FISSION BARRIER OF A ROTATING NUCLEUS

G. A. PIK-PICHAK

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The liquid drop model is used to calculate the Coulomb and the surface energy of a nucleus for small arbitrary deformations of a spherical shape. Expressions are derived for the fission barrier and for the critical rotational energy at which the nucleus becomes unstable. The range of applicability of these expressions is determined.

EXPERIMENTS on the fission of the compound nuclei formed in reactions with heavy ions and therefore having large angular momentum have aroused interest in the fission properties of rotating nuclei. In earlier work [1] an expression was derived for the fission barrier of a rotating nucleus having small angular momentum. Hiskes [2] recently determined the fission barrier in the same approximation as in [1] but without expanding in a series with respect to the ratio of the rotational energy to the surface energy. Sitenko [3] has also investigated the effect of nuclear rotation on fission, assuming that the moment of inertia of a nucleus equals that of a liquid drop. In this way even for relatively small angular momentum the rotational energy becomes large enough to make fission possible, in disagreement with experiment.

The limits of applicability of the different expressions for the fission barrier were determined in none of the aforementioned publications; an unambiguous comparison of theory and experiment has thus been difficult to achieve. The present work has two purposes: first, to determine the applicability range of an expansion of the total nuclear energy in small deformations of a sphere; secondly, to determine the characteristics of a rotating nucleus that can be compared with experiment. Exact expressions will be derived, which will be compared with other, approximate, calculations. The influence of rotation on fission has been discussed in [1]; we proceed here directly to determine the total nuclear energy, the fission barrier, deformations etc.

We shall regard a nucleus as a drop of uniformly charged incompressible fluid whose surface is described by

$$r(\theta, \varphi) = R\Big(1 + \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \alpha_{lm} D_{lm}(\theta, \varphi)\Big). \qquad (1)$$

Here r is the radius of the sphere as a function of the spherical angles θ and φ; $D_{lm}(\theta,\varphi) = D^{0}_{m0}(\varphi, \theta, 0)$ are spherical functions [4,5] with the normalization $D^{l}_{m0}(0, 0, 0) = 1$; α_{lm} are parameters of the nuclear deformation relative to a sphere. In the general case we have $\alpha_{lm} = \beta_{lm} + i\gamma_{lm}$, where β_{lm} and γ_{lm} are real.

From the reality of r and the relation $D^{l*}_{m0}(\varphi, \theta, 0) = (-1)^m D^{l}_{-m0}(\varphi, \theta, 0)$, where the asterisk denotes the complex conjugate, it follows that

$$\beta_{l-m} = (-1)^m \beta_{lm}, \quad \gamma_{l-m} = (-1)^{m+1}\gamma_{lm}. \qquad (2)$$

The change of Coulomb energy of a nucleus having charge Ze for an arbitrarily small deformation represented in (1), while the volume and position of the center of gravity of the deformed nucleus remain unchanged, is given by

$$\begin{aligned}\Delta u_C = \frac{3}{10}\frac{Z^2e^2}{R}\Big\{&\sum \alpha_{st}\alpha_{s-t}\frac{(-1)^t}{2s+1}\Big(\frac{15}{2s+1}-5\Big)\\
&+\sum \alpha_{st}\alpha_{r\nu}\alpha_{pi} I^{p-i}_{str\nu}\frac{(-1)^i}{2p+1}\Big(-\frac{35}{6}-\frac{15}{2}\frac{p-1}{2p+1}+\frac{15}{2}\frac{s+2}{2s+1}\Big)\\
&+\sum \alpha_{st}\alpha_{s-t}\alpha_{r\nu}\alpha_{r-\nu}\frac{(-1)^{t+\nu}}{(2s+1)(2r+1)}\Big(5-\frac{45}{2r+1}\Big)\\
&+\sum \alpha_{st}\alpha_{r\nu}\alpha_{pi}\alpha_{\delta f} I^{\omega\gamma}_{str\nu} I^{\omega-\gamma}_{pi\delta f}\frac{(-1)^\gamma}{2\omega+1}\\
&\times\Big[-\frac{5}{2}+\frac{5}{12}(18\delta^2+18\delta-9\omega^2-9\omega-6p^2-6p)\\
&+\frac{5}{2}\frac{3\omega^2+3\omega-3p^2-3p-2\delta}{2\delta+1}-\frac{15}{4}\frac{(\omega-1)(\omega+2)}{2\omega+1}\\
&+\frac{5}{2}\frac{(s+1)(s+2)}{2s+1}\Big]+\sum \alpha_{st}\alpha_{r\nu}\alpha_{pi}\alpha_{\delta f} I^{1-\beta}_{str\nu} I^{1\beta}_{pi\delta f}(-1)^\beta\\
&\times\Big(\frac{35}{4}+\frac{15}{2}\frac{\delta-1}{2\delta+1}-\frac{15}{4}\frac{4p+5}{2p+1}\Big)\Big\},\end{aligned} \qquad (3)$$

where $I^{lm}_{str\nu} = C^{lm}_{str\nu} C^{l0}_{s0r0}$ ($C^{lm}_{str\nu}$ are Clebsch-Gordan coefficients). [4,5] Summations will be carried out over all values of the indices of $\alpha_{\mu\nu}$ and $I^{\omega i}_{\mu\nu st}$ allowed by the Clebsch-Gordan coefficients.

The change of surface energy for unaltered volume and position of the center of gravity is

(see the Appendix)

$$\Delta u_S = 4\pi R^2 O \Big\{ \sum \alpha_{st}\alpha_{s-t} \frac{(-1)^t}{2s+1} \frac{s(s+1)-2}{2}$$
$$- \frac{2}{3} \sum \alpha_{st}\alpha_{r\nu}\alpha_{pi} \frac{(-1)^i}{2p+1} I^{p-i}_{str\nu}$$
$$+ \sum \alpha_{st}\alpha_{s-t}\alpha_{r\nu}\alpha_{r-\nu} \frac{(-1)^{t+\nu}}{(2s+1)(2r+1)}$$
$$+ \sum \alpha_{st}\alpha_{r\nu}\alpha_{pi}\alpha_{\delta f} (-1)^{\beta} I^{1-\beta}_{str\nu} I^{1\beta}_{pi\delta f}$$
$$- \sum \alpha_{st}\alpha_{r\nu}\alpha_{pi}\alpha_{\delta f} I^{\omega\gamma}_{st-1r\nu+1} I^{\omega-\gamma}_{pi-1\delta f+1} \frac{(-1)^{\gamma}}{8(2\omega+1)}$$
$$\times \sqrt{s(s+1)-t(t-1)}\sqrt{r(r+1)-\nu(\nu+1)}$$
$$\times \sqrt{p(p+1)-i(i-1)}\sqrt{\delta(\delta+1)-f(f+1)}$$
$$- \sum \alpha_{st}\alpha_{r\nu}\alpha_{pi}\alpha_{\delta f} I^{\omega\gamma}_{st-1r\nu+1} I^{\omega-\gamma}_{pi\delta f} \frac{(-1)^{\gamma}}{4(2\omega-1)} if$$
$$\times \sqrt{s(s+1)-t(t-1)}\sqrt{r(r+1)-\nu(\nu+1)}$$
$$- \sum \alpha_{st}\alpha_{r\nu}\alpha_{\delta f}\alpha_{pi} I^{\omega\gamma}_{str\nu} I^{\omega-\gamma}_{pi\delta f} \frac{(-1)^{\gamma}}{2\omega+1} \frac{t\nu if}{8} \Big\}; \quad (4)$$

O is the coefficient of surface tension.

The change of nuclear rotational energy is taken to be the change of rotational energy of a solid body having the same shape as the nucleus for constant angular momentum independent of the deformation. Let α_{lm} in (1) describe the nuclear shape with respect to coordinate axes coinciding with the principal axes of inertia of the nucleus. The products of inertia are now zero:

$$J_{XY} = \int XY dv = 0, \qquad J_{XZ} = J_{YZ} = 0.$$

From this condition we obtain relations for α_{lm}:

$$\alpha_{2\pm1} = -2 \sum \alpha_{lm}\alpha_{st} I^{2\mp1}_{lmst} + \ldots, \quad (5)$$

$$\alpha_{2-2} + 2 \sum \alpha_{lm}\alpha_{st} I^{2-2}_{lmst} + 2 \sum \alpha_{lm}\alpha_{st}\alpha_{r\nu} I^{\omega\gamma}_{lmst} I^{2-2}_{\omega\gamma r\nu} + \ldots$$
$$= \alpha_{22} + 2 \sum \alpha_{lm}\alpha_{st} I^{22}_{lmst} + 2 \sum \alpha_{lm}\alpha_{st}\alpha_{r\nu} I^{\omega\gamma}_{lmst} I^{22}_{\omega\gamma r\nu} + \ldots \quad (6)$$

The principal moments of a solid body having the shape (1) with conservation of volume are

$$\frac{1}{J_{ZZ}} = \frac{1}{J_0} \Big\{ 1 + \alpha_{20} + \alpha_{20}^2 - 5 \sum \alpha_{lm}\alpha_{st} I^{00}_{lmst}$$
$$+ 2 \sum \alpha_{lm}\alpha_{st} I^{20}_{lmst} + \ldots \Big\}, \quad (7)$$

$$J_{ZZ} = \int (X^2 + Y^2) \rho_m dv, \quad J_0 = \frac{2}{5} MR^2, \quad M = \frac{4}{3}\pi R^3 \rho_m;$$

$$\frac{1}{J_{XX}} = \frac{1}{J_0} \Big[1 - \frac{1}{2}\alpha_{20} \mp \frac{3}{4}(\alpha_{22} + \alpha_{2-2}) - 5 \sum \alpha_{lm}\alpha_{st} I^{00}_{lmst}$$
$$- \sum \alpha_{lm}\alpha_{st} I^{20}_{lmst} \mp \frac{3}{2} \sum \alpha_{lm}\alpha_{st} I^{22}_{lmst} \mp \frac{3}{2} \sum \alpha_{lm}\alpha_{st} I^{2-2}_{lmst}$$
$$+ \frac{1}{4}\alpha_{20}^2 \pm \frac{3}{4}(\alpha_{22} + \alpha_{2-2})\alpha_{20} + \frac{9}{16}(\alpha_{22} + \alpha_{2-2})^2 \Big]. \quad (8)$$

Here the upper sign pertains to

$$J_{XX} = \int \rho_m (Y^2 + Z^2) dv,$$

and the lower sign to

$$J_{YY} = \int \rho_m (X^2 + Z^2) dv.$$

Rotational equilibrium occurs when the angular momentum is along a principal axis. If j is the total angular momentum of the nucleus the rotational energy is

$$u_{rot} = j^2/2J_{AA}.$$

The change of total nuclear energy is

$$\Delta u = \Delta u_S + \Delta u_C + \Delta u_{rot}. \quad (9)$$

Before writing out an explicit expression for Δu, we shall discuss the consequences of (5), (6), and the equations (A.3) and (A.4) of the Appendix.

Since we shall determine the positions and properties of the extremal points of the energy surface, i.e., of the points for which

$$\partial u / \partial \alpha_{lm} = 0 \quad (10)$$

and $\alpha_{lm} \ll 1$, many of the parameters α_{lm} can be disregarded. Indeed, Δu contains only even powers of α_{lm} with odd l; therefore (10) yields $\alpha_{lm} = 0$ for odd l. Other roots of (10) for α_{lm} with odd l do not contain a small parameter since Δu includes only one quadratic term having a small parameter—the coefficient of $\alpha^2_{2\mu}$. The same holds true for α_{lm} with even l but odd m, since it follows from (5) that $l \geq 4$ for the quadratic terms in α^2_{lm} with odd m.

Therefore Δu is left with α_{lm} having only even l and m; these are complex quantities in general. Since Δu is real it contains only even powers of γ. Equation (6) is satisfied identically for β_{lm} in virtue of (2). For γ_{lm} in conjunction with (2) we obtain an equation determining γ_{22} in terms of $\gamma_{l\mu}$ with $l \geq 4$:

$$\gamma_{22} = -\frac{4}{3}\sqrt{\frac{10}{7}}\beta_{22}\gamma_{44} - \frac{4}{7}\sqrt{\frac{5}{3}}\beta_{20}\gamma_{42}.$$

Substituting this expression in Δu, we find that (10) leads to $\gamma_{l\mu} = 0$, since the expansion in powers of γ_{lm} begins with quadratic terms having $l \geq 4$ whose coefficients are of the order of unity. Thus only β_{lm} with even l and m play an important part in determining the extremal points of the total energy surface.

The results are, of course, independent of the axial orientation of the angular momentum. However, Eq. (10) is easily solved when the angular momentum j is along the Z axis. This equation is considerably more complicated when the angular momentum is along the X or Y axis. (In first approximation we then obtain two second-order equations in two unknowns or a single fourth-order equation in a single variable.)

It is convenient to introduce the parameters

$$x = \frac{3(Ze)^2/10R}{4\pi R^2 O} = \frac{Z^2/A}{(Z^2/A)_{cr}}, \quad y = \frac{j^2/2J_0}{4\pi R^2 O},$$

where $R = r_0 A^{1/3}$ and A is the nuclear mass number; the energy is determined in units of $4\pi R^2 O$.

If $z = 1 - x$ is used as the small expansion parameter, it will be seen subsequently that y is of the order z^2, with $\beta_{2\mu} \sim z$. Then for j along the Z axis we have

$$\Delta u = \frac{2}{5} z(\beta_{20}^2 + 2\beta_{22}^2) + \frac{4}{105}(-3 + 2z)(\beta_{20}^3 - 6\beta_{20}\beta_{22}^2)$$
$$+ \frac{48}{1225}(\beta_{20}^4 + 4\beta_{20}^2\beta_{22}^2 + 4\beta_{22}^4) - \frac{16}{105}(3\beta_{20}^2\beta_{40} + \beta_{22}^2\beta_{40})$$
$$- \frac{32}{21}\sqrt{\frac{3}{5}}\beta_{20}\beta_{22}\beta_{42} - \frac{16}{3}\sqrt{\frac{2}{35}}\beta_{22}^2\beta_{44}$$
$$+ \frac{17}{27}(\beta_{40}^2 + 2\beta_{42}^2 + 2\beta_{44}^2) + y\beta_{20} + \frac{4}{7}y\beta_{20}^2 - \frac{22}{7}y\beta_{22}^2. \quad (11)$$

We obtain four extremal points in first approximation from (10):

$$\beta_{20} = -\frac{7}{6}z, \quad \beta_{22} = \pm\sqrt{\frac{49}{24}\left(z^2 - \frac{5}{7}y\right)}, \quad (12)$$

$$\beta_{22} = 0, \quad \beta_{20} = \frac{7}{6}\left(z + \sqrt{z^2 + \frac{15}{7}y}\right), \quad (13)$$

$$\beta_{22} = 0, \quad \beta_{20} = \frac{7}{6}\left(z - \sqrt{z^2 + \frac{15}{7}y}\right). \quad (14)$$

The properties of these points are determined, as we know, by the sign of the second derivative of u with respect to β_{20} and by the sign of the determinant Δ composed of the second derivatives of u with respect to β_{20} and β_{22}:

$$\Delta = \frac{32}{25}\left(z^2 - \frac{36}{49}\beta_{20}^2 - \frac{72}{49}\beta_{22}^2\right).$$

The solution (12) has $\Delta < 0$, i.e., these are saddle points. For j close to zero ($y \to 0$) these saddle points are located on the X and Y axes; they thus correspond to fission along the X and Y axes. For nonvanishing angular momentum the nucleus is not axisymmetric at these points.

The solution (13) also has $\Delta < 0$. Since $\beta_{22} = 0$ the nucleus is axisymmetric at this point; it is elongated ($\beta_{20} > 0$), and its axis of symmetry lies in the Z direction. This is a saddle point for fission along the Z axis (the direction of the angular momentum). Since for this direction of fission the fission barrier rises with the angular momentum (centrifugal forces now prevent increasing deformation), this saddle point will hereafter be disregarded.

At the point (14) we have $\beta_{22} = 0$ and $\beta_{20} < 0$. The nucleus is thus an oblate ellipsoid of rotation with its axis of symmetry in the direction of j. At this point we have

$$\Delta = \frac{32}{25}\sqrt{z^2 + \frac{15}{7}y}\left(2z - \sqrt{z^2 + \frac{15}{7}y}\right),$$

$$\partial^2 u/\partial\beta_{20}^2 = \frac{4}{5}\left(z - \frac{6}{7}\beta_{20}\right) > 0.$$

For $y = \frac{7}{5}z^2 = y_{cr}$ the determinant vanishes. For $y < y_{cr}$ we have $\Delta > 0$; therefore a minimum is located at the point (14). In other words, this is a point of stable equilibrium for $y < y_{cr}$. As y increases and approaches y_{cr}, the saddle points (12) approach the minimum (14), and all three points coincide for $y = y_{cr}$. The solution (12) does not exist for $y > y_{cr}$, since β_{22} here becomes imaginary. At the same time (14) does exist, but $\Delta < 0$, i.e., a saddle point appears at (14).

The nucleus therefore has no position of stable equilibrium for $y > y_{cr}$ (the fission barrier vanishes), and fission occurs immediately.

Having considered the principal fission properties of rotating nuclei in the zeroth approximation, we shall now determine corrections proportional to z^4. The extremum of Δu with respect to $\beta_{4\mu}$ leads to the following relations (in first approximation $\beta_{4\mu}$ depends on y only indirectly, through β_{20} and β_{22}):

$$\beta_{40} = \frac{216}{595}\beta_{20}^2 + \frac{72}{595}\beta_{22}^2, \quad \beta_{42} = \frac{72}{119}\sqrt{\frac{3}{5}}\beta_{20}\beta_{22},$$
$$\beta_{44} = \frac{36}{17}\sqrt{\frac{2}{35}}\beta_{22}^2. \quad (15)$$

Substituting these values in (11), we obtain Δu as a function of only β_{20} and β_{22}. In the next approximation with respect to z the extremal points will remain the same as previously; only the magnitudes of the deformation and energy will change.

For the position of the minimum we have

$$\beta_{22} = 0, \quad \beta_{20} = \frac{7}{6}z - \frac{7}{6}\sqrt{z^2 + 15y/7} + \frac{469}{765}\frac{z^3}{\sqrt{z^2 + 15y/7}}$$
$$- \frac{469}{765}z^2 - \frac{9}{34}\frac{zy}{\sqrt{z^2 + 15y/7}} + \frac{47}{51}y,$$

and for the energy at this point, considering only terms $\sim z^3$, we have

$$\Delta u_{min} = \frac{49}{135}z^3 - \frac{49}{135}z^2\sqrt{z^2 + 15y/7}$$
$$+ \frac{7}{6}zy - \frac{7}{9}y\sqrt{z^2 + 15y/7}.$$

The nuclear energy at equilibrium must be known in order to determine the excitation energy since the total excitation energy is the sum of the excitation energy of a spherical nucleus and Δu_{min}.

The positions of the saddle points are determined from the following magnitude of the deformation:

$$\beta_{20} = -\frac{7}{6} z + \frac{469}{765} z^2 + \frac{261}{68} y,$$

$$\beta_{22}^2 = \frac{49}{24}\left(z^2 - \frac{5}{7} y - \frac{268}{255} z^3 - \frac{1244}{357} yz\right). \qquad (16)$$

The fission barrier E_f is defined as the difference between the nuclear energy at the saddle point (12) and the equilibrium energy at (14), which in the case $y \sim y_{cr}$ contributes considerably to the fission barrier. Including terms in z^4, we have

$$E_f = \Delta u_{s.p.} - \Delta u_{min} = \frac{49}{135} z^3 - \frac{7}{3} zy - \frac{5684}{34425} z^4 + \left(\frac{49}{135} z^2 + \frac{7}{9} y\right)\sqrt{z^2 + \frac{15}{7} y} - \frac{18263}{3060} z^2 y + \frac{35}{12} y^2 - \frac{5684}{34425} z^3 \sqrt{z^2 + \frac{15}{7} y} + \frac{2569}{2295} zy \sqrt{z^2 + \frac{15}{7} y}. \qquad (17)$$

Equation (16) is used to determine the correction for y_{cr}, because the properties of the extremal points remain the same as in the zeroth approximation. Setting (16) equal to zero, we therefore obtain

$$y_{cr} = \frac{7}{5} z^2 \left(1 - \frac{504}{85} z\right). \qquad (18)$$

For this value of y_{cr} the coefficient of β_{22}^2 vanishes in the expression for the total energy, i.e., there is no stable nuclear state for this rotational energy (for a small deformation, in any event). From this expression we determine the range in which the total energy of a rotating nucleus can be expressed by a series expansion in the small deformation of a sphere

$$z \ll 85/504. \qquad (19)$$

Thus in the case of rotating nuclei the range of applicability of this expansion is narrower than for a nonrotating nucleus, where

$$z \ll 0.45. \qquad (20)$$

At first glance the foregoing result seems unexpected, since at the top of the fission barrier the deformation is smaller for a rotating nucleus than for a nucleus at rest. However, when we consider the origin of the large coefficient of z^3 in (18), which determines the applicability range of our equations, we find that the value comes mainly from the large numerical coefficients in the expansion of the rotational energy with respect to the deformation. This expansion becomes unsuitable for relatively small $\alpha_{2\mu}$, since for $1 - x \sim 0.1$ and $y \sim y_{cr}$ we obtain $\alpha_{20} \sim 0.1$. We also note that $\beta_{4\mu}$ does not depend explicitly on the rotational energy (15).

We can expect on the basis of the foregoing that a different zeroth approximation, i.e., a nuclear shape with respect to which the computed deformation can be regarded as small, will give better results. A more suitable zeroth approximation would, of course, be a nuclear shape taking rotation into account. This requirement is entirely reasonable and evidently necessary, since in studying the behavior of the extremal points of the total energy surface one must know the expansion of the energy with respect to the deformation near these points. It is therefore obvious that with a sphere as the zeroth approximation correct results cannot be obtained for sufficiently large values of $1 - x$ and of the rotational energy. It is possible that with an ellipsoid as the zeroth approximation the properties of the extremal points can be studied over a broader range of x than for a sphere. However, in the case of an ellipsoidal nucleus results can be obtained only by numerical calculation.

It follows from (19) that (17) has such a small range of applicability that it is invalid for most nuclei resulting from reactions with heavy ions. An expression suitable over a broader range of z is therefore required. For small rotational energy ($y < y_{cr}$) the expansion of the fission barrier in a small parameter must be valid for a broader range of z than in the case $y \sim y_{cr}$, and in the limit $y \to 0$ this region must coincide with (20).

The series expansion of (17) in terms of y gives

$$E_f = \frac{98}{135} z^3 - \frac{11368}{34425} z^4 + \left(-\frac{7}{6} z - \frac{15379}{3060} z^2\right) y + \left(-\frac{5}{8}\frac{1}{z} + \frac{859}{204}\right) y^2. \qquad (21)$$

This expression differs from the analogous formula in [1], where an axisymmetric deformation was considered only in first approximation. Moreover, Δu_{min} was neglected in [1].

The influence of rotation on the equilibrium configuration appears only in the $\sim y^2$ term, since the nuclear deformation itself at equilbrium is $\sim y$. The coefficient of y contains the first two terms of the expansion in z. The next two terms of the expansion are easily obtained. For this purpose it is more convenient to consider rotation of the nucleus around the X axis (or, equivalently, around the Y axis) instead of around the Z axis. Then, if the nucleus fissions along the Z axis, it appears from (8) and (9) that the deformation α_{22}

for a non-axisymmetric nucleus is proportional to the rotational energy y. Terms proportional to y^2 are therefore obtained when α_{22} is taken into account in (9). In order to obtain the succeeding terms in the expansion of the coefficient of y^2 in (21), in addition to those already given, terms of the order $\alpha_{20}^3\alpha_{22}^2$ must be included in the total energy (9). We can therefore assume $\alpha_{22} = 0$, and also that α_{20} is independent of y, since such dependence determines the next terms of the expansion in y.

The coefficient of y in this case therefore equals the rotational energy of the nucleus, which has the same shape as a nonrotating nucleus at the top of the fission barrier. The rotational energy when the angular momentum is along the X axis is proportional to (8):

$$\frac{1}{J_{XX}} = \frac{1}{J_0}\Big(1 - \frac{1}{2}\beta_{20} - \frac{29}{28}\beta_{20}^2 + \frac{551}{840}\beta_{20}^3 - \frac{4}{7}\beta_{20}\beta_{40} - \frac{5}{9}\beta_{40}^2 - \frac{102}{77}\beta_{20}^2\beta_{40} + \frac{73611}{43120}\beta_{20}^4\Big). \qquad (22)$$

The nuclear deformation at the top of the barrier has been given with sufficient accuracy for the present case in [6]:

$$\beta_{20} = 2.33z - 1.23z^2 + 9.5z^3 - 8.05z^4 + \ldots,$$
$$\beta_{40} = 1.94z^2 - 1.69z^3 + \ldots.$$

With these values substituted in (22), we obtain from (21):

$$E_f = 0.726z^3 - 0.330z^4 + 1.92z^5 + \ldots + y(-1.17z - 5.03z^2 + 6.87z^3 - 18.8z^4 + \ldots) + y^2(-0.625/z + 4.21).$$

In order to determine the angular distribution of fission fragments and to calculate the dependence of the fission width Γ_f on the angular momentum [7,8] we must know the moment of inertia with respect to the axis of symmetry:

$$\frac{1}{J_{\parallel}} = \frac{1}{J_{ZZ}} = \frac{1}{J_0}(1 + 2.33z + 5.44z^2 + 0.83z^3).$$

The moment of inertia with respect to an axis perpendicular to the axis of symmetry is

$$\frac{1}{J_{\perp}} = \frac{1}{J_{XX}} = \frac{1}{J_0}(1 - 1.17z - 5.03z^2 + 6.87z^3).$$

APPENDIX

First-approximation terms in the Coulomb and surface energies (i.e., terms of the order $\alpha_{2\mu}^2$ and $\alpha_{2\mu}^3$, which were considered in [1,2]) can be obtained from the dependence of the respective energies on the deformation of an axisymmetric nucleus. In this approximation the energy has the form $a\alpha_{20}^3 + b\alpha_{22}^2 + c\alpha_{20}^3 + d\alpha_{20}\alpha_{22}^2$. Since the energy is, of course, independent of the coordinate system, the coefficients are determined uniquely by the condition that for the deformation represented by

$$\alpha_{20} = -\tfrac{1}{2}\alpha_2, \qquad \alpha_{22} = \pm\tfrac{1}{2}\sqrt{\tfrac{3}{2}}\,\alpha_2$$

(in which case the nucleus is axisymmetric with its axis of symmetry along either the X or Y axis) the energy must be that of an axisymmetric nucleus with its axis of symmetry along the Z axis ($\alpha_{20} = \alpha_2$, $\alpha_{22} = 0$).

For terms of the order $\alpha_{2\mu}^n$ ($n \geq 4$) or $\alpha_{s\mu}^2$ ($s \geq 4$) the foregoing procedure for determining coefficients of the energy expansion in terms of $\alpha_{l\mu}$ is unsuitable. The symmetry condition enables us to obtain only two relations between the coefficients, whereas in this case more than two coefficients are required.

The Coulomb energy of the nucleus is

$$u_C = \frac{1}{2}\int \varphi_i \rho\, dv, \qquad (A.1)$$

where φ_i, the internal electric potential of the nucleus, satisfies the equation $\Delta\varphi_i = -4\pi\rho$ (where $\rho = Ze/(4\pi/3)R_0^2$ is the charge density). The potential φ_e external to the nucleus satisfies the equation $\Delta\varphi_e = 0$. We shall derive the potential in the form

$$\varphi_i(r,\theta,\varphi) = \frac{2}{3}\pi\rho R^2\Big[-\frac{r^2}{R^2} + \sum_{l,m} A_{l,m}\Big(\frac{r}{R}\Big)^l D_{lm}(\theta,\varphi)\Big],$$
$$\varphi_e(r,\theta,\varphi) = \frac{2}{3}\pi\rho R^2 \sum_{l,m} C_{l,m}\Big(\frac{R}{r}\Big)^{l+1} D_{lm}(\theta,\varphi), \qquad (A.2)$$

where the coefficients A_{lm} and C_{lm} are determined from the boundary conditions for the potentials on the nuclear surface S. Assuming, furthermore, that all $\alpha_{lm} \ll 1$, we can obtain A_{lm} and C_{lm} as expansions in $\alpha_{l\mu}$ ($A_{lm} = A_{lm}^{(0)} + A_{lm}^{(1)} + \ldots$, $A_{lm}^{(i)}$ is the i-th approximation).

We present here all coefficients C_{lm} determining the extranuclear potential required to derive the probability of charged particle emission:

$$C_{00}^{(0)} = 2, \qquad C_{lm}^{(0)} = 0 \quad \text{for } l \neq 0,$$
$$C_{lm}^{(1)} = \frac{6}{2l+1}\alpha_{lm}, \qquad C_{lm}^{(2)} = 3\frac{l+2}{2l+1}\sum \alpha_{st}\alpha_{r\nu} I_{str\nu}^{lm}.$$

Substituting (A.2) in (A.1) and using the expression for the product of D functions,

$$D_{r\nu}(\theta,\varphi) D_{st}(\theta,\varphi) = \sum_{\omega=|r-s|}^{s+r} I_{str\nu}^{\omega\mu=s+\nu} D_{\omega\mu}(\theta,\varphi),$$

integration over the volume of the nucleus now gives the Coulomb energy as a function of the deformations α_{lm} with $l \geq 0$. Volume conservation enables us to eliminate α_{00}:

$$\alpha_{00} = -\sum_{l,\, s\neq 0} \alpha_{lm}\alpha_{st} I_{lmst}^{00} - \frac{1}{3}\sum_{l,\, sr\neq 0} \alpha_{lm}\alpha_{st}\alpha_{rv} I_{lmst}^{\omega\gamma} I_{\omega\gamma rv}^{00}. \quad \text{(A.3)}$$

Here the term proportional to α^4 vanishes. The conservation of the position of the center of gravity enables us to determine $\alpha_{1\mu}$ in terms of the other α_{lm}:

$$\alpha_{1\mu} = -\frac{3}{2}\sum \alpha_{st}\alpha_{rv} I_{strv}^{1-\mu} + \ldots \quad \text{(A.4)}$$

Equation (3) is obtained with the aid of (A.3) and (A.4).

The surface energy is proportional to the nuclear surface area S:

$$u_S = OS,$$

where O is the coefficient of surface tension. The surface area in spherical coordinates is

$$S = \iint r\, d\theta\, d\varphi \{[r^2 + (\partial r/\partial\theta)^2] \sin^2\theta + (\partial r/\partial\varphi)^2\}^{1/2}. \quad \text{(A.5)}$$

We use the expression for the angular derivatives of the D function:[9]*

$$\left(i\frac{\partial}{\partial\theta} + \operatorname{ctg}\theta\frac{\partial}{\partial\varphi}\right) D_{lm}(\theta, \varphi) = ie^{i\varphi}\sqrt{(l(l+1) - m(m-1)} D_{l,\, m-1}(\theta, \varphi),$$

$$\left(-i\frac{\partial}{\partial\theta} + \operatorname{ctg}\theta\frac{\partial}{\partial\varphi}\right) D_{lm}(\theta, \varphi) = ie^{-i\varphi}\sqrt{l(l+1) - m(m+1)} D_{l,\, m+1}(\theta, \varphi)$$

and integrate the radical in (A.5) after expanding it in a series in α_{lm}; this gives the change of surface energy. The final expression for the change of surface energy (4) is obtained with the aid of (A.3) and (A.4).

*ctg = cot.

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