ANNIHILATION OF AN ELECTRON-POSITRON PAIR INTO TWO GRAVITONS

Yu. S. VLADIMIROV

Moscow State University

Submitted to JETP editor November 22, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **43**, 89-91 (July, 1962)

The effective cross section for the annihilation of an electron-positron pair into two transverse quanta of a weak gravitational field, i.e., into two gravitons, is calculated.

THE application of quantum field theory to gravitation leads to the existence of transverse quanta of the gravitational field ("gravitons"), which in analogy to the quanta of other fields can be transformed into particle-antiparticle pairs and vice versa. The latter circumstance has been first pointed out by D. Ivanenko and the relevant ideas have been afterwards developed in a number of papers.[1-4]

Thus, for the annihilation of two scalar particles into two gravitons one obtains in the first approximation the expression[5]

$$\sigma_1 \sim r_g^2 \frac{c}{v}\left(\frac{E}{mc^2}\right)^2, \qquad r_g = \frac{\varkappa m}{c^2}. \tag{1}$$

In the second approximation one obtains an additional contribution of the form $\sigma_2 \sim (cp/E)^4\sigma_1$.

The probability of the occurrence of such processes is insignificant because of the smallness of κ and because of the quadrupole nature of the gravitational interaction, nevertheless the possibility of these transmutations is of fundamental significance both for the understanding of the nature of the gravitational field and for cosmological considerations.[6]

In order to derive an exact expression for the differential effective cross section for the annihilation of a pair of spinor particles (electron-positron) into two gravitons we shall make use of the linearized gravitational potential[7]

$$g_{\mu\nu} = \varepsilon_{\mu\nu} + \sqrt{\varkappa}\left(h_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}h_{\lambda\lambda}\right)$$

and of the coordinate condition $\partial h_{\mu\nu}/\partial x_\nu = 0$. After second quantization these conditions are taken in the weaker form:

$$\frac{\partial h_{\mu\nu}^{(+)}}{\partial x_\mu}\Psi = 0, \qquad (h_{\mu\mu}^{(+)} - h^{(+)})\Psi = 0, \qquad \Psi^+\frac{\partial h_{\mu\nu}^{(-)}}{\partial x_\mu} = 0,$$

$$\Psi^+(h_{\mu\mu}^{(-)} - h^{(-)}) = 0,$$

where $h_{\mu\nu}^{(+)}$, $h^{(+)}$, $h_{\mu\nu}^{(-)}$, $h^{(-)}$ are respectively the positive and negative frequency parts of $h_{\mu\nu}$ and h. It turns out that for real gravitons only two states are possible, corresponding to the operators $a_{12}(\mathbf{k})$ and $[a_{11}(\mathbf{k}) - a_{22}(\mathbf{k})]/2$ ($\mathbf{k}$ is the graviton momentum; it is taken to lie along x_3).

The Lagrangian of the linearized gravitational field in interaction with usual matter has the form

$$L = -\frac{1}{4}\left(\frac{\partial h_{\mu\nu}}{\partial x_\lambda}\frac{\partial h_{\mu\nu}}{\partial x_\lambda} - \frac{1}{2}\frac{\partial h}{\partial x_\lambda}\frac{\partial h}{\partial x_\lambda}\right) - \frac{\sqrt{\varkappa}}{2}\left(h_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}h\right)T_{\mu\nu}.$$

After substitution of the energy tensor of the spinor field and making use of the symmetry of $h_{\mu\nu}$ as well as of the fact that we are interested in the interaction with transverse gravitons we obtain for the interaction Hamiltonian the expression

$$H = \frac{i}{4}\sqrt{\varkappa}h_{\mu\nu}\left(\overline{\Psi}\gamma_\mu\frac{\partial\Psi}{\partial x_\nu} - \frac{\partial\overline{\Psi}}{\partial x_\nu}\gamma_\mu\Psi\right).$$

The calculation will be carried out in the barycentric frame using the perturbation theory methods of relativistic quantum field theory. The differential effective cross section for the annihilation process is of the form[8]

$$d\sigma = (2\pi)^2\frac{k_0^3}{4p}\sum_\nu |F|^2 d\Omega \qquad (p = |\mathbf{p}|), \tag{2}$$

where p is the momentum of the electron.

The rules for the construction of diagrams corresponding to the matrix element for the process in question are analogous to the Feynman rules. In second order we obtain for F the expression

$$F = \frac{i\varkappa h_{\mu\nu}^{(2)}h_{\lambda\rho}^{(1)}}{64(2\pi)^2 k_0}\Big\{\bar{v}^-(\mathbf{p})\gamma_\mu\frac{4p_\nu p_\rho(m - \hat{p} + \hat{k}_2)}{2pk_2}\gamma_\lambda v^-(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{p})$$
$$+ \bar{v}^-(\mathbf{p})\gamma_\lambda\frac{4p_\nu p_\rho(m - \hat{p} + \hat{k}_1)}{2pk_1}\gamma_\mu v^-(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{p})\Big\}.$$

Squaring this expression and averaging over the electron and positron spins we find

$$\sum_\nu |F|^2 = \frac{\varkappa^2}{2^{12}(2\pi)^4 k_0^4 (pk_1)^2(pk_2)^2}\,\mathrm{Sp}\,\{(\hat{p} - m)\,[(pk_1)(-2h^{(2)}\hat{h}^{(1)}$$
$$+ \hat{h}^{(2)}\hat{k}_2\hat{h}^{(1)}) + (pk_2)(-2h^{(1)}\hat{h}^{(2)} + \hat{h}^{(1)}\hat{k}_1\hat{h}^{(2)})]$$
$$\times(\hat{k}_1 + \hat{k}_2 - \hat{p} + m)\,[(pk_1)(-2h^{(2)}\hat{h}^{(1)} + \hat{h}^{(1)}\hat{k}_2\hat{h}^{(2)})$$
$$+ (pk_2)(-2h^{(1)}\hat{h}^{(2)} + \hat{h}^{(2)}\hat{k}_1\hat{h}^{(1)})]\}. \tag{3}$$

Here we have introduced the abbreviations

$$h^{(1,2)} = h^{(1,2)}_{\mu\nu} p_\mu p_\nu, \qquad \hat{h}^{(1,2)} = h^{(1,2)}_{\mu\nu} p_\mu \gamma_\nu,$$

$$\hat{k} = k_\alpha \gamma_\alpha, \qquad \hat{p} = p_\alpha \gamma_\alpha.$$

Making use of the relations

$$h^{(i)}_{\mu\nu} k^{\mu}_{(m)} = 0 \qquad (i, m = 1, 2),$$

$$k_1^\nu k_2^\nu = 2k_0^2, \qquad k_1^2 = k_2^2 = 0$$

and denoting by A the quantity whose trace is to be taken in Eq. (3) we find after rather tedious calculations

$$\mathrm{Sp}\, A = 32\,(pk_1)^2 (pk_2)^2 \times \left\{ \frac{k_0^4 (h_2^\mu h_2^\mu)(h_1^\rho h_1^\rho)}{(pk_1)(pk_2)} - \frac{4k_0^4 h_2^2 h_1^2}{(pk_1)^2 (pk_2)^2} - \frac{4k_0^2 h_1 h_2 (h_1^\mu h_2^\mu)}{(pk_1)(pk_2)} - (h_2^\mu h_1^\mu)^2 \right\}. \tag{4}$$

Summation over the polarization states of the gravitons yields

$$\sum (h_2^\mu h_1^\mu)^2 = 2p^4 \sin^4\theta, \qquad \sum h_1^2 h_2^2 = p^8 \sin^8\theta,$$

$$\sum (h_1^\nu h_1^\nu)(h_2^\mu h_2^\mu) = 4p^4 \sin^4\theta, \qquad \sum h_1 h_2 (h_1^\mu h_2^\mu) = -p^6 \sin^6\theta. \tag{5}$$

After substituting Eq. (5) into Eq. (4) and then Eq. (3) into Eq. (2) we obtain the final expression for the differential effective cross section for the annihilation of an electron-positron pair into two gravitons:

$$d\sigma = \frac{\varkappa^2}{4(4\pi)^2} \frac{p^3}{16k_0} \sin^4\theta \left\{ 1 + \frac{2p^2}{k_0^2 - p^2\cos^2\theta} - \frac{2p^4 \sin^4\theta}{(k_0^2 - p^2\cos^2\theta)^2} \right\} d\Omega. \tag{6}$$

The total cross section is of the form

$$\sigma = \frac{\varkappa^2}{64\pi} \frac{p^2}{8a} \left[\frac{(a^2-1)^2 (7a^4 - 4a^2 - 1)}{a^3} \ln\left|\frac{1+a}{1-a}\right| + \frac{1}{15a^2}(210a^6 - 470a^4 + 266a^2 - 30) \right], \tag{7}$$

where $a = k_0/p$. For the annihilation of a positron and an electron at rest we have

$$\sigma' = \frac{\varkappa^2}{64\pi} \frac{m^2 (\gamma-1)^{3/2}}{16(\gamma+1)^{1/2}} \left[\frac{8(\gamma^2 + 8\gamma + 5)}{(\gamma-1)^{5/2}(\gamma+1)^{3/2}} \ln(\gamma + \sqrt{\gamma^2 - 1}) + \frac{6\gamma^2 - 112\gamma + 946}{15(\gamma-1)^2} - \frac{2(\gamma-1)}{\gamma+1} \right] \tag{8}$$

($\gamma = k_0'/mc^2$, k_0' is the energy of the positron in the electron rest frame).

In the ultrarelativistic case ($a \to 1$) we obtain from Eq. (7) $\sigma \sim (E/mc^2)^2$, i.e. the cross section increases with increasing energy in contrast to the cross section for two-photon annihilation. The ratio of the gravitational cross section to the photon cross section

$$\sigma_g/\sigma_{ph} \sim (r_g/r_e)^2 (k_0/mc^2)^4 / \ln(k_0/mc^2)$$

becomes larger than unity at energies $k_0 \sim 10^{12} mc^2$. It should be noted, however, that at such energies the method of linearizing the gravitational field becomes inapplicable, and the result obtained here is only of qualitative significance.

Another peculiarity of the gravitational annihilation consists of the fact that as a consequence of the factor $\sin^4\theta$ that appears in Eq. (6) the maximum of the radiation of gravitons should occur at $\theta = \pi/2$.

In the nonrelativistic approximation Eq. (6) agrees accurate to within a coefficient with the result of the scalar case, Eq. (1), and the calculation of Wheeler-Brill.[3]

[1] D. D. Ivanenko and A. A. Sokolov, Vestnik, Moscow State University 8, 103 (1947).

[2] J. A. Wheeler, Rendiconti Scuola, Varenna, Corso 12 (1960).

[3] J. A. Wheeler and D. Brill, Revs. Modern Phys. **29**, 465 (1957).

[4] I. Piir, Trudy, Physics and Astronomy Institute, Academy of Sciences, Estonian SSR, 1957.

[5] A. Sokolov and D. Ivanenko, Kvantovaya teoriya polya (Quantum Field Theory), AN SSSR, 1952, p. 678.

[6] D. Ivanenko, Tezisy Pervoĭ sovet-skoĭ gravitatsionnoĭ konferentsii (Proceedings of the Fist Soviet Conference on Gravitation), MGU, 1961, p. 98.

[7] S. N. Gupta, Proc. Phys. Soc. **A65**, 161 (1952) (Russ. Transl., IIL, 1961).

[8] N. N. Bogolyubov and D. V. Shirkov, Vvedenie v teoriyu kvantovannykh poleĭ (Introduction to Quantum Field Theory), Gostekhizdat, 1957, p. 194.

Translated by A. M. Bincer
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