CRITICISM OF AN ATTEMPT TO DISPROVE THE UNCERTAINTY RELATION BETWEEN TIME AND ENERGY

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In Sec. 1, a review of the various interpretations of the uncertainty relation between energy and time is presented. In Sec. 2 an attempt by Aharonov and Bohm to disprove the relation $\Delta(E'-E)\Delta t > h$, and to subject to doubt the interpretation of this relation given by Fock and Krylov, is criticized. In the example used by Aharanov and Bohm to refute the uncertainty relation there appear in the Hamiltonian operator discontinuous functions of time (instantaneous switching on and off of the interaction); but this implies the introduction of a field which does not obey the uncertainty relation and so taking as a premise in the discussion the very proposition to be proven. Aharonov and Bohm's argument thus contains a logical error known as petitio principii (begging the question). The example in the criticized article, when treated correctly, only serves to support the uncertainty relation.

1. TWO TYPES OF RELATIONS FOR ENERGY AND TIME

THE interpretation of the Heisenberg uncertainty relation for energy and time was discussed in detail by Krylov and Fock in 1947.[1] They showed that it is necessary to distinguish between two types of uncertainty relations. Relations referring to the act of measurement (to which the Schrödinger equation is inapplicable) belong to the first type, whereas those referring to the unperturbed state, which develops according to the Schrödinger equation, belong to the second type.

The relation of the first type might be called the Heisenberg-Bohr relation. It is best written in the form

$$\Delta(E'-E)\,\Delta t > h, \tag{1}$$

where $\Delta(E'-E)$ and Δt stand for the absolute values of the uncertainty in the energy change $E'-E$ and in the instant of time t, when this change occurred.

The derivation of relation (1) requires, beside the application of the apparatus of quantum mechanics, additional considerations that allow one to reduce the uncertainties in the instant of time and in the energy difference to uncertainties in the coordinate and momentum; the necessity for additional considerations is due on the one hand to the fact that one may not use in the derivation the Schrödinger equation, and on the other hand to the fact that time is not an operator. For this reason the derivation is not as direct and formal as the derivation of the Heisenberg relations for coordinates and momenta.

The relations of the second type (referring to the unperturbed state) may in turn be subdivided into two kinds: one relates the decay half-life of a quasi stationary state to the width of the level, the other, found by Mandel'shtam and Tamm,[2] relates the shift time of a wave packet to the uncertainty in energy.

The first of these relations may be viewed as a consequence of the general theorem of Fock and Krylov on the connection between the decay law and the energy distribution function.* According to this theorem, if the energy distribution function for the initial state is given by

$$dW(E) = w(E)\,dE, \tag{2}$$

then the decay law for that state is expressed by the formula

$$L(t) = \left|\int e^{-iEt/h}\,dW(E)\right|^2. \tag{3}$$

The quantity $L(t)$ is the probability that at the time t the system has not yet decayed. For an energy distribution given by the dispersion formula

*This theorem is contained in an indirect form in the course of lectures given by V. A. Fock at the Leningrad University in 1936-1937;[3] its first explicit formulation and proof was given by Krylov and Fock in 1947.[1] In the years following 1959 the theorem has been rediscovered in connection with problems in the theory of unstable particles by a number of authors, in particular by Levy,[4] Matthews and Salam,[5] and Petzold.[6]

$$w(E) = \frac{1}{\pi} \frac{\Gamma}{(E - E_0)^2 + \Gamma^2} \tag{4}$$

it follows, with sufficient accuracy, that

$$L(t) = \exp\left(-\frac{2}{h}\Gamma t\right), \tag{5}$$

so that if the half-life $T_{1/2}$ is defined by the condition $L(T_{1/2}) = 1/2$ we obtain

$$\Gamma T_{1/2} = \frac{h}{2} \ln 2, \tag{6}$$

where Γ is the level width appearing in the dispersion formula.

The uncertainty relations of the second kind for the unperturbed state are based on the expression for the time derivative of the operator for a certain quantity R. If in the given state the quantity R has a standard (mean square deviation from the mean) equal to ΔR, and if the mathematical expectation (mean) value of the quantity R is $\bar{R}$, then following Mandel'shtam and Tamm we can introduce the time interval ΔT given by

$$\Delta T = \Delta R / (\partial \bar{R} / \partial t). \tag{7}$$

This is the time in which the wave packet for the quantity R will be shifted by ΔR. The time interval ΔT is, in a well-known sense, a measure of the stationarity of the system with respect to the quantity R. According to Mandel'shtam and Tamm this time interval is related to the standard (mean square deviation) of the energy ΔH by

$$\Delta H \, \Delta T \geqslant h/2, \tag{8}$$

no matter what the state of the system and the quantity R might be.

This relation looks superficially like the Heisenberg-Bohr relation, Eq. (1), however its meaning is different. The relation (8) refers to the unperturbed motion of the wave packet and the time ΔT does not refer to the duration of the measurement [as Δt does in Eq. (1)] but rather gives the time delay in the beginning of the measurement—the time delay necessary for the wave packet for R to be shifted by the amount ΔR. A detailed comparison of the relations (1) and (8) may be found in [1].

On the other hand, although both relations (6) and (8) refer to an unperturbed system they do not have the same physical meaning. The relation (8) is, generally speaking, inapplicable to the decay law since the energy distribution in (8) is characterized by too gross an energy standard ΔH, which in problems of this type either does not exist at all or does not represent a characteristic magnitude.

2. CRITICISM OF THE WORK OF AHARONOV AND BOHM

Aharonov and Bohm [7] attempt to refute the Heisenberg-Bohr uncertainty relation, written in the form of Eq. (1) and interpreted in the sense of Krylov and Fock [$\Delta(E' - E)$ and Δt are the uncertainties in the magnitude and in the instant of energy change, see above]. Aharonov and Bohm arrive at the conclusion that the interpretation of Krylov and Fock is incorrect, and that in the general case the relation (1) does not hold.

At the same time Aharonov and Bohm acknowledge the validity of the relation (8), since it follows from the Schrödinger equation and belongs to the second type. The rejection of the uncertainty relation of the first type is motivated by a (incorrectly understood) statement by Bohr, that no limitations must be imposed on individual measurements except those that follow from the apparatus and interpretation of quantum mechanics. (These words of Bohr, clearly, must not be interpreted as is done by Aharonov and Bohm to forbid the use of additional considerations that allow one to reduce the uncertainties in the energy difference and in time to those in momentum and coordinate; such additional considerations are constantly used by Bohr in, for example, his well known "Discussions with Einstein." [8])

We shall not analyze here all of the considerations given by Aharonov and Bohm in their attempt to refute the relation (1), but will limit ourselves to a discussion of the example, which in the opinion of the named authors violates the relation. The authors, apparently, consider this example decisive.

Aharonov and Bohm consider the collision of two particles (x) and (y), each with one degree of freedom, which compose a system with a Hamiltonian operator

$$H = \frac{1}{2m} p_x^2 + \frac{1}{2m} p_y^2 + y p_x g(t), \tag{9}$$

where $g(t)$ is a discontinuous function of time with the constant value $g(t) = g_0$ during a certain time interval of width Δt, and the value $g(t) = 0$ outside that time interval. The authors assert that if the interaction between the particles is of the form indicated in Eq. (9), then it is possible in the time Δt to precisely measure the increase in energy $E' - E$ of particle (x), thus violating the uncertainty relation (1).

Before entering into the analysis of the considerations of Aharanov and Bohm let us call attention to the fact that the introduction of an explicitly time-dependent interaction is equivalent to the introduction of a field which can be described classically (in the correspondence principle sense). However the classical fields and test bodies utilized in the equations of quantum mechanics are not free of the limitations imposed by the Heisenberg uncertainty relations. On the contrary, the use of classical fields and test bodies is based on the premise that the uncertainty relations are satisfied for them so strongly that it is unnecessary to introduce these relations explicitly. If however it is (illegitimately) assumed that the classical field or test particle may violate the Heisenberg relations it should come as no surprise that these relations will also be violated for the quantum mechanical particle serving as object.

Just such an illegitimate assumption was made by Aharonov and Bohm. Indeed, the introduction into the Hamiltonian operator of discontinuous functions of time implies admitting the possibility that the field which carries the interaction between the particles may be instantaneously switched on and off. But such a possibility violates relation (1) for this field. For the mechanical system under consideration the instantaneous switching on and off of the interaction means an instantaneous change in the energy by a prescribed finite amount [so that $\Delta(E'-E) = 0$] in a given instant of time (so that $\Delta t = 0$). The possibility of such a change amounts to a hypothesis that is even stronger than the assertion the authors wish to prove [they only claim that it is possible to have $\Delta(E'-E) = 0$ with $\Delta t \neq 0$].

On the formal side, the reasoning of Aharonov and Bohm is an example of an error in logic, long known under the name petitio principii (begging the question): assuming in the premise that which is to be proved.

Having clarified the logical nature of the error of Aharonov and Bohm let us show that the operator (9) leads to the uncertainty relation (1), provided that one does not introduce discontinuous functions but assumes instead that during the time under consideration (of the order of Δt) the interaction keeps its order of magnitude. For the operator (9) the Hamilton equations take on the form

$$\dot{x} = \frac{1}{m} p_x + yg, \qquad \dot{p}_x = 0$$

$$\dot{y} = \frac{1}{m} p_y, \qquad \dot{p}_y = -g p_x. \tag{10}$$

The energy of particle (x) is given by

$$E = \tfrac{1}{2} m\dot{x}^2. \tag{11}$$

[It is clear that one may not take as energy the quantity $p_x^2/(2m)$, since the canonical variable p_x does not coincide with the kinetic momentum $m\dot{x}$.] Making use of the expression for $\dot{x}$ it is easy to see that the uncertainty in the energy change will be of the order of

$$\Delta(E'-E) = \Delta y \cdot |g p_x|, \tag{12}$$

where Δy is the uncertainty in y during the interaction (collision) time. Further, the duration of the collision Δt is equal to the uncertainty in the instant of time at which the energy transfer took place. The uncertainty in the value of p_y should satisfy the inequality

$$\Delta p_y < \Delta t\, |\dot{p}_y| = \Delta t\, |g p_x|, \tag{13}$$

since it should be less than the predictable part of the change in p_y, which is equal to $\dot{p}_y \Delta t$. Making use of Eq. (13) and the Heisenberg relation for y and p_y we obtain

$$\Delta y > h/\Delta p_y > h/(\Delta t\, |g p_x|), \tag{14}$$

and on introducing this inequality for Δy into Eq. (12) we arrive at the relation (1).

Consequently the example of Aharonov and Bohm, when treated correctly, does not refute but confirms the Heisenberg-Bohr uncertainty relation, Eq. (1).

The same considerations are also valid for systems described by a Hamiltonian operator of a more general form provided that we write everywhere in place of the product $g p_x$ the quantity $\partial H/\partial y$.

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