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DIAMAGNETIC PLASMA INSTABILITY FOR LARGE ION LARMOR RADIUS

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The criterion for diamagnetic instability of a plasma with an anisotropic ion velocity distribution is derived for an arbitrary ratio of perturbation wavelength to ion Larmor radius.

A number of authors [1-3] have considered the diamagnetic instability of a plasma with an anisotropic velocity distribution for wavelengths λ appreciably greater than the mean ion Larmor radius ρ: $\lambda \gg \rho$. However, the maximum possible wavelength, which is of the order of the characteristic dimensions of the experimental device, can be comparable with the ion Larmor radius. This is the case, for example, in a magnetic trap in which the plasma is formed by injection of a high-energy ion beam. Under these conditions the only perturbations that are possible are those with wavelengths of the order of, or smaller than, the ion Larmor radius. For this reason it is of interest to analyze the diamagnetic instability for arbitrary wavelengths, including wavelengths much smaller than the ion Larmor radius. It is clear that the most important differences from the usual results pertaining to long-wave instabilities would be expected in the case in which a possible instability is related to anisotropy in the ion distribution function rather than in the electron distribution function. For this reason, the electron velocity distribution is assumed hereinafter to be isotropic.

The equation that relates the oscillation frequency ω and the wave vector $\mathbf{k}$ in a medium with a tensor dielectric constant $\varepsilon_{\alpha\beta}$ is of the familiar form

$$\mathrm{Det}\left(\varepsilon_{\alpha\beta} + N^2 \frac{k_\alpha k_\beta}{k^2} - \delta_{\alpha\beta} N^2\right) = 0, \qquad N^2 = \frac{c^2 k^2}{\omega^2}. \tag{1}$$

Expressions for the components of $\varepsilon_{\alpha\beta}$ in a plasma with an anisotropic particle velocity distribution are given, for example, by Sagdeev and Shafranov.[4] These expressions are in the form of infinite sums. In the case of interest here, i.e., frequencies appreciably below the ion-cyclotron frequency, these infinite sums can be summed partially by making use of an expansion in powers of the small parameter $|\omega - k_z v_\parallel|/\omega_c$, where k_z is the component of the wave vector along the constant magnetic field $\mathbf{B}_0$ ($\mathbf{B}_0 \parallel \mathbf{z}$), $v_\parallel$ is the particle velocity in the z direction, and ω_c is the cyclotron frequency of the ions or electrons. It is assumed that the majority of particles do not have very high longitudinal velocities so that $\bar{v}_\parallel k_z \ll \omega_c$; resonance particles, which can produce an instability with exponentially small growth rates (such as that considered in [4]), are neglected. Under these conditions $\varepsilon_{\alpha\beta}$ assumes the following form:

$$\varepsilon_{xx} = 1 + \sum \frac{m\omega_0^2}{k_x^2} \int d\mathbf{v}\,(1 - J_0^2)\left[-\frac{\partial f}{\partial \varepsilon_\perp} + \frac{k_z^2 v_\parallel^2}{\omega^2}\left(\frac{\partial f}{\partial \varepsilon_\parallel} - \frac{\partial f}{\partial \varepsilon_\perp}\right)\right],$$

$$\varepsilon_{xy} = -\varepsilon_{yx} = i \sum \frac{m\omega_0^2 \omega_c}{\omega k_x^2} \int d\mathbf{v}\, v_\perp^2 \frac{\partial f}{\partial \varepsilon_\perp} \frac{dJ_0^2}{dv_\perp^2},$$

$$\varepsilon_{yy} = \varepsilon_{xx} + \sum \frac{m\omega_0^2}{\omega} \int d\mathbf{v}\, v_\perp^2 \left\{ J_1^2 \left[\frac{\partial f/\partial \varepsilon_\parallel}{\omega - k_z v_\parallel} + \frac{1}{\omega}\left(\frac{\partial f}{\partial \varepsilon_\perp} - \frac{\partial f}{\partial \varepsilon_\parallel}\right)\right]\right.$$

$$+ 2 \sum_{n=1}^{+\infty} \frac{\omega}{n^2 \omega_c^2}\left[-\frac{\partial f}{\partial \varepsilon_\perp} + \frac{k_z^2 v_\parallel^2}{\omega^2}\left(\frac{\partial f}{\partial \varepsilon_\parallel} - \frac{\partial f}{\partial \varepsilon_\perp}\right)\right]$$

$$\left.\times \left[\frac{1}{2}\frac{d^2 J_n^2}{dx^2} - J_n^2 - \frac{J_n J_n'}{x}\right]\right\},$$

$$\varepsilon_{xz} = \varepsilon_{zx} = \sum \frac{m\omega_0^2}{\omega^2} \frac{k_z}{k_x} \int d\mathbf{v}\, v_\parallel^2 (1 - J_0^2)\left(\frac{\partial f}{\partial \varepsilon_\perp} - \frac{\partial f}{\partial \varepsilon_\parallel}\right),$$

$$\varepsilon_{yz} = -\varepsilon_{zy} = -i \sum \frac{m\omega_0^2}{\omega} \int d\mathbf{v} \left\{ \frac{v_\parallel v_\perp J_0 J_0'}{\omega - k_z v_\parallel} \frac{\partial f}{\partial \varepsilon_\parallel}\right.$$

$$\left. + 2 \sum_{n=1}^{+\infty} \frac{k_z v_\parallel^2 v_\perp J_n J_n'}{n^2 \omega_c^2}\left(2\frac{\partial f}{\partial \varepsilon_\perp} - \frac{\partial f}{\partial \varepsilon_\parallel}\right)\right\},$$

$$\varepsilon_{zz} = 1 + \sum \frac{m\omega_0^2}{\omega} \int d\mathbf{v} \left\{ \frac{v_\parallel^2 J_0^2}{\omega - k_z v_\parallel} \frac{\partial f}{\partial \varepsilon_\parallel} - \frac{v_\parallel^2 (1 - J_0^2)}{\omega}\left(\frac{\partial f}{\partial \varepsilon_\perp} - \frac{\partial f}{\partial \varepsilon_\parallel}\right)\right\}, \tag{2}$$

where

$$\varepsilon_\parallel = \frac{m v_\parallel^2}{2}, \qquad \varepsilon_\perp = \frac{m v_\perp^2}{2}, \qquad \omega_0^2 = \frac{4\pi n_0 e^2}{m}, \qquad \int f\, d\mathbf{v} = 1,$$

and n_0 is the plasma density. The coordinate system is chosen so that the vector $\mathbf{k}$ lies in the xz plane. The argument of the Bessel functions is $x = k_x v_\perp/\omega_c$. The summation is carried out over charge species (electrons and ions) and also over n in ε_{yy} and ε_{yz}.

It is well-known (cf. [3]) that the diamagnetic instability is related to ion anisotropies and corresponds to "oscillations" characterized by small phase velocity, in which case $|\omega| \ll k_z v_{\parallel i}$, where $v_{\parallel i}$ is the mean longitudinal ion velocity. Under these conditions and when the velocity anisotropy is not too small (so that $|\partial f/\partial \varepsilon_\parallel - \partial f/\partial \varepsilon_\perp| \gtrsim \partial f/\partial \varepsilon_\perp$) we need retain only the terms containing the difference $\partial f/\partial \varepsilon_\parallel - \partial f/\partial \varepsilon_\perp$ in the expression for ε_{xx}. Then

$$\varepsilon_{xx} = -\varepsilon_{xz}\frac{\cos\theta}{\sin\theta} = \varepsilon''_{zz}\frac{\cos^2\theta}{\sin^2\theta},$$

where ε''_{zz} is the part of ε_{zz} due to the anisotropy, i.e., the part containing $\partial f/\partial \varepsilon_\perp - \partial f/\partial \varepsilon_\parallel$, and θ is the angle between $\mathbf{k}$ and $\mathbf{B}_0$. Having made use of this relation between the elements of $\varepsilon_{\alpha\beta}$ and neglecting the quantity ε_{xy}^2 compared with $(\varepsilon_{yy} - N^2)(\varepsilon_{xx} - N^2\cos^2\theta)$ we can write (1) in the form

$$N^2 - \varepsilon_{yy} - (\varepsilon_{yz} - \varepsilon_{xy}\sin\theta/\cos\theta)^2/\varepsilon'_{zz} = 0, \qquad (3)$$

where $\varepsilon'_{zz} = \varepsilon_{zz} - \varepsilon''_{zz}$. Formally (3) differs from the corresponding equation for long-wave oscillations given by Kitsenko and Stepanov [3] only in the presence of the term containing ε_{xy}, which is not small at short wavelengths ($k\rho \gtrsim 1$).

The stability condition can be found easily from (2) and (3) by a method similar to that used by Kitsenko and Stepanov.[3] We present the result for the case of a low-pressure plasma ($p_\perp \ll B_0^2/8\pi$) or a strong velocity anisotropy; the electron distribution function is assumed to be Maxwellian with temperature T_e while the ion distribution function is equal to the product of "longitudinal" and "transverse" Maxwellian functions with corresponding temperatures $T_\parallel$ and $T_\perp$ ($T_\parallel \ll T_\perp$). The stability condition for arbitrary ρ/λ is given by

$$\frac{2T_\perp^2}{Mv_A^2 T_\parallel}\Psi(z)\left[1 - \frac{1}{2}\frac{\Psi(z)}{T_\parallel/T_e + I_0 e^{-z}}\right] < 1;$$

$$v_A^2 = B_0^2/4\pi n_0 M c^2, \qquad z = k_x^2\rho^2/2, \qquad \rho^2 = 2T_\perp/M\omega_c^2,$$

$$\omega_c = eB_0/Mc, \qquad \Psi(z) = e^{-z}(I_0 - I_1). \qquad (4)$$

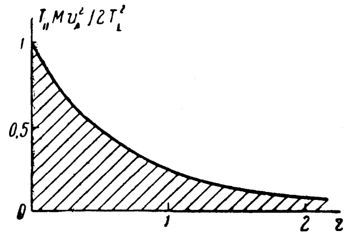
Here, M and e are the mass and charge of the ion while $I(z)$ represents the modified Bessel function. The condition in (4) is written under the assumption that $k_z \ll k_x$.

The stability region in the plane of the variables z and $T_\parallel M v_A^2/2T_\perp^2$ for the case $T_e \to 0$ is shown in the figure (the instability region is shown cross-hatched).

The growth rate γ for unstable waves is given by

$$\gamma = \sqrt{\frac{2T_\parallel}{\pi M}}\,k_z\left\{1 - \frac{1}{2}\frac{\Psi}{T_\parallel/T_e + I_0 e^{-z}} - \left(1 + \frac{k_z^2}{k_x^2}\right)\frac{1}{\Psi}\frac{T_\parallel M v_A^2}{2T_\perp^2}\right\}$$

$$\times\left\{1 + \frac{\Psi}{T_\parallel/T_e + I_0 e^{-z}}\left(1 - \frac{1}{2}\frac{\Psi}{T_\parallel/T_e + I_0 e^{-z}}\right)\right\}^{-1}. \qquad (5)$$



In particular, with $T_e \ll T_\parallel$ and $z \ll 1$ the growth rate reaches a maximum when

$$\frac{k_z^2}{k_x^2} = \frac{3}{2}k_x^2\rho^2 = \frac{1}{3}\left(\frac{2T_\perp^2}{T_\parallel M v_A^2} - 1\right), \qquad (6)$$

furthermore, if $(2T_\perp^2/T_\parallel M v_A^2) - 1 \ll 1$ then

$$\frac{\gamma_{max}}{\omega_c} = \frac{2}{9\sqrt{3\pi}}\left(\frac{T_\parallel}{T_\perp}\right)^{1/2}\left(\frac{2T_\perp^2}{T_\parallel M v_A^2} - 1\right)^2. \qquad (7)$$

When $z \gg 1$ the growth rate as a function of k_z reaches a maximum when

$$\frac{k_z^2}{k_x^2} = \frac{1}{6z\sqrt{2\pi z}}\left(1 - 2z\sqrt{2\pi z}\,\frac{T_\parallel M v_A^2}{2T_\perp^2}\right) \qquad (8)$$

and, under these conditions, depends on k_x in the following way:

$$\frac{\gamma}{\omega_c} = \left(\frac{2}{9\pi}\right)^{3/4}\left(\frac{T_\parallel}{T_\perp}\right)^{1/2} z^{-1/4}\left(1 - 2z\sqrt{2\pi z}\,\frac{T_\parallel M v_A^2}{2T_\perp^2}\right)^{3/2}. \qquad (9)$$

The inequality in (4) is satisfied worst at long wavelengths $\lambda \to \infty$ (for fixed values of $T_\perp$, $T_\parallel$ and V_A). Hence, writing $z = 0$ in (4) we obtain the condition for diamagnetic stability of a plasma with respect to perturbations of arbitrary wavelength:[3]

$$\frac{2T_\perp^2}{Mv_A^2 T_\parallel}\left(1 - \frac{1}{2(T_\parallel/T_e + 1)}\right) < 1. \qquad (10)$$

If the condition in (10) is not satisfied instabilities will arise in a range of wave numbers $\Delta k_\perp$ limited from above by some value $k_{\perp max}$ which can be found by means of (4). The interval $\Delta k_\perp$ is wider the greater the ratio $2T_\perp^2/T_\parallel M v_A^2$ compared with unity. For example, for a very pronounced anisotropy, where $2T_\perp/T_\parallel M v_A^2 \gg 1$, the minimum wavelength for instabilities is given by the relation

$$z_{max} = [2T_\perp^2/Mv_A^2 T_\parallel\,(8\pi)^{1/2}]^{2/3}. \qquad (11)$$

If, however, the anisotropy is such that the ratio $T_\perp^2/T_\parallel M v_A^2$ is of the order of unity (but the inequality in (10) is not satisfied) then the only unstable perturbations are those with wavelength λ of the order of, or greater than, ρ. However, one hopes that in systems whose characteristic dimensions are comparable with the ion Larmor radius this large-scale instability can be stabilized by means of external conductors. It is reasonable to assume that in these traps the most dangerous instability is the fine-scale instability ($k\rho \gg 1$);

however, this instability appears only for a very strong anisotropy $T_{\|}/T_{\perp} \ll 8\pi p_{\perp}/B_0^2$.

In conclusion I wish to thank V. D. Shafranov for his continued interest in this work and for valuable comments.

[1] L. I. Rudakov and R. Z. Sagdeev, Fizika plazmy i problema upravlyaemykh termoyadernykh reaktsiĭ (Plasma Physics and the Problem of a Controlled Thermonuclear Reaction), **3**, AN SSSR, 1958, p. 268.

[2] A. A. Vedenov and R. Z. Sagdeev, ibid. p. 278.

[3] A. B. Kitsenko and K. N. Stepanov, JETP **38**, 1840 (1960), Soviet Phys. JETP **11**, 1323 (1960).

[4] R. Z. Sagdeev and V. D. Shafranov, JETP **39**, 181 (1960), Soviet Phys. JETP **12**, 130 (1961).

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