CONVECTIVE INSTABILITY SPECTRUM OF A CONDUCTING MEDIUM IN A MAGNETIC FIELD

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The conditions for vibrational convective instability of a conducting medium in a magnetic field are determined. It can be seen from the example considered that vibrational instability arises if $4\pi\sigma\chi/c^2 > 1$ and if the field is greater than some critical value.

As is well known, the small perturbations of the equilibrium of a liquid heated from below are generated monotonically.[1] For a conducting liquid in a magnetic field, this principle does not generally hold. In a research of one of the authors,[2] it was noted that the perturbations are monotonic only for weak fields. Sorokin and Sushkin[3] obtained solutions of the equations for the perturbations in the form of power series in M^2 (M is the Hartmann number). These expansions were found to be real, whence it follows that the perturbations are monotonic, at any rate for weak fields. Chandrasekhar[4] investigated the equilibrium of a plane horizontal layer heated from below. From his work it is clearly evident that vibrational excitations are possible. However, the conditions for which vibrational instability takes place were essentially not fully explained; in particular, a conclusion was drawn that vibrational instability is possible even for weak fields. This conclusion, which is discussed in the literature (see [5]) contradicts the results pointed out above. We present an example which seems to us simpler than that considered by Chandrasekhar, and which makes it possible to clarify the situation.

We consider the stability of a plane vertical layer of a conducting medium heated from below and located in a transverse magnetic field H_0. The temperature gradient at equilibrium is equal to $\nabla T_0 = -A\boldsymbol{\gamma}$ ($\boldsymbol{\gamma}$ is a unit vector directed upwards). Let the perturbations of the equilibrium be such that the velocity **v** and the perturbation of the field **H** be vertical, and let the perturbation of temperature be $T = T(x, y)$, where x is the transverse coordinate of the layer, measured from its middle. Assuming the pressure gradient to be zero (free convection) and taking all quantities to depend on the time as $e^{\lambda t}$, we get from the usual equations of the magnetohydrodynamics the following equation for the perturbations:

$$\lambda v = v'' + RT + M^2 H',$$
$$\lambda PT = v + T'', \quad \lambda P_m H = v' + H''. \qquad (1)$$

All quantities in (1) are dimensionless; the units of length, time, velocity, temperature, and field are chosen to be d (half thickness of the layer), d^2/ν, χ/d, Ad, and $4\pi\sigma\chi H_0/c^2$. The dimensionless parameters are $R = g\beta A d^2/\nu\chi$ (the Rayleigh number), $M = (\sigma/\eta)^{1/2} H_0 d/c$ (the Hartmann number), $P = \nu/\chi$ (the Prandtl number), and $P_m = 4\pi\sigma\nu/c^2$.

On the boundaries of the layer, $x = \pm 1$, the velocity v, and the temperature perturbation T vanish. We assume the boundary to be an ideal conductor; then, in accord with the Fermi condition, $H' = 0$ (current vanishes). Under such boundary conditions, the solution of the problem is simply

$$v = v_0 \sin \pi x, \quad T = T_0 \sin \pi x, \quad H = H_0 \cos \pi x. \qquad (2)$$

A simple solution also exists for the boundary conditions $v' = 0$, $T' = 0$ (free, thermally insulated boundaries), $H = 0$. The solution in this case is

$$v = v_0 \sin (\pi x/2), \quad T = T_0 \sin (\pi x/2),$$
$$H = H_0 \cos (\pi x/2). \qquad (3)$$

Any change in the boundary conditions is reflected only in the numerical coefficients of the characteristic equation; we therefore take in the following the solution in the form (2). The eigenvalues λ of the perturbations (2) are determined from the equation

$$A\lambda^3 + B\lambda^2 + C\lambda + D = 0;$$
$$A = PP_m, \quad B = \pi^2 (P + P_m + PP_m),$$
$$C = \pi^4 (1 + P + P_m) + \pi^2 PM^2 - RP_m,$$
$$D = \pi^2 (\pi^4 + \pi^2 M^2 - R). \qquad (4)$$

Setting $\lambda = a + ib$, we find the equations for the real and imaginary parts:

$$A(a^3 - 3ab^2) + B(a^2 - b^2) + Ca + D = 0, \quad (5)$$

$$b[A(3a^2 - b^2) + 2aB + C] = 0. \quad (6)$$

It is seen from (6) that $b = 0$ is possible. This corresponds to real λ, i.e., to perturbations that attenuate or grow monotonically. On the boundary of the instability we have $a = 0$, and the motion is stationary. The critical Rayleigh number for monotonic perturbations, R_1, is found from (5), where it is necessary to set $a = 0$ and $b = 0$, i.e., $D = 0$. We then find*

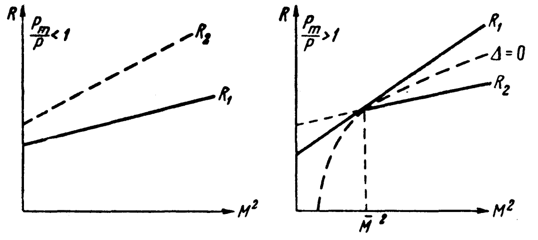
$$R_1 = \pi^4 + \pi^2 M^2. \quad (7)$$

We now investigate the vibrational perturbations that correspond to $b = 0$. In (6), it is necessary to set the expression in square brackets equal to zero. On the boundary of the stable region we have $a = 0$, and the motion is a vibration with frequancy b. The critical Rayleigh number R_2 for vibrational excitations and the frequency b are obtained from (5) and (6) for $a = 0$:

$$R_2 = \pi^4 \frac{(P + P_m)(1 + P_m)}{P_m^2} + \pi^2 \frac{1 + P_m}{1 + P} \frac{P^2}{P_m^2} M^2, \quad (8)$$

$$b^2 = \pi^4 \frac{P}{P_m}\left(\frac{M^2}{\pi^2}\frac{P_m - P}{1 + P} - 1\right). \quad (9)$$

Equations (7) and (8) describe the branches of the stability curves for monotonic and vibrational perturbations, respectively. In the (R, M^2) plane, both (7) and (8) are straight lines. For $M = 0$ we have $R_2 > R_1$ and the slopes of the straight lines depend on the ratio P_m/P: when $P_m/P < 1$ the lines do not cross; when $P_m/P > 1$, the lines cross for some $M^2 = \overline{M}^2$ (see the drawing). The only parts of the stability curves $R_2(M^2)$ that have meaning are those for which $b^2 > 0$. It is evident from (9) that when $P_m/P < 1$ we have $b^2 < 0$ along the entire curve R_2; i.e., when $4\pi\sigma\chi/c^2 < 1$ instability sets in only under the action of monotonic perturbations (for liquid metals under normal conditions, $4\pi\sigma\chi/c^2 < 1$). When $P_m/P > 1$, the sign of b^2 changes along R_2 at the point of intersection M^2: to the left of the point of intersection, $b^2 < 0$ and, consequently, for weak fields, the equilibrium crisis is produced only by monotonic perturbations. For $M > \overline{M}$, there are two stability branches which emanate from a single point $\overline{M^2}$ (bifurcation): the lower of these (and, consequently, the more "dangerous") corresponds to



the vibrational perturbations, the upper to monotonic.

V. S. Sorokin kindly directed our attention to the following curious circumstance. The curve $\Delta = 0$ [Δ is the discriminant of Eq. (4)] passes between the curves R_1 and R_2 (see the drawing). Both complex roots of Eq. (4) exist in the region below the curve $\Delta = 0$. Therefore, if we continue to increase the Rayleigh number after crossing the vibration threshold R_2, then the vibrational perturbations disappear before the monotonic threshold R_1 is reached.

Thus, vibrational stability in our example takes place under certain properties of the medium $(4\pi\sigma\chi/c^2 > 1)$ and for sufficient field $(M > \overline{M})$. The critical field $\overline{M}$ is determined from the condition $R_1 = R_2$ (or $b^2 = 0$):

$$\overline{M}^2 = \pi^2(1 + P)/(P_m - P). \quad (10)$$

In the problem of Chandrasekhar, it is also necessary to investigate the frequency of "neutral" vibrations along the line R_2, and to discard those parts of the branch on which this frequency is imaginary. Then the spectrum is qualitatively the same as in the example considered.

The described situation evidently takes place in the case of a cavity of arbitrary shape. If the conditions $v = 0$, $T = 0$, $H_n = j_\tau = 0$ (the Fermi condition) or $\partial T/\partial n = 0$, $v_n = \partial v_\tau/\partial n = 0$ (free, thermally insulated boundaries), and $\mathbf{H} = 0$ are satisfied on the boundary of the cavity, then one can get [2,3] expressions for the real and imaginary parts of λ from the general equations:

$$(\lambda - \lambda^*)\int(|\mathbf{v}|^2 + RP|T|^2 - M^2 P_m |\mathbf{H}|^2)\,dV = 0, \quad (11)$$

$$\lambda + \lambda^* = \frac{2\int(|\mathrm{rot}\,\mathbf{v}|^2 + M^2|\mathrm{rot}\,\mathbf{H}|^2 - R|\nabla T|^2)\,dV}{\int(RP|T|^2 - |\mathbf{v}|^2 - M^2 P_m|\mathbf{H}|^2)\,dV}. \quad (12)^*$$

For monotonic excitations, $\lambda - \lambda^* = 0$, and the critical Rayleigh number R_1 is found from the condition $\lambda + \lambda^* = 0$. The vibrational branch of R_2 is determined by the conditions $\lambda - \lambda^* \neq 0$ and $\lambda + \lambda^* = 0$. It is seen from (11) that $\lambda - \lambda^*$ can be different from zero only in the case when the integral in (11) is equal to zero. This can be

*The same dependence of the critical R on the field was obtained by Regirer[6] for other boundary conditions.

*rot = curl.

the case, as is seen, only for sufficiently large fields; for weak fields ($M^2 \to 0$) the integral is always positive; this means that $\lambda - \lambda^* = 0$ and only monotonic excitations are possible. The critical field is determined by the integral equation derived from $R_1 = R_2$:

$$\overline{M^2} = \frac{\int |\mathbf{v}|^2 dV \int |\nabla T|^2 dV + P \int |T|^2 dV \int |\mathrm{rot}\, \mathbf{v}|^2 dV}{P_m \int |\mathbf{H}|^2 dV \int |\nabla T|^2 dV - P \int |T|^2 dV \int |\mathrm{rot}\, \mathbf{H}|^2 dV}. \quad (13)$$

It is then seen that the necessary condition for the existence of vibrational stability is

$$P_m / P > \int |T|^2 dV \int |\mathrm{rot}\, \mathbf{H}|^2 dV \Big/ \int |\mathbf{H}|^2 dV \int |\nabla T|^2 dV, \quad (14)$$

where the right side is clearly of the order of unity.

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