi_\lambda | = (-1)^{J-M} O | \chi_\lambda (-M) \rangle$$

and the condition for the invariance of the Hamiltonian under time reversal  $O^+ H^* O = H$  (see [10]), the consequence of which is the condition  $O^+ \Gamma^* O = \Gamma$ , we can establish the equality

$$b_{\lambda c} = b_{c\lambda}^+. \quad (3.14)$$

The expression for  $\Gamma_{\lambda c}$  from (2.13) can be transformed after slight manipulation, using (3.13) and (3.11) and assuming that  $N_c(E, r)$  is weakly dependent on  $E$  (see the last footnote), into

$$\begin{aligned} \Gamma_{\lambda c}/2 &= |\text{Im}_e \mathcal{H}_{\lambda\lambda}| = \Pi_c(E, R) b_{\lambda c}^2, \\ \text{Im}_e \mathcal{H}_{\lambda\lambda} &= -\Pi_c(E, R) b_{\lambda c}^2 \Omega_c^2 \Theta_c^{02} / \Omega_c^{02}. \end{aligned} \quad (3.15)$$

The expression for the diagonal element of the  $S_{cc}^J$  matrix from (2.11) assumes upon substitution of (3.12), (3.13), and (3.15) the following form

$$S_{cc}^J = S_{cc}^{0J} - 2i \Pi_c \Omega_c^2 \Theta_c^{02} \sum_\lambda b_{\lambda c} Z_{\lambda c} + \frac{2\pi}{D_J} \bar{b}_{\lambda c}^2 \Pi_c \Omega_c^2 \Theta_c^{02}, \quad (3.16)$$

where  $Z_{\lambda c}$  is determined by the system (2.12), transformed into

$$\begin{aligned} \sum_\lambda \left[ \delta_{\mu\lambda} - \frac{\pi}{D_J} \bar{b}_{\lambda c} \bar{b}_{\mu c} \Pi_c \left( \frac{\Omega_c \Theta_c^0}{\Omega_c^0} \right)^2 \right] \left( E - E_\lambda - \Delta_{\lambda r} + i \frac{\Gamma_{\lambda r}}{2} \right) \\ + i b_{\lambda c} b_{\mu c} \Pi_c \left( \frac{\Omega_c \Theta_c^0}{\Omega_c^0} \right)^2 \Big\} Z_{\lambda c} = b_{\mu c}, \end{aligned} \quad (3.16a)$$

with the sought quantities  $Z_{\lambda c}$  denoting

$$Z_{\lambda c} = -i (2\hbar \Pi_c)^{-1/2} (\Omega_c \Theta_c^0)^{-1} Y_{\lambda c};$$

the expression for  $Y_{\lambda c}$  is given in (2.12).

#### 4. AVERAGE CROSS SECTIONS

With the aid of well known formulas<sup>[3,4]</sup> we can relate the cross sections for different reactions (total cross section  $\sigma_t$ , elastic scattering cross section  $\sigma_s$ , absorption or total-reaction cross section  $\sigma_a$ ) with the diagonal elements of the  $S_{cc}^J$  matrix

$$\sigma_t(E) = \sum_J \sigma_t^J(E) = \pi \lambda^2 \sum_{J,I} 2g_J (1 - \text{Re } S_{cc}^J), \quad (4.1)$$

$$\sigma_s(E) = \sum_J \sigma_s^J(E) = \pi \lambda^2 \sum_{J,I} g_J |1 - S_{cc}^J|^2, \quad (4.2)$$

$$\sigma_a(E) = \sum_J \sigma_a^J(E) = \pi \lambda^2 \sum_{J,I} g_J (1 - |S_{cc}^J|^2), \quad (4.3)$$

where  $g_J$  is the statistical weight of the state of the compound nucleus with spin  $J$ .

In addition, in accordance with [4], we have for the average cross sections respectively:

$$\bar{\sigma}_t^J = 2(1 - \text{Re } S_{cc}^{0J}), \quad (4.4)$$

\*In units of  $g_J$  and  $\pi \lambda^2$ .

$$\bar{\sigma}_s^J = |1 - S_{cc}^{0J}|^2 + (|S_{cc\text{res}}^J|^2 - |\bar{S}_{cc\text{res}}^J|^2), \quad (4.5) \quad \text{and accordingly}$$

$$\bar{\sigma}_{se}^J = |1 - S_{cc}^{0J}|^2 \quad (4.6) \quad b_{c\lambda}^+ = E_R^{-1/2} \langle cN_c | \Gamma^+ | \chi_\lambda \rangle = b_{\lambda c}^*$$

—cross section for elastic scattering on the optical potential,

$$\bar{\sigma}_{ee}^J = |\overline{S_{cc\text{res}}^J}|^2 - |\bar{S}_{cc\text{res}}^J|^2 \quad (4.7)$$

—cross section for elastic scattering via the compound nucleus,

$$\bar{\sigma}_c^J = 1 - |S_{cc}^{0J}|^2 \quad (4.8)$$

—cross section for production of the compound nucleus

$$\bar{\sigma}_a^J = \bar{\sigma}_c^J - \bar{\sigma}_{ce}^J \quad (4.9)$$

—cross section for the total reaction.

We put for brevity

$$S_{cc}^J = S_{cc}^{0J} + S_{cc\text{res}}^J - \bar{S}_{cc\text{res}}^J;$$

$$S_{cc\text{res}}^J = -2i\Omega_c^2 \theta_c^{02} \Pi_c \sum_{\lambda} b_{\lambda c} Z_{\lambda c}, \quad \bar{S}_{cc\text{res}}^J = -\frac{2\pi}{D_J} \bar{b}_{\lambda c}^2 \Omega_c^2 \theta_c^{02} \Pi_c. \quad (4.10)$$

Thus, for example, for  $\bar{\sigma}_{ce}^J$  we can obtain, in accord with (4.7) and (4.10), the expression

$$\bar{\sigma}_{ce}^J = \frac{2\pi}{D_J} \frac{(\bar{\Gamma}_{\lambda n}^J)^2}{\bar{\Gamma}_{\lambda}^J} - \pi^2 \left( \frac{\bar{\Gamma}_{\lambda n}^J}{D_J} \right)^2 \quad \text{for } \frac{\bar{\Gamma}_{\lambda n}}{D} \ll 1, \quad (4.11)$$

where  $\bar{\Gamma}_{\lambda n}^J = 2b_{\lambda c}^2 \Pi_c$  —average neutron width of the levels  $\lambda$  with spin  $J$ ,  $\bar{\Gamma}_{\lambda}^J$  —average total width of levels  $\lambda$  with spin  $J$ .

For  $\bar{\sigma}_c^J$  we can obtain from (4.8) an expression in terms of the imaginary part of the optical potential  $W_c$ , if we use the pair of complex conjugate equations

$$(E - H_c^0) | u_c^{0(+)} \rangle = 0, \quad (E' - H_c^{0*}) \langle u_c^{0(+)} | = 0. \quad (4.12)$$

Integrating over a volume of radius  $R$  we obtain with the aid of the Green's theorem the relation

$$1 - |S_{cc}^{0J}|^2 = 2\hbar^{-1} \langle u_c^{0(+)} | \text{Im} V_c^0 | u_c^{0(+)} \rangle, \quad (4.13)$$

from which, using the expression for  $\text{Im} V_c^0$  and  $|u_c^{0(+)}\rangle$  from (2.10) and (3.3),

$$1 - |S_{cc}^{0J}|^2 = 4W_c E_R^{-1} \Pi_c(E, R), \quad (4.14)$$

and for  $W_c$  we have

$$W_c = \langle cN_c | \text{Im} V_c^0 | cN_c \rangle = -\frac{\pi}{D_J} E_R |\bar{b}_{\lambda c}|^2 - \frac{E_R}{2\hbar} \sum_r |b_{cr}|^2.$$

Here

$$b_{rc} = E_R^{-1/2} \langle u_r^{(+)} | \Gamma_{rc} | cN_c \rangle$$

—amplitude of reduced single-particle half-width of the direct transition with redistribution of the particles. Here  $b_{rc} = b_{cr}^*$ , where

$$b_{cr}^+ = E_R^{-1/2} \langle cN_c | \Gamma_{cr}^+ | u_r^{(+)} \rangle$$

## 5. CALCULATION RESULTS

To check the theory we made actual calculations of the cross sections for the interaction between neutrons and the nuclei  $\text{Mn}^{55}$ ,  $\text{U}^{238}$ , and  $\text{Th}^{232}$  in the region below 1 MeV. To estimate the logarithmic derivatives  $L_c^0$  we used the rectangular complex well model, following FPW,<sup>[4]</sup> with the following parameters:

$$\text{Re}V_c^0 = 42 \text{ MeV}, \quad \xi = 0.05, \quad W_c = 2.13 \text{ MeV}, \\ R = 1.25 A^{1/3} \cdot 10^{-13} \text{ cm}.$$

For the case of  $\text{Mn}^{55}$  we calculated the total cross section  $\sigma_t$  and the radiation capture cross section  $\sigma_\gamma$ , which in this case coincides with the total absorption cross section  $\sigma_a$ . It has been assumed in the computational scheme that the parameters for the nuclei  $\text{Mn}^{55}$  and the energies under consideration are

$$\text{Re}L_0^0 = 0, \quad \gamma_2^{02} \text{Re}L_2^0 + \Delta_2^0 = 0, \quad E_0^0 = E_2^0 = 0, \\ \gamma_0^{02} \approx \gamma_2^{02} = 1.38 \text{ MeV}, \quad R = 5.5 \cdot 10^{-13} \text{ cm}.$$

For  $\text{Th}^{232}$  in  $\text{U}^{238}$ , in approximately the same energy region, best agreement between theory and experiment<sup>[11]</sup> is obtained for the following parameters: for  $\text{Th}^{232}$

$$\text{Re}V^0 = 42 \text{ MeV}, \quad W = 1.26 \text{ MeV}, \\ \gamma_1^{02} E_1^0 + \Delta_1^0 = 10 \text{ keV}, \quad E_0^0 \approx 5 \text{ MeV}, \\ \gamma_1^{02} = 0.58 \text{ MeV}, \quad R = 8.3 \cdot 10^{-13} \text{ cm};$$

and for  $\text{U}^{238}$

$$\text{Re}V^0 = 42 \text{ MeV}, \quad W = 1.26 \text{ MeV}, \\ \gamma_1^{02} E_1^0 + \Delta_1^0 = 10 \text{ keV}, \quad E_0^0 \approx 5 \text{ MeV}, \\ \gamma_1^{02} = 0.58 \text{ MeV}, \quad R = 8.5 \cdot 10^{-13} \text{ cm}.$$

## 6. CONCLUSION

The nuclear-reaction theory developed in the present paper starts out with the possibility of two types of interaction between neutrons and nuclei, single-particle and many-particle. Unlike the earlier theories,<sup>[1-4,8]</sup> the present theory enables us to obtain an exact expression for the diagonal S-matrix element in terms of the diagonal element of the ordinary optical scattering  $S^0$  matrix and contributions from the resonances of the compound nucleus [see (3.16)]. Knowledge of the exact S matrix of the problem enables us to obtain different cross sections for the interaction between neutrons and nuclei (Sec. 4). The theory is most

effective in describing the average cross sections.

Knowing the fluctuating part of the S matrix, we can determine the cross section for compound-elastic scattering and consequently separate the neutron absorption cross section from the cross section for the production of the compound nucleus [see (4.11) and (4.14)]. The theory makes it possible to separate in the "force function"  $\overline{\gamma_{\lambda C}^2 / \overline{D}_J}$  of the earlier theories<sup>[1-3]</sup> the "large resonance structure" connected with the optical properties of the penetration factor  $\Pi_C(E, R)$  (Sec. 3). We also establish a connection between the imaginary part of the optical potential  $W_{C \text{ res}}$ , due to the resonances of the compound nucleus, and the true "force function"  $\overline{b_{\lambda C}^2 / \overline{D}_J}$  [see (4.14)].

Numerical calculations of the total cross sections for the interaction between neutrons and  $\text{Mn}^{55}$ ,  $\text{Th}^{232}$ , and  $\text{U}^{238}$  and of the cross section for radiative capture of neutrons by  $\text{Mn}^{55}$  show satisfactory agreement between theory and experiment.

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