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CERENKOV RADIATION AND INELASTIC SCATTERING OF PARTICLES

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The effective cross sections for the scattering of neutrons and electrons in condensed media due to radiation (absorption) of bosons (phonons, photons and excitons) are calculated by classical methods.

In calculating the effective cross sections for inelastic scattering of fast particles in condensed media one usually makes use of quantum-mechanical methods, computing the appropriate transition probabilities (matrix elements). However, if the scattering is due to the production (absorption) of "quasi-particles" that obey Bose statistics (photons, phonons, rotons etc.), a classical method can be used to analyze scattering at small angles; this method is used to compute the appropriate energy loss.

This approach to the problem is the subject of the present analysis. In the present work we compute the following differential scattering cross sections at small angles: neutron scattering with the production of phonons and electron scattering with the production of Cerenkov photons and excitons. The differential neutron scattering cross section computed in the present work coincides with expressions obtained earlier in a number of particular cases.[1-3] However, the method used here exhibits the rather general nature of the results obtained at small scattering angles.

In the electron case the small-angle differential scattering cross sections are apparently computed for the first time and particular emphasis is given to the relation between the angular dependence of the cross section and the nature of the elementary excitation. The differential cross sections obtained in the work allow us to use electron scattering to investigate spectra in solids.

Assume that the spectral density of the energy loss $(dE/dt)_\omega$ is known, i.e.,

$$\frac{dE}{dt} = \int \left(\frac{dE}{dt}\right)_\omega d\omega. \tag{1}$$

It must first be noted that the particle energy loss is a difference effect in the cases being considered here. The particle loses an energy

$$\left(\frac{dE}{dt}\right)_- = \int \left(\frac{dE}{dt}\right)_\omega (N_\omega + 1)\, d\omega \tag{2}$$

and gains an energy

$$\left(\frac{dE}{dt}\right)_+ = \int \left(\frac{dE}{dt}\right)_\omega N_\omega\, d\omega, \tag{3}$$

where N_ω is the distribution function for the quasi-particles with energy $\hbar\omega$.

The probability that the particle in flight radiates a quasi-particle per unit time in the frequency range $(\omega, \omega + d\omega)$ is given by

$$dW_\omega^{(-)} = (\hbar\omega)^{-1} (dE/d\omega)_\omega (N_\omega + 1). \tag{4}$$

Correspondingly, the absorption probability in the frequency range $(\omega, \omega + d\omega)$ is

$$dW_\omega^{(+)} = \hbar\omega^{-1} (dE/dt)_\omega N_\omega\, d\omega. \tag{5}$$

Equations (2) and (4) correspond to Stokes lines while (3) and (5) correspond to anti-Stokes lines.

Once the probability of a given process is known the cross section can be obtained easily:

$$d\sigma^{(\pm)} = \left(\frac{1}{\hbar\omega v \mathfrak{N}}\right) \begin{Bmatrix} N_\omega \\ N_\omega + 1 \end{Bmatrix} d\omega \left(\frac{dE}{dt}\right)_\omega. \tag{6}$$

Here v is the particle velocity while $\mathfrak{N}$ is the number of atoms per unit volume of the scattering material.

Inasmuch as a quasi-particle is either produced or absorbed in the scattering process its energy must be related uniquely to the scattering angle; hence the cross sections can be written in the following form:[1)]

$$d\sigma^{(\pm)} = \left(\frac{1}{\hbar\omega v \mathfrak{N}}\right) \left(\frac{dE}{dt}\right)_\omega \begin{Bmatrix} N_\omega \\ N_\omega + 1 \end{Bmatrix} \frac{d\omega}{d\chi} \delta(E' - E \mp \hbar\omega)\, dE' d\chi. \tag{7}$$

Here E′ is the energy of the scattered particle and χ is the scattering angle (the angle between the particle momentum before and after scattering). The derivative $d\omega/d\chi$ must be determined taking account of both energy and momentum conservation:

[1)]Equation (6) represents the integral over E′ of the expression in (7).

$$E' = E \pm \hbar\omega, \qquad \mathbf{p}' = \mathbf{p} \pm \hbar \mathbf{k} \tag{8}$$

($\mathbf{k}$ is the wave vector associated with the quasi-particle).

The expression in (7) shows that in scattering of a monoenergetic particle beam particles with a given energy are to be associated with a definite direction. The dissipative processes in the scattering medium may reasonably be assumed to be linear. The broadening may be described qualitatively by a resonance formula with a width of order $\hbar/\tau$, where τ is the lifetime of the quasi-particle. Exact calculations carried out earlier for inelastic scattering of neutrons in crystals [1] and scattering of neutrons in He^3 [2] lead to precisely the same result.

The spectral density of the energy loss $(dE/dt)_\omega$ computed by the classical method is valid only for fast particles; in the fast-particle case radiation (absorption) of the quasi-particle has only a small effect on the motion of the radiating (absorbing) particle. Hence, in calculating $d\omega/d\chi$ we can use an approximation based on the fact that the scattering angle is small. In this approximation[2)]

$$\chi = (\hbar\omega/2E)\{v^2/v_{\mathrm{ph}}^2 - 1\}^{1/2}, \tag{9}$$

where

$$v_{\mathrm{ph}} = \omega/k \tag{10}$$

is a function of frequency: the wave vector $\mathbf{k}$ is expressed in terms of the frequency ω through the dispersion relation[3)]

$$\omega = \omega(\mathbf{k}). \tag{11}$$

The relation in (9) applies in those frequency regions for which $\chi \ll 1$. If $v > v_{\mathrm{ph}}$ for all frequencies then (9) can be applied only at low frequencies. For example, in scattering due to absorption (radiation) of a phonon, when $v > s$ (s is the velocity of sound) it is possible that

$$\chi = \frac{\hbar\omega}{2E}\left\{\frac{v^2}{s^2} - 1\right\}^{1/2} \tag{12}$$

and

$$\frac{d\omega}{d\chi} \approx \frac{2E}{\hbar}\left\{\frac{v^2}{s^2} - 1\right\}^{-1/2}.$$

We use (7) and (12) to compute the effective cross section for neutron absorption (radiation) of a phonon. It has been shown earlier [14] that the energy lost per unit time by the neutron is[4)]

$$\frac{dE}{dt} = -\frac{iU^2}{8\pi^3\rho}\int \frac{k^2\omega\, d\tau_{\mathbf{k}}}{\omega^2 - s^2k^2}, \qquad d\tau_{\mathbf{k}} = dk_x\, dk_y\, dk_z. \tag{13}$$

Here, $\omega = \mathbf{k}\cdot\mathbf{v}$ and U is a constant, with the dimensions of energy, which appears in the expression for the force $\mathbf{F}$ exerted by the neutron on a unit mass:

$$\mathbf{F} = -(U/\rho)\nabla\delta(\mathbf{r} - \mathbf{v}t),$$

ρ is the density of the material.

Using (13) and noting that the contribution to the integral comes only from the residue at the point $(\mathbf{k}\cdot\mathbf{v})^2 = s^2k^2$ (whence it follows that v must be greater than s), we have

$$\frac{dE}{dt} = \frac{U^2}{4\pi s^4 v\rho}\int \omega^3 d\omega \equiv \int \frac{U^2}{4\pi s^4 v\rho}\{N_\omega + 1 - N_\omega\}\,\omega^3 d\omega, \tag{14}$$

where the integration is taken only over small values of ω. Thus,

$$(dE/dt)_\omega = U^2\omega^3/4\pi s^4 v\rho,$$

$$d\sigma_{\mathrm{ac}}^{\pm} = \frac{a^2}{\pi}\left(\frac{UE}{\Theta^2}\right)^2 \frac{m}{M}\left(\frac{v^2}{s^2} - 1\right)^{-3/2} \times \left\{\begin{matrix} N_\omega \\ N_\omega + 1\end{matrix}\right\}\delta(E' - E \mp \hbar\omega)\, dE'\chi^2 d\chi. \tag{15}$$

For convenience we have introduced the notation

$$\Theta = \hbar s/a, \qquad a = (M/\rho)^{1/3}, \qquad M = \rho/\mathfrak{N}.$$

The expression given here naturally coincides with the effective scattering cross section[5)] for neutrons in He^{II} [3] and in He^3 [2] computed by quantum-mechanical methods. It has been shown by A. Akhiezer and Pomeranchuk[5] that in scattering in helium the constant U must be related to the usual constant A that appears in the expression for the cross section for neutron scattering on the free He nucleus σ_0 (the quantity σ_0 does not take account of the spin interaction):

$$U = \frac{\rho}{M}A, \qquad \sigma_0 = \frac{A^2}{\pi\hbar^4}\left(\frac{Mm}{M+m}\right)^2.$$

It is evident from the derivation that (15) applies in all cases in which scattering is due to radiation (absorption) of acoustic quanta-phonons.

We now consider the scattering of electrons by

[2)] In obtaining (9) we have neglected the quantity $\hbar^2k^2/2m$ compared with $\hbar\omega$ (m is the particle mass). For a phonon ($\omega = sk$) this procedure sets a limit on the radiation (absorption) frequency: $\hbar\omega \ll 2ms^2$.

[3)] The entire analysis given here applies to the isotropic case so that the frequency ω depends only on the modulus of the wave vector. The generalization to the anisotropic case is not difficult.

[4)] In [4] the electron case has been treated but no use has been made of the fact that this particle possesses a charge. Furthermore, it is easy to show that the aggregate state of the scattering material is not important (the formulas are the same for an isotropic solid or a liquid).

[5)] The formulas obtained in [2] and [3] obviously give a much more complete description of the scattering process; scattering at arbitrary angles is considered in [3] while the additional effects due to dissipative processes are considered in [2].

virtue of electrical interaction with the medium. The energy lost per unit time by an electron is given by the familiar expression

$$\frac{dE}{dt} = \frac{ie^2}{2\pi^2}\int \frac{\omega}{\varepsilon} \frac{1 - v^2\varepsilon/c^2}{\omega^2\varepsilon/c^2 - k^2} d\tau_{\mathbf{k}}, \tag{16}$$

where ε is the dielectric constant of the medium and is, in general, a function of frequency and wave vector. We shall be interested in scattering due to the production (absorption) of photons and exitons. Hence we must consider the region in which the dielectric is transparent. At frequencies outside this region the only contributions come from the residues at the zeroes of the denominator.

The specific loss can be written as a sum of two terms:

$$dE/dt = (dE/dt)_{\text{Cer}} + (dE/dt)_{\text{pol}}. \tag{17}$$

The first term is the loss due to Cerenkov radiation

$$\left(\frac{dE}{dt}\right)_{\text{Cer}} = \frac{e^2 v}{c^2}\int\left(1 - \frac{c^2}{\varepsilon v^2}\right)\omega d\omega. \tag{18}$$

The integration is taken only over those frequencies for which $c^2/\varepsilon v^2 < 1$. The second term represents the polarization loss and will be considered below.

To compute the electron scattering cross section we use (9), (18) and the relation $v_{ph} = c/\sqrt{\varepsilon(\omega)}$. Two cases are possible (we neglect spatial dispersion):

a) $v > c/\sqrt{\varepsilon(\omega)}$ for arbitrarily low frequencies. In this case it follows from (7), (9) and (18) that

$$d\sigma^{\pm}_{\text{opt}} = \left(\frac{e^2}{\hbar c}\right)^2 \frac{a^3}{\varepsilon r_0}\left(1 - \frac{c^2}{\varepsilon v^2}\right)^{1/2} \delta(E' - E \mp \hbar\omega) \begin{Bmatrix} N_\omega \\ N_\omega + 1 \end{Bmatrix} dE' d\chi, \tag{19}$$

where $\varepsilon = \varepsilon(0)$ and $r_0 = e^2/mc^2$ is the classical radius of the electron.

b) The inequality $v > c/\sqrt{\varepsilon(\omega)}$ is satisfied starting at some frequency ω_1. In this case, using (9) we have

$$\frac{d\omega}{d\chi} \cong 8\left(\frac{E}{\hbar\omega_1}\right)^2 \frac{c^2}{v^2} \frac{\chi}{(d\varepsilon/d\omega)_1}, \qquad \left(\frac{d\varepsilon}{d\omega}\right)_1 \equiv \left(\frac{d\varepsilon(\omega)}{d\omega}\right)_{\omega=\omega_1},$$

while

$$d\sigma^{(\pm)}_{\text{opt}} = 8\left(\frac{e^2}{\hbar c}\right)^2 \frac{a^3}{r_0} \frac{1}{\varepsilon_1 (d\varepsilon/d\omega)_1 \omega_1} \frac{mc^2E^2}{(\hbar\omega_1)^3} \times \chi^3 \delta(E' - E \mp \hbar\omega) \begin{Bmatrix} N_\omega \\ N_\omega + 1 \end{Bmatrix} dE' d\chi. \tag{20}$$

Thus, if scattering at zero angle ($\chi \to 0$) corresponds to radiation (absorption) of a photon of finite energy ($\hbar\omega_1$) the scattering must approach zero ($\sim \chi^3$). Furthermore, account must be taken of the fact that there is an important difference in the energy dependence of the cross sections (in the first case the cross section depends on energy only through the factor $(1 - c^2/\varepsilon v^2)^{1/2}$; in the second case it is proportional to E^2).

We now consider the second term in (17). The quantity $(dE/dt)_{\text{pol}}$ is given by the contributions in the integral in (16) associated with the zeroes of $\varepsilon(\omega, \mathbf{k})$. If the attenuation is small the energy lost by the electron can be treated as the production of an exciton (cf. [6]) and the dispersion relation is determined from the condition

$$\varepsilon(\omega, \mathbf{k}) = 0. \tag{21}$$

We limit ourselves to terms of order $(a/\lambda)^2$ where a is the interatomic distance while λ is the wavelength, thereby obtaining

$$\varepsilon(\omega, \mathbf{k}) = \varepsilon(\omega) + B(\omega) k^2. \tag{22}$$

If ω_0 is used to denote a zero of $\varepsilon(\omega)$, we have from (22)

$$\varepsilon(\omega, \mathbf{k}) = A(\omega - \omega_0) + B(\omega_0) k^2, \tag{23}$$

while the exciton dispersion relation assumes the form

$$\omega = \omega_0 - \frac{B}{A}k^2 \equiv \omega_0 + \frac{\hbar k^2}{2m_e}, \qquad m_e = -\frac{\hbar A}{2B}. \tag{24}$$

Substituting the expansion in (23) in (16) and integrating over $k_\perp$ we have

$$\frac{dE}{dt} = \frac{e^2}{v|A|}\int_{\omega_l} \frac{\omega d\omega}{\omega - \omega_0}, \tag{25}$$

where

$$\omega_l = \omega_0 + \left|\frac{B}{A}\right| \frac{\omega_0^2}{v^2}, \tag{26}$$

and we have made use of the fact that $|B/A|\,\omega_0^2/v^2 \ll \omega_0$ i.e., $\omega_l \approx \omega_0$. Thus at sufficiently small scattering angles ($\omega \sim \omega_l$)

$$(dE/dt) = e^2 v/|B|\,\omega_0. \tag{27}$$

Further, proceeding as before, we find the differential scattering cross section due to the production (absorption) of an exciton:

$$d\sigma^{(\pm)}_{e} = \frac{2}{q} a^2 \frac{m}{m_e} \frac{U_c E}{(\hbar\omega_0)^2} \begin{Bmatrix} N_\omega \\ N_\omega + 1 \end{Bmatrix} \delta(E' - E \mp \hbar\omega)\, dE' \chi d\chi.$$

We have used the notation

$$q = (B/a^2) \qquad (q \sim 1), \qquad U_c = e^2/a \qquad (U_c \sim 10^{-12}\,\text{erg}).$$

A comparison of the formulas obtained here shows that the dependence of the inelastic scattering cross section on angle and energy of the incident particle depends on the scattering mechanism even for scattering at small angles.

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