ADIABATIC ONE-DIMENSIONAL MOTIONS OF AN ULTRARELATIVISTIC GAS

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The equations for isentropic one-dimensional motions of an ultrarelativistic gas are extended to the case of adiabatic motions. The number of particles can be variable or constant. Investigation of shock waves turns out to be possible in this case. In particular, consideration is given to the attenuation of a traveling shock wave which interacts with a simple rarefaction wave overtaking it.

IN the study of multiple particle production, one of the methods of investigation is that of relativistic gas dynamics, where the motions of an ultrarelativistic gas are usually considered. One particular and very thoroughly studied case is the study of the one-dimensional motions of a medium under the assumption that the chemical potential $\mu = 0$, inasmuch as the total number of particles is a variable.

As has already been shown by Khalatnikov[1], exact solutions of the one-dimensional motion of the ultrarelativistic gas can be found for $\mu = 0$. The problem reduces to the solution of the Riemann equation

$$3\frac{\partial^2\chi}{\partial\eta^2} = 2\frac{\partial\chi}{\partial y} + \frac{\partial^2\chi}{\partial y^2}, \qquad y = \ln T, \qquad \eta = \text{arctha}, \quad (1)^*$$

where T is the temperature, a the velocity; the function $\chi = \chi(\eta; y)$ is connected with x and $\tau = t$ by the relations (the velocity of light is taken to be unity)

$$\tau = e^{-y}\left(\frac{\partial\chi}{\partial y}\operatorname{ch}\eta - \frac{\partial\chi}{\partial\eta}\operatorname{sh}\eta\right), \qquad x = e^{-y}\left(\frac{\partial\chi}{\partial y}\operatorname{sh}\eta - \frac{\partial\chi}{\partial\eta}\operatorname{ch}\eta\right). \quad (2)\dagger$$

When $\mu = 0$, $p = \text{const}\cdot T^4$, where p is the pressure.

Equation (1) was recently obtained by the author (independently of Khalatnikov) in another way.[2] Khalatnikov considered the isentropic motion of the medium and investigated the properties of the rarefaction wave reflected from the plane of symmetry. Traveling (simple) waves, as is well known, are studied on the basis of the particular solution of the equation of relativistic gas dynamics, which can be written in simple analytic form.

One can easily show that the set of equations (1) and (2), with $\mu = 0$, describes not only isentropic but also adiabatic motions of the medium, inasmuch as the law $d\sigma/d\tau = 0$ holds, where σ is the entropy per particle; here, $p = \text{const}\cdot T^4$. It can also be easily proved that the set (1) and (2) describes isentropic motions for $\mu \neq 0$, inasmuch as $p = \text{const}\cdot T^4$ in this case, too.

In the collision of a nucleon with a nucleus, a situation can arise in which a simple traveling rarefaction wave, moving from the position of the light particle, overtakes the stationary shock front moving along the heavier particle and begins to weaken the latter. In this case the shock wave becomes nonstationary. Its subsequent motion requires investigation.

Belen'kiĭ and Milekhin[3] considered the problem of the entropy change in the region of a nonstationary shock wave for $\mu = 0$, i.e., in the case when the Khalatnikov equation describes adiabatic motions. The entropy change was calculated from the moment of approach to it of the rarefaction wave up to the moment of its escape to the edge of the heavier particle. The problem of the motion of the shock wave was not considered by them in full.

We shall show below that the Khalatnikov equation permits extension to the case in which $\mu \neq 0$, which permits us to solve the problem for both a variable and a constant number of particles by the same method. This is significant, for example, in the study of the motion of particles after their formation, and in a whole series of other cases. We shall also show how it is possible to solve the problem of the motion of a shock wave attenuated by a rarefaction wave.

By furnishing the equation $p = (k-1)\epsilon$ for the ultrarelativistic gas, and using the equation $\partial T_{ik}/\partial x_k = 0$, where $T_{ik} = (p+\epsilon)u_i u_k + \delta_{ik}p$, we get a complete set of equations in which the value of μ does not appear. In the case of adiabatic one-dimensional motions, we quickly arrive at Eqs. (1) and (2) if we set

*arctha = $\tanh^{-1}$a.

†ch = cosh; sh = sinh.

$$y = \ln p^{(k-1)/k}, \qquad k = 4/3.$$

We note that the law $d\sigma/d\tau = 0$ will hold both for $\mu = 0$ and for $\mu \neq 0$.

Inasmuch as the particular solutions have the form

$$y = \pm \frac{\sqrt{3}}{3}\eta + \text{const}, \qquad x = \frac{\operatorname{th}\eta \pm \sqrt{3}/3}{1 \pm \sqrt{3}\operatorname{th}\eta/3}\tau + F(\eta), \tag{3*}$$

it is easy to show that on the line of intersection (on the characteristic) of the general solution (compound wave) with the particular solution (simple wave)

$$d\chi = -F(\eta)\, e^{y} \operatorname{ch}\eta\, [1 \pm \sqrt{3}\operatorname{th}\eta/3]\, d\eta. \tag{4}$$

If

$$F(\eta) = 0, \tag{5}$$

we can then set $\chi = 0$ without loss of generality, since $d\chi = 0$. By introducing $\varphi = \chi e^{y}$, Eq. (1) can be written in the form

$$\frac{\partial^2 \varphi}{\partial \alpha_1\, \partial \alpha_2} = -\varphi, \tag{6}$$

where

$$\alpha_{1,2} = \frac{1}{2}\left[\frac{\sqrt{3}}{3}\eta \pm y\right]. \tag{7}$$

We proceed to the study of shock waves. Two cases of shock wave propagation are possible. In the first, the shock wave moves along the usual unheated gas; in the second, the wave is propagated along the heated relativistic gas. In this case, we consider the effect of attenuation of the shock wave moving along the ultrarelativistic gas in the interaction of it with the overtaking (simple) rarefaction wave. We note that even the traveling shock wave cannot be described by the particular solution, inasmuch as the conditions on its front $[p_n = p(a_0)]$ are not consistent with the first equation of the solution (3).

In a coordinate system in which the medium is at rest behind the shock front we have[4]

$$p_2/\varepsilon_1 = (3a_0^2 + 1)/3(1 - a_0^2), \qquad D_y = dx/d\tau = 1/3a_0, \tag{8}$$

where a_0 is the rate of flow of the undeformed medium at the shock front (shock velocity), D_y is the velocity of the shock front (the indices 1 and 2 refer to parameters in front of and behind the front).

Let the rarefaction wave begin in the cross section $x = 0$ at the time $\tau = 0$, and let the front of the shock have the coordinate $x = l$. The simple wave is described by the equations

$$\frac{p}{p_2} = \left(\frac{1-a}{1+a}\right)^{2\sqrt{3}/3}, \qquad x = \frac{a + \sqrt{3}/3}{1 + \sqrt{3}a/3}\tau; \tag{9}$$

the rarefaction wave front will move according to the law $x = \sqrt{3}\tau/3$. This front overtakes the shock wave front in the cross section $\bar{x} = \sqrt{3}l/(\sqrt{3} - 1/a_0)$ at

$$\tau = 3l/(\sqrt{3} - 1/a_0). \tag{10}$$

Inasmuch as $a_0 = 1 - \Delta$ and we are considering a strong wave in which $\Delta \ll 1$, we have in the limit when $a_0 = 1$,

$$\bar{x} = (3 + \sqrt{3})\, l/2, \qquad \bar{\tau} = 3(\sqrt{3} + 1)\, l/2. \tag{11}$$

We now formulate the conditions under which it is necessary to seek a solution of the equation $\partial^2\varphi/\partial\alpha_1\partial\alpha_2 = -\varphi$, which describes the nonstationary shock wave formed after the rarefaction wave overtakes the front. In the coordinate system in which the medium ahead of the shock front is at rest to the left of the characteristic

$$\frac{dx}{d\tau} = \frac{a - \sqrt{3}/3}{1 - \sqrt{3}a/3}$$

we have

$$\frac{p}{p_2} = \left(\frac{1+a}{1-a}\,\frac{1-a_0}{1+a_0}\right)^{2\sqrt{3}/3}, \qquad \frac{x}{\tau} = \frac{a + \sqrt{3}/3}{1 + \sqrt{3}a/3}, \tag{12}$$

where a_0 is the rate of the flow behind the shock front; here, $\varphi = 0$. Consequently,

$$y = y_0 + \sqrt{3}(\eta - \eta_0)/3, \qquad \varphi = 0,$$

where $y_0 = y_2$. On the shock front, the motion of which is still unknown, we have

$$dx/d\tau = D_y = (3a_0^2 + 1)/4a_0 = (3\operatorname{th}^2\eta_0 + 1)/4\operatorname{th}\eta_0, \tag{13}$$

$$p_2/\varepsilon_1 = (3a_0^2 + 1)/3(1 - a_0^2) = \tfrac{1}{3}[4\operatorname{sh}^2\eta_0 + 1] = e^{4y_0}/\varepsilon_1. \tag{14}$$

As is well known, the simplest solution of the equation $\partial^2\varphi/\partial\alpha_1\partial\alpha_2 = -\varphi$ in quadratures can be obtained if the line $\alpha_1 = \alpha_1^*(\eta^*)$, $\alpha_2 = \alpha_2^*(\eta^*)$ is given, where η^* is a parameter, and along this line we are given

$$\varphi = \varphi^*(\eta^*), \qquad d\varphi/d\eta^* = \varphi^{*\prime}(\eta^+). \tag{15}$$

Let $\eta^* = \eta_0$ and the function $\varphi = \varphi^*(\eta_0)$ also be given. Then, inasmuch as

$$\alpha_{1,2} = \tfrac{1}{2}[\sqrt{3}\eta/3 \pm y],$$

we have on the shock front

$$y_0 = \frac{1}{4}\ln\left[\tfrac{1}{3}\varepsilon_1(4\operatorname{sh}^2\eta_0 + 1)\right],$$

$$\alpha_{1,2}^* = \frac{1}{2}\left[\sqrt{3}\eta_0/3 \pm \frac{1}{4}\ln\left[\tfrac{1}{3}\varepsilon_1[(4\operatorname{sh}^2\eta_0 + 1)\right]\right]. \tag{16}$$

Let the motion of the shock wave be given in the form $\tau = \tau_0(\eta_0)$. Then, since

$$dx/d\tau = (3\operatorname{th}^2\eta_0 + 1)/4\operatorname{th}\eta_0,$$

we have

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*th = tanh.

$$x_0 = \int \frac{3\,\mathrm{th}^2\,\eta_0 + 1}{4\,\mathrm{th}\,\eta_0}\,\frac{d\tau_0}{d\eta_0}\,d\eta_0 = x_0(\eta_0). \quad (17)$$

Further, since

$$\chi = \varphi e^{-y}, \qquad \partial\chi/\partial y = e^{-y}(\partial\varphi/\partial y - \varphi),$$

then

$$\tau = e^{-2y_0}[(\partial\varphi/\partial y - \varphi)\,\mathrm{ch}\,\eta_0 - \mathrm{sh}\eta_0\partial\varphi/\partial\eta] = \tau_0(\eta_0),$$
$$x = e^{-2y_0}[(\partial\varphi/\partial y - \varphi)\,\mathrm{sh}\,\eta_0 - \mathrm{ch}\eta_0\partial\varphi/\partial\eta] = x_0(\eta_0).$$

Since

$$\frac{d\varphi}{d\eta} = \frac{\partial\varphi}{\partial\eta} + \frac{\partial\varphi}{\partial y}\frac{dy}{d\eta} = \varphi',$$

where

$$dy/d\eta = y_0' = 2\,\mathrm{sh}\eta_0\,\mathrm{ch}\eta_0/(4\,\mathrm{sh}^2\,\eta_0 + 1),$$

then

$$\tau_0 = e^{-2y_0}[(\mathrm{ch}\,\eta_0 + y_0'\,\mathrm{sh}\,\eta_0)\,\partial\varphi/\partial y - (\varphi\,\mathrm{ch}\,\eta_0 + \varphi'\,\mathrm{sh}\,\eta_0)],$$
$$x_0 = e^{-2y_0}[(\mathrm{sh}\,\eta_0 + y_0'\,\mathrm{ch}\,\eta_0)\,\partial\varphi/\partial y - (\varphi\,\mathrm{sh}\,\eta_0 + \varphi'\,\mathrm{ch}\,\eta_0)].$$

Hence we have

$$\frac{\partial\varphi}{\partial y} = \frac{\tau_0 e^{2y_0} + \varphi\,\mathrm{ch}\,\eta_0 + \varphi'\,\mathrm{sh}\,\eta_0}{\mathrm{ch}\,\eta_0 + y_0'\,\mathrm{sh}\,\eta_0} = \frac{x_0 e^{2y_0} + \varphi\,\mathrm{sh}\,\eta_0 + \varphi'\,\mathrm{ch}\,\eta_0}{\mathrm{sh}\,\eta_0 + y_0'\,\mathrm{ch}\,\eta_0}. \quad (18)$$

After simple transformations, we get for φ the linear ordinary equation

$$\frac{d\varphi}{d\eta_0} - \varphi\frac{dy_0}{d\eta_0} = e^{2y_0}\Big[\tau_0\Big(\mathrm{sh}\,\eta_0 + \frac{dy_0}{d\eta_0}\mathrm{ch}\,\eta_0\Big) - x_0\Big(\mathrm{ch}\,\eta_0 + \frac{dy_0}{d\eta_0}\mathrm{sh}\,\eta_0\Big)\Big]. \quad (19)$$

Solving this equation, we find

$$\varphi = \varphi^*(\eta_0), \qquad d\varphi/d\eta_0 = \varphi'^*(\eta_0).$$

Further, we find $\partial\varphi/\partial y_0$ and $\partial\varphi/\partial\eta_0 = d\varphi/d\eta_0 - y_0'\partial\varphi/\partial y_0$ and thus determine

$$\frac{\partial\varphi}{\partial\alpha_1} = \frac{\partial\varphi}{\partial y_0} + \sqrt{3}\,\frac{\partial\varphi}{\partial\eta_0}, \qquad \frac{\partial\varphi}{\partial\alpha_2} = -\frac{\partial\varphi}{\partial y_0} + \sqrt{3}\,\frac{\partial\varphi}{\partial\eta_0}. \quad (20)$$

It is now easy to write the solution of the equation $\partial^2\varphi/\partial\alpha_1\partial\alpha_2 = -\varphi$ in the form

$$\varphi(\xi_1;\xi_2) = \frac{1}{2}[(\varphi\bar{\varphi})_{Q_1} + (\varphi\bar{\varphi})_{Q_2}] + \frac{1}{4}\int_{\theta_1}^{\theta_2}\Big[\Big(\varphi\frac{\partial\bar{\varphi}}{\partial\alpha_2} - \bar{\varphi}\frac{\partial\varphi}{\partial\alpha_2}\Big)\frac{\partial\alpha_2}{\partial\eta_0} - \Big(\varphi\frac{\partial\bar{\varphi}}{\partial\alpha_1} - \bar{\varphi}\frac{\partial\varphi}{\partial\alpha_1}\Big)\frac{d\alpha_1}{d\eta_0}\Big]d\eta_0,$$

$$\bar{\varphi} = J_0[2\sqrt{(\alpha_1 - \xi_1)(\alpha_2 - \xi_2)}], \qquad \xi_1 \to \alpha_1, \quad \xi_2 \to \alpha_2. \quad (21)$$

Here J_0 is the Bessel function of order zero. $\xi_2 = \alpha_2$ at the point Q_1 on the line $\alpha_2 = \alpha_2(\alpha_1)$; and $\xi_1 = \alpha_1$ at the point Q_2; the line $\alpha_2 = \alpha_2(\alpha_1)$ is given; by the point Q_1 is meant the initial point on the shock wave; the coordinates of the point Q_2 are arbitrary.

This solution must be continued from the shock front until we get a junction with the simple wave. Inasmuch as, for the simple wave,

$$y = y_0 + \sqrt{3}(\eta - \eta_0)/3$$

or

$$\alpha_2 = \alpha_{20}, \quad (22)$$

we get, substituting $\alpha_2 = \xi_2 = \alpha_{20}$ in the solution thus found,

$$\varphi = \varphi_1(\xi_1) = \varphi_1(\alpha_1). \quad (23)$$

It then follows that

$$\chi = \varphi_1 e^{-y} = \varphi_1 e^{\alpha_1 - \alpha_{20}} = \varphi_1\,\exp(-2\alpha_{20} + \sqrt{3}\,\eta/3). \quad (24)$$

Further, we find

$$d\chi = e^{-y}[d\varphi_1/d\eta + \sqrt{3}\,\varphi_1/3]\,d\eta. \quad (25)$$

Comparing this expression with the condition on the characteristic

$$d\chi = F(\eta)\,e^{-y}\,\mathrm{ch}\,\eta[1 + \sqrt{3}\,\mathrm{th}\,\eta/3]\,d\eta, \quad (26)$$

we find that

$$F(\eta) = \frac{d\varphi_1/d\eta + \sqrt{3}\,\varphi_1/3}{\mathrm{ch}\,\eta\,[1 + \sqrt{3}\,\mathrm{th}\,\eta/3]}, \quad (27)$$

which determines the arbitrary function $F(\eta)$ for the simple wave. It is always possible to choose $\tau = \tau_0(\eta_0)$ so that $F(\eta) = 0$, which also solves the stated problem completely.

We note that the case worked out here of the exact solution for a nonstationary shock wave, and in general, the possibility of solving the problem of adiabatic motion of the medium exactly, do not have a classical analogue. Classical shock waves in gases and adiabatic motions do not yield exact solutions.

It is not difficult to generalize the method thus developed to a medium with the more general equation of state $p = (k-1)\epsilon$, where $1 < k \leq 4/3$.

In conclusion, we consider the possibility of studying continuous adiabatic currents. Since $(d\sigma/d\tau)_{x_0} = 0$ in Lagrangian coordinates, then

$$\sigma = \sigma(x_0), \quad (28)$$

where x_0 is the Lagrangian coordinate and σ is the entropy. By knowing $\Phi(a, x, \tau) = 0$, where $a = (dx/d\tau)_{x_0}$, and integrating this equation for the condition that $x = \bar{x}(\tau) = x_0$ in the front of any wave, we get

$$\psi(x, x_0, \tau) = 0, \quad (29)$$

whence, for (28), by eliminating x_0, we find $f(\sigma, x, \tau) = 0$. Knowing $\Phi_2(p, x, \tau) = 0$, we find $\Phi_2(V, x, \tau) = 0$, which also solves the problem at hand completely.

For example, in the case of simple waves, it is easy to get the final solution by simple procedures.

Inasmuch as here

$$x = \frac{a \pm \sqrt{k-1}}{1 \pm \sqrt{k-1}\,a}\,\tau + F(a),$$

then, by differentiating with respect to τ, we get the equation

$$\frac{\pm\sqrt{k-1}\,(1-a)^2}{1 \pm \sqrt{k-1}\,a} = -\frac{\partial a}{\partial \tau}\left[\tau\frac{2-k}{(1 \pm \sqrt{k-1})^2} + \frac{dF}{da}\right],$$

whence

$$-\frac{d\tau}{da} = \frac{2-k}{\pm\sqrt{k-1}}\,\frac{\tau}{(1-a^2)(1 \pm \sqrt{k-1}\,a)} - \frac{dF}{da}\,\frac{1 \pm \sqrt{k-1}\,a}{\pm\sqrt{k-1}\,(1-a^2)},$$

$$\tau = (1 \pm \sqrt{k-1}\,a)(1+a)^{[\pm(k-1)^{-1/2}+1]/2}(1-a)^{[\pm(k-1)^{-1/2}-1]/2} \times \Big[\Phi(x_0) - \int \frac{dE}{da}(1 \pm \sqrt{k-1}\,a)^{-1} \times (1+a)^{[\pm(k-1)^{-1/2}+1]/2}(1-a)^{[\pm(k-1)^{-1/2}-1]/2}\,da\Big]. \quad (30)$$

In the special case in which $F(a) = 0$, we get

$$\tau = \frac{1 \pm \sqrt{k-1}\,a}{a \pm \sqrt{k-1}} = (1 \pm \sqrt{k-1}\,a)(1+a)^{[\pm(k-1)^{-1/2}+1]/2} \times (1-a)^{[\pm(k-1)^{-1/2}-1]/2}\Phi(x_0). \quad (31)$$

Since $x = \pm\sqrt{k-1}\,\tau = x_0$ and $a = 0$ on the characteristic, we have $\Phi(x_0) = \pm x_0/\sqrt{k-1}$; we introduce $z = x/\tau = (a \pm \sqrt{k-1})/(1 \pm \sqrt{k-1}\,a)$, and then (31) takes the form

$$\tau^2 = \left(\frac{1-z}{1+z}\,\frac{1 \pm \sqrt{k-1}}{1 \mp \sqrt{k-1}}\right)^{\pm(k-1)^{-1/2}}\frac{2-k}{k-1}\,\frac{x_0^2}{1-z^2}, \quad (32)$$

whence

$$x_0^2 = \frac{k-1}{2-k}(\tau^2 - x^2)\left[\frac{\tau+x}{\tau-x}\,\frac{1 \pm \sqrt{k-1}}{1 \pm \sqrt{k-1}}\right]^{\pm(k-1)^{-1/2}} \quad (33)$$

Knowing $\sigma = \sigma(x_0)$, we find

$$\sigma = \sigma(x, \tau), \quad (34)$$

which solves the given problem. Furthermore, it is easy to find $V = V(x, \tau)$, inasmuch as

$$p = \text{const}\,[(1+a)/(1-a)]^{\pm k/2\sqrt{k-1}}, \qquad pV^k = \sigma(x, \tau).$$

It remains to determine the number of particles. For all cases, if the number of particles in the entire system is constant, then $n \sim 1/V$. If the number of particles is variable, then we have the equation

$$\frac{dn}{d\tau} + \frac{n}{\theta^2}\left(\frac{\partial a}{\partial x} + a\frac{\partial a}{\partial \tau}\right) = \frac{dN}{V^* d\tau}, \quad (35)$$

where N is the total variable number of particles in the entire volume V^*. This number of particles must be determined from the "kinetic" conditions of their creation and annihilation.

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