SOLUTION OF THE LINEAR EQUATIONS OF THE DISPERSION METHOD IN THE TWO-PARTICLE APPROXIMATION

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A method is proposed for solving the linear integral equations which arise when the effect of the nearest singularities is calculated by the dispersion technique (two-meson approximation). It is shown that in this case only one solution can be obtained in the low energy region, which does not depend on the behavior of the amplitude under consideration at infinity. This behavior is defined by a number of mass states in the unitarity condition (distant singularities). The condition of independence of the behavior in the high-energy region, which is equivalent to the requirement of stability under disturbances at infinity, is a necessary condition for the solution of the problem and ensures moreover uniqueness of the solution. The ambiguity previously encountered [2,3] is due to an additional family of solutions which are unstable against perturbations at infinity and are meaningless in this problem. The methods proposed by Frazer and Fulco [4] and by Bowcock [5] for calculating $\pi\pi$-interaction effects are based on the application of unstable solutions and lead in general to erroneous results. A detailed analysis is carried out for the simplest amplitude, the pion electromagnetic form factor.

1. INTRODUCTION

METHODS connected with the use of the unitarity condition in the dispersion technique occupy an important place in the investigation of various amplitudes at low values of the invariants (see, for example, [1]). This makes it necessary to solve the linear integral equation (for example, in the invariant t), resulting from substitution of the absorptive part, with respect to t in the two-particle approximation. [2,3] Confining ourselves in the unitarity condition to the lower mass states, we naturally expect an adequate description in the region of small values of the invariant, since the heavier mass states do not make any contribution to the absorptive part on the initial portion of the cut. Thus, the basis of the two-particle approximation is the rather obvious general-theoretical consideration that the behavior of the amplitude at small values of t is determined only by the nearest singularities in the complex plane of t, and that the contributions from the farther regions of the cuts, due to a whole series of intermediate states in the unitarity condition, are smooth functions and can with sufficient accuracy be approximated by subtractional polynomials. The behavior of the amplitude at large t is determined, to the contrary, by a whole series of intermediate states and cannot be determined in principle within the framework of the two-particle approximation.

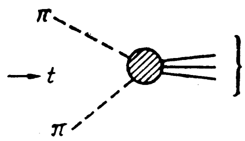

FIG. 1

For amplitudes F(t) of the type shown in Fig. 1, to the consideration of which we confine ourselves in the present paper, the kernel of the equation is the partial amplitude of $\pi\pi$ scattering[1)] $\lambda(t) = \sin\delta(t)\exp[i\delta(t)]$, and the problem under consideration is equivalent to calculating the effect of $\pi\pi$ interaction neglecting the contribution from the four-meson and heavier mass states in the unitarity condition. The solution of the obtained equation turns out to be, however, multiple-valued and various supplementary cri-

[1)]This property is possessed, strictly speaking, only by the simplest amplitudes, which depend on a single invariant (3-point diagrams). For amplitudes dependent on several invariants t, s, ..., the functions F(t) are certain integrals with respect to s, ..., (see, for example, the paper of Frazer and Fulco[4] for the amplitude of $\pi + \pi \to N + \bar{N}$.

teria were proposed to separate the true solution, for example, the requirement of correct asymptotic approximation [2] or that there be no zeros [3] in the obtained solution.

It seems obvious to us that for this purpose one cannot use the asymptotic behavior of the solutions obtained, since it does not have, generally speaking, any relation to reality. There are no grounds for assuming that an equation which by definition is valid only for small t has at least one solution which simultaneously has the correct asymptotic behavior and is a good approximation for large t. Such a formulation of the problem can be compared with an approximation of any function in the region of small t by polynomials under the additional requirement that the resultant expression have a correct asymptotic behavior. Obviously such a problem has generally speaking no solution.

That the criterion which stipulates the absence of zeros is not justified was noted by Federbush, Goldberger, and Treiman [3], but they identified it erroneously with the requirement of correspondence to the iterational method of solving the equation. This misunderstanding is based on the use of the additional expansion of the kernel in powers of the phase $\delta(t) \approx \lambda(t)$. It is obvious that this expansion is not valid when $\delta \gtrsim \pi/2$, whereas the iterational solution is meaningful and even gives good results at small integral contributions ($\int|\lambda(t)|\,dt/t_{\text{eff}} \ll 1$), for example, in the most interesting case of narrow resonance.

The procedure proposed by Frazer and Fulco [4] and by Bowcock et al [5] leads to a unique solution of the problem, but is based on an implicit account of the region of large energies, which is equivalent to an equivalent choice of the solution (see below). For this reason the contributions from the cut $t \geq 4\mu^2$ (μ is the pion mass), obtained by this method, are not the result of an account of the nearest singularities, i.e., the effect of $\pi\pi$ interaction.

A correct method for calculating the effect of the $\pi\pi$ interaction in the $\pi + \pi \to N + \bar{N}$ amplitude was used by Galanin and Grashin [6], but they actually did not demonstrate the uniqueness of the solution nor did they give sufficiently detailed analysis of the earlier errors. We shall show in the present paper that the previously obtained ambiguity is the consequence of the logical inconsistency in the analysis of the problem of calculating the effect of nearest singularities, and the correctly formulated problem has either no solutions whatever, or a unique solution. This solution is contained among the set of solutions obtainable by the usual method [2,3] and is unstable against perturbations of the equation at infinity.[2)] Thus, in the two-meson approximation one can obtain only one solution describing correctly, in the region of small t, the amplitude under consideration and independent of the conditions at infinity, which are not known to us. A general procedure is proposed for calculating the quadratures obtained during the course of the solution, and this procedure is used to analyze in detail the simplest amplitude, the electromagnetic form factor of the pion.

2. GENERAL METHOD

We write down the dispersion relation for $F(t)$ with $N \geq N_0$ subtractions (N_0 is the degree of polynomial that majors $F(t)$ at infinity), separating out explicitly the integral along the cut $t \geq 4\mu^2$:

$$F(t) = \widetilde{F}(t) + \frac{t^N}{\pi} \int_{4\mu^2}^{\infty} \frac{\operatorname{Im} F(t')\,dt'}{(t'-t)\,t'^N}. \tag{1}$$

The function $\widetilde{F}(t)$, which contains the subtractional $(N-1)$-st degree polynomial $P_{N-1}(t)$ and possible contributions from other cuts, is considered in this problem to be known in the region of small t (the calculation of $F(t)$, for example, for the amplitude $\pi + \pi \to N + \bar{N}$ was carried out earlier [6]). The unitarity condition for the absorptive part with $t \geq 4\mu^2$ has the form

$$\operatorname{Im} F(t) = \lambda^*(t)\,F(t) + f(t), \tag{2}$$

where the function $f(t)$ is the contribution from the four-meson and heavier intermediate states and differs from zero only in the region $t > 16\mu^2$. At certain values $t \gtrsim T > 16\mu^2$, the contribution from both terms in (2) becomes comparable and it becomes meaningless at all to separate the two-meson term from the exact absorptive part. Near the inelastic threshold, $f(t)$ is very small, since it has a singularity of type $(t-16\mu^2)^{7/2}$, so that we can expect $T \gtrsim 4m\mu$ (where m is the nucleon mass) and $4\mu^2/T \lesssim \epsilon = \mu/m = 0.15 \ll 1$, i.e., there is a sufficiently broad initial portion of the cut, where we can confine ourselves to the two-meson approximation for the absorptive part.

The initial section of the cut $4\mu^2 \leq t \lesssim T$ can

2) In other words, arbitrary changes in the order of magnitude of the kernel and of the inhomogenous term of the equation when $t \gtrsim \tau$ is as large as desired, change the obtained solution in the region of small energies by small amounts, $\lesssim t/\tau$.

be arbitrarily called the nearest singularity, and the region $t > T$ can be called the more remote singularities. By majoring the absorptive part in the region $t \gtrsim T$ by means of the polynomial $|\operatorname{Im} F(t)| \leq C(t/4\mu^2)^{N_0}$ we can verify that the far singularities give rise to corrections $\lesssim C(t/4\mu^2)^{N-N_0}$ in (1), and that this additional contribution is a smooth function in the region $|t| \lesssim T$, which varies over distances $\sim T$. Smallness of these corrections can be ensured by increasing the number of subtractions. We think that for this purpose it is necessary to have at least $N - N_0 \approx 1$, and we shall consider from now on only the case $N \geq N_0 + 1 \geq 1$.

We now formulate the following problem: Calculate $F(t)$ in the vicinity of $|t| \lesssim T$ with accuracy $\sim |t/T|$, using the foregoing properties of the amplitude. This method of calculation will be called the two-meson approximation, taking this to mean the calculation of the effective nearest singularities without any additional information concerning the behavior of the amplitude in the region $|t| > T$. We note that this point of view differs somewhat from the generally accepted one, where the two-meson approximation is defined as the conservation of the dependence $\operatorname{Im} F = \lambda^* F$ over the entire cut $t \geq 4\mu^2$.

Since an account of the region $t > T$ is actually beyond the framework of the two-meson approximation, we shall approximate the absorptive part by means of an arbitrary function of the form

$$\Psi\lambda^* F \equiv \Lambda^* F, \qquad \Lambda = e^{i\Delta}\sin\Delta, \tag{3}$$

where $\Psi(t) = 1$ in the region $4\mu^2 \leq t \leq T$ and $\Psi(\infty) = 0$, while the phase $\Delta(t)$ has the following properties:

$$\Delta(t) = \delta(t) \text{ for } 4\mu^2 \leqslant t \leqslant T,$$
$$|\Delta(t)| \lesssim |\delta(T)| \quad \text{for} \quad t > T, \quad \Delta(\infty) = 0.$$

We call attention to the fact that with the aid of the functions Ψ and Δ we gradually "eliminate" the relation $\operatorname{Im} F = \lambda^* F$, which becomes meaningless when $t \gtrsim T$. Equations (1)—(3) are equivalent to the integral equation

$$F(t) = \widetilde{F}_\Delta(t) + \frac{t^N}{\pi}\int_{4\mu^2}^{\infty} \frac{\Lambda^*(t') F(t')}{(t'-t)\,t'^N}\,dt', \tag{4}$$

in which the inhomogeneous term

$$\widetilde{F}_\Delta(t) = \widetilde{F}(t) + \frac{t^N}{\pi}\int_{16\mu^2}^{\infty} \frac{f(t') + [\lambda^*(t') - \Lambda^*(t')]\,F(t')}{(t'-t)t'^N}\,dt' \tag{5}$$

is specified in the region $t \lesssim T$ with accuracy $\sim |t/T|^{N-N_0}$. Since the second term in (5) is unknown (within the limits of the indicated error), the function $\widetilde{F}(t)$ need also be specified only with accuracy $\sim |t/T|^{N-N_0}$. If $\widetilde{F}_\Delta(t)$ contains a large contribution from known functions, for example pole functions, as is the case for certain invariant πN amplitudes[6], the accuracy of the method is correspondingly increased.

Equation (4) has a zero index[7] and for a phase $\Delta(t)$ satisfying the Hölder condition, its only solution can be written in the form

$$F(t) = \widetilde{F}_\Delta(t) + \varphi(t)\frac{t^N}{\pi}\int_{4\mu^2}^{\infty} \frac{\widetilde{F}_\Delta(t')\,\Lambda(t')}{(t'-t)\,t'^N\,\varphi(t')}\,dt', \tag{6}$$

$$\varphi(t) = \exp\left[\frac{t-t_0}{\pi}\int_{4\mu^2}^{\infty} \frac{\Delta(t')\,dt'}{(t'-t-i0)(t'-t_0)}\right]. \tag{7}$$

Equations (5), (6), and (7) are some identical transformations of the initial dispersion equation (1), in which the effect of the nearest singularities is expressed in the form of an integral of smooth functions that result from the more remote singularities and are contained in the inhomogeneous term (5). It is obvious that the problem formulated can be solved only when the effective contribution to the integral (6) comes from the region $t_{\rm eff} \ll T$, where the integrand is known with good accuracy.

We thus have the following alternatives.

1. An appreciable contribution to (6) for all values of $\Delta(t)$ and for any number of subtractions comes from the region $t \gtrsim T$, and therefore the effect of the nearest singularities cannot be calculated without simultaneous account of the more remote singularities, which determine the behavior in the region $t \gtrsim T$.

2. The high-energy region makes no contribution to the integral (6), in analogy with the integral (1), i.e., an effective contribution is made to both integrals by the region $t_{\rm eff} \ll T$ when the number of subtractions is sufficient and for certain values of the phase $\Delta(t)$.

In the second case the integral equations which we obtain from the two-meson approximation should be stable against perturbations at infinity in the sense that a change of the order of magnitude of the kernel and of the inhomogeneous term in the region $t \gtrsim \tau > T$ should give rise to errors $\lesssim t/\tau$. For this reason it is sufficient to verify that the integrand in (6) decreases rapidly even in the range of small energies, for example, that it has the form

$$\frac{\widetilde{F}_\Delta(t)\,\Lambda(t)}{\varphi(t)\,t^N} \sim \frac{1}{t} \quad \text{for} \quad t \leqslant T. \tag{8}$$

If condition (8) is satisfied, which can be verified by direct substitution of the known function $\widetilde{F}(t) = \widetilde{F}_\Delta(t)$, or more accurately by the lowest term of its expansion in t/T, then the calculation of the integral (6) can be carried out by means of an arbitrary smooth continuation of the integrand in the region $t \gtrsim T$, and particularly for arbitrary $\Delta(t)$ having the properties indicated above. We can dispense here with the use of the previously stated condition $\Delta(\infty) = 0$, if (and only if) the integral (6) still converges at infinity with sufficient speed.

This possibility is due to the good convergence of the integral (7), which causes the behavior of $\varphi(t)$ at $t \ll T$ to be independent of the values of $\varphi(t)$ in the region $t \gtrsim T$, which determines the behavior of $\varphi(t)$ at $t \gtrsim T$. It is obvious that we can even forego the equality $\Delta(t) = \delta(t)$ when $t \le T$, and we need merely require that the phases $\Delta(t)$ and $\delta(t)$ coincide with accuracy $\sim t/T$ in the region $t \lesssim T$. The auxiliary phase $\Delta(t)$ can, in particular, be regarded as real over the entire cut $t \ge 4\mu^2$.

Taking the foregoing remarks into account, we can easily carry out further calculations by numerical integration, but in order to understand the analytical properties of the amplitudes and to use them in other problems it is desirable to consider for the phase some model that ensures the calculation of the quadratures (6) and (7). In this respect an interesting model is[3)]

$$\sqrt{x}\,\mathrm{ctg}\,\Delta(x) = X(x)/Q(x); \qquad x = t/4\mu^2 - 1, \tag{9}*$$

where $X(x)$ and $Q(x)$ are arbitrary polynomials; when the number of parameters is not very large this model permits, at least qualitatively, to approximate practically any possible behavior of $\delta(t)$ in the region of small energies $t \lesssim T$. The model (9) corresponds to the following functions contained in (6):

$$\varphi(t) = \frac{\prod\limits_{k=1}^{n}(x - x_k)}{X(x) + Q(x)\sqrt{-x}}, \tag{10}$$

$$\Lambda(t)/\varphi(t) = i\sqrt{-x}\,Q(x)/\prod_{k=1}^{n}(x - x_k), \tag{11}$$

where x_k $(k = 1, 2, \ldots, n)$ are the roots of the equation $X(x) + Q(x)\sqrt{-x} = 0$, $\mathrm{Re}\sqrt{-x} \ge 0$, and we choose for $x \ge 0$ the value $\sqrt{-x} = -i\sqrt{x}$. The number n can be readily expressed in terms of the powers of the polynomials X and Q, if one takes account of the fact that the asymptotic behavior of $\varphi(t)$ has the form

$$\varphi(t) \sim x^{-\Delta(\infty)/\pi}.$$

The calculation of the quadrature (6) after making the substitution $\widetilde{F}_\Delta \to \widetilde{F}$ must be carried out in each specific case depending on the form of $\widetilde{F}$, but it is appreciably facilitated by the substitution of (11)[6]. For the case of the polynomial $\widetilde{F}(t) = P(t)$ we obtain

$$\varphi(t)\frac{t^N}{\pi}\int_{4\mu^2}^{\infty}\frac{P(t')\Lambda(t')\,dt'}{(t'-t)\,t'^N\varphi(t')} = i\Lambda(t)\left\{P(t) + \frac{L_{N+n-1}(x)}{Q(x)\sqrt{-x}}\right\}, \tag{12}$$

where the polynomial of the $(N+n-1)$-st degree $L_{N+n-1}(x)$ is determined from the condition that the expression in the curly bracket vanish at the points $x = -1$, x_k $(k = 1, 2, \ldots, n)$, and that its first, second, ... $(N-1)$-st derivatives also vanish at $x = -1$. The boundary conditions at $x = -1$ $(t = 0)$ are equivalent to the presence of N subtractions in the initial equations.

3. ELECTROMAGNETIC FORM FACTOR OF THE PION

By way of an example of the use of the general procedure, and also for a detailed analysis of the earlier errors[3-5] let us consider the simplest case of electromagnetic form factor of the pion, $F(t)$ ($\pi + \pi \to \gamma$ amplitude), which depends on the P phase of the $\pi\pi$ scattering.

In a scheme with a single subtraction $(N = 1)$, the lowest term of the expansion of $\widetilde{F}_\Delta(t)$ in powers of t/T has the form $\widetilde{F}(t) = 1$, and (12) goes over into the equation

$$F_\pi(t) = [X(x) - L_n(x)]/[X(x) + Q(x)\sqrt{-x}]. \tag{13}$$

Convergence of the integrals (6) and (12) in the region of small energies will be assured if

$$Q(x)/\Pi_{k=1}^{n}(x - x_k) \sim \mathrm{const} \quad \text{for } t \gtrsim 16\mu^2, \tag{14}$$

which guarantees a calculation accuracy $\sim \sqrt{t/T}$.

The substitution $X(x) = \alpha^{-3} > 0$, $Q(x) = x$, which corresponds to the model of positive scattering length, leads to the previously obtained[3] expression

$$F_\pi(t) = \frac{\varphi(t)}{\varphi(0)} = \frac{(1+\alpha+\alpha^2)(1+\alpha^2 x)}{(1+\alpha)(1+\alpha^3 x\sqrt{-x})}. \tag{15}$$

In the case of a Breit-Wigner (dynamic) resonance (curve a on Fig. 2) we have $X(x) = x_r - x$, $Q(x) = \gamma \ll 1$, $n = 0$

$$F_\pi(t) = (x_r + \gamma - x)\,\varphi(t) = \frac{x_r + \gamma - x}{x_r - x + \gamma\sqrt{-x}}. \tag{16}$$

We point out that we are using the functional

[3)] A particular case of this model, $Q(x) = x^l$, was considered earlier[6], but the results obtained there are valid if the substitution $x^l \to Q(x)$ is made. In the final formulas (23), (24), (27), and (28) of the cited paper, the function x^{-l} (with $l = 0, 2, 1, 1$, respectively) is contained exclusively only in the form of a factor for the polynomials

$$L_{n+1}^{(S)}(x),\ L_{n+1}^{(D)}(x),\ L_{n+1}^{(P)}(x),\ M_{n+1}^{(P)}(x).$$

*ctg = cot.

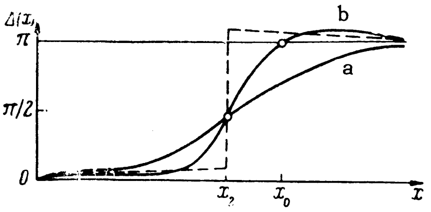

FIG. 2

form for the total width $\Gamma \sim \sqrt{x}$. The substitution $Q(x) = \gamma x$ ($\Gamma \sim x^{3/2}$) would lead to a violation of the condition (14), for in the wide interval $0 \leq x \lesssim \gamma^{-2}$ we would have $Q(x)\,(x - x_1) \sim x\,(x_1 \approx -\gamma^{-2})$. Condition (14) is also violated in the model (15) at small scattering lengths, and therefore expression (15) can be used only if $-x_1 = \alpha^{-2} \lesssim T/4\mu^2 \sim 1/\epsilon$ ($\alpha^2 \gtrsim 0.15$).

The behavior of $\Gamma \sim x^{3/2}$ can be considered in the model $X(x) = (x_r - x)(x + b)$, $Q(x) = \gamma' x$; $1 \lesssim b < 1/\epsilon$, for in this case condition (14) is satisfied. This model leads to the previous functional dependence of the form factor

$$F_\pi(t) = (\xi - x)\,\varphi(t) \tag{16'}$$

and to an inessential difference from formula (16). The position of the zero $\xi \approx x_r + \gamma$ can be found from the normalization condition $F_\pi(0) = 1$ after substituting (10) in (16′).

For the model of the kinematic[4)] resonance (curve b on Fig. 2) $X(x) = (x_r - x)(x + x_0)$, $Q(x) = a(x_0 - x)$, $x_0 > x_r$ (ρ-meson effect) we have

$$F_\pi(t) = (\xi - x)\,\varphi(t) = \frac{(\xi - x)(x - x_1)}{(x_r - x)(x + x_0) + a\sqrt{-x}\,(x_0 - x)},$$
$$\xi = -(x_r x_0 + x_0 - x_r + x_1 + a(1 + x_0))/(1 + x_1). \tag{17}$$

When $x_0 = \infty$ this model is equivalent to (16), and when $x_0 \to x_r$ it describes a superposition of an infinitely narrow resonance and a potential scattering in the approximation of the effective interaction radius (dashed curve in Fig. 2).

For the resonance models, the form factors differ from the previously known expression[3-5]

$$F_\pi(t) = \varphi(t)/\varphi(0) \tag{18}$$

by additional multiplication by a certain polynomial of the first degree, which has a zero near resonance. For the dynamic resonance model the zero is located at a distance $\Delta x = \gamma$ beyond the resonance, as a result of which $|F_\pi(t)| \lesssim 1$ near the resonance. For the kinematic resonance model, the zero is located ahead of the resonance at a distance $\Delta x \gg \gamma = a(x_0 - x_r)/(x_0 + x_r)$, where the quantity γ is analogous to the width in the model (16), as a result of which $|F_\pi(t)| \gg 1$ near resonance. This difference makes it possible to obtain for the model (17) a large effect in the electromagnetic form factors of the nucleon, proportional to $|F_\pi(t)| \gg 1$, whereas the model of the dynamic resonance leads to small absorptive parts, which do not agree with experiments.[8]

Let us see now why the method used by us leads to a "loss" of the other previously obtained solutions[3,4]. If we start from the equation

$$F_\pi(t) = 1 + \frac{t}{\pi}\int_{4\mu^2}^{\infty} \frac{\lambda^*(t')\,F(t')}{(t' - t)\,t'}\,dt', \tag{19}$$

which is obtained by substituting the two-meson absorptive part on the entire cut $t \geq 4\mu^2$, then for the case of resonance $\delta(\infty) = \pi$ we obtain, in addition to solutions such as (16) and (17) the supplementary single-parameter family

$$Ct\varphi(t). \tag{20}$$

The supplementary contribution (20) is a general solution of the corresponding homogeneous equation and can be obtained also by the method considered in Sec. 2 as a result of the integration in (6) at infinity. In fact, we introduce an auxiliary phase $\Delta(t)$, which differs from $\delta(t)$ only when $t > \tau \gg T$, and which decreases to zero on the segment $\tau < t < \infty$. In this case the region of integration in (6) breaks up into two: The region near resonance $t \sim t_r$, which gives the solution which we have obtained, and the region $t \geq \tau$, which yields at $\tau \to \infty$ a contribution

$$\varphi(t)\,\frac{t}{\pi}\int_{\tau}^{\infty} \frac{\widetilde{F}_\Delta(t')\,\Lambda(t')\,dt'}{(t' - t)\,t'\varphi(t')} \xrightarrow[\tau\to\infty]{} \varphi(t)\,t\cdot\text{const.} \tag{21}$$

Thus, the additional family (20) can arise only if the fundamental requirement of the formulated problem $t_{eff} \ll T$ is violated, and is for this reason incompatible with the two-meson approximation. This means that (19) has only one solution which is independent of the conditions on infinity and which is meaningful in the problem dealing with the calculation of the effect of the nearest singularities. This solution is a particular solution of the inhomogeneous equation. The solution of the corresponding homogeneous equation, to the contrary, is completely determined by conditions at infinity that are unknown to us. This manifests itself in the instability of the solution (20) against perturbations of the equation at infinity in the sense that changes on the order of magnitude of the kernel and of the inhomogeneous term will lead, in a region $t \geq \tau$ ($\tau \to \infty$) as remote as de-

4)We use the concept of dynamic and kinematic resonances as an indication of the absence or presence of a zero of the $\pi\pi$ amplitude near resonance.

sired, to slight changes in the equation in the region of small energies. It is precisely because of this "correlation" between the regions of small and large energies that a solution such as (20) is meaningless in our problem.

Let us call attention to the fact that the result (21) indicates a non-uniform convergence of the integral (6) with respect to the values of $\Delta(\infty)$ in the vicinity $\Delta(\infty) = \pi$ and therefore a change in the value of $\Delta(\infty)$ after calculation of the integral is not equivalent to the change of $\Delta(\infty)$ under the integral sign, used in Sec. 2. Thus, for example, assuming $\Delta(\infty) = 0$ and calculating formally the integral (6) for $\widetilde{F}_\Delta(t) = 1$ without investigating the effective region of integration, we would have arrived at the result (18) which is incorrect in the general case. The expression (18) obtained in this manner is no longer dependent, in the low energy region, on the value of $\Delta(\infty)$, and therefore it might appear that it can be used for resonance models. However, it is easy to verify that this is equivalent to adding to (16) and (17) solutions of the type (20) as a result of an implicit account of the infinitely remote region of integration on changing the orders of integration in (6) and on going over to the limit $\Delta(\infty) \to \delta(\infty) = \pi$.

An analogous error was made by Frazer and Fulco[4], who started from the exact relation

$$F_\pi(t) = \frac{\varphi(t)}{\varphi(0)} + \varphi(t)\frac{t}{\pi}\int_{16\mu^2}^{\infty}\frac{\operatorname{Im}[F_\pi(t')/\varphi(t')]}{(t'-t)\,t'}\,dt', \tag{22}$$

which can be converted by changing the contour of integration into the relation (6) which we have used. They have discarded the second term of (22), which contains in fact a large contribution of the type (20), and which should cancel out an analogous contribution in the first term. The need for such a cancellation is evident from an examination of the limiting case of an infinitesimally narrow resonance

$$\delta(t) = \begin{cases} 0 & \text{for } t < t_r, \\ \pi/2 & \text{for } t = t_r, \\ \pi & \text{for } t > t_r, \end{cases} \tag{23}$$

which is obtained in models (16) and (17) when $\gamma \to 0$ and $x_0 \to x_r$, $a \to 0$. The $\pi\pi$ interaction corresponding to this case is experimentally unobservable, and therefore it should not make any contribution to $F_\pi(t)$, with the possible exception of the single point $t = t_r$. This is satisfied for solutions (16) and (17), which lead for the limiting case (23) to a value $F_\pi(t) = 1$ when $t \neq t_r$, whereas expression (18) gives the nonvanishing effect

$$F_\pi(t) = t_r/(t_r - t). \tag{24}$$

We were unable to obtain a general proof for the need for the vanishing of the effect for the case (23), since for this purpose it is apparently necessary to go beyond the framework of the two-meson approximation, since this requirement is based essentially on intuitive considerations. We note that a vanishing effect is produced by the iteration method of solving (19), which is equivalent to a successive consideration of diagrams made of chains of $\pi\pi$ amplitudes. The same result is arrived at by the general formulas derived by Gell-Mann and Zachariasen[9] on the basis of field theory of the intermediate vector ρ meson, if it is considered that the case (23) is equivalent to turning off all the interactions of the ρ meson with other particles and the vanishing of the renormalization of the ρ-meson mass $m_{\rho 0}^2 - m_\rho^2 \to 0$.

A generalization of (22) to the case of arbitrary amplitudes $F(t)$ can be readily obtained by writing out the dispersion relation with N subtractions for $F(t)/\varphi(t)$.[5)] In the general case this equation can likewise not be used to obtain an approximate expression, as is done by the Frazer-Fulco method, where the integral over the cut $t \geq 16\mu^2$ is discarded[4]. For resonant $\pi\pi$ interaction, a transformation of relation (6) to an equality of the type (22) is equivalent to the above-indicated change in the orders of integration and transition to the limit $\Delta(\infty) \to \delta(\infty) = \pi$. This leads to an artificial and meaningless separation of two mutually cancelling contributions of the type (20), one of which is discarded together with the integral over $t \geq 16\mu^2$, and the other is retained in the Frazer-Fulco method. We emphasize that an account of the non-valid contributions from the region of large energies is carried out in this case for any number of subtractions, and consequently in the calculations of the $\pi + \pi \to N + \bar{N}$ amplitude carried out by Ball and Wong[10] with $N = 1$, contains the same error as in the calculations of Frazer and Fulco at $N = 0$.[4]

Inasmuch as the functional form of the solution of the type (20) is incompatible with the two-meson approximation, it cannot be used for a phenomenological construction of two-meson contributions, as was done in [5]. If it turns out that the solution (13) which we have obtained does not agree with experiment for the "true" phase $\Delta(t)$, for example if it does not give a correct first derivative

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5)This method is widely used at the present time to "eliminate" the cut $4\mu^2 \leq t \leq 16\mu^2$.

$$[dF_\pi(t)/dt]_{t=0} = \frac{1}{6}\langle r_\pi^2\rangle,$$

where $\langle r_\pi^2\rangle$ is the mean square of the pion radius, then this should indicate that an appreciable contribution is made to the first derivative by the farther singularities. In this case we cannot "correct" the solution (13) by adding a term (20), but it is necessary to use the computation scheme given in Sec. 2 with two subtractions ($N = 2$), substituting in (6) and (12) the inhomogeneous term

$$\widetilde{F}(t) = 1 + \frac{1}{6}t\langle r_\pi^2\rangle. \tag{25}$$

Introduction of additional subtraction does not lead to qualitative changes in the properties of $F_\pi(t)$ indicated above and in the results obtained with the aid of formula (13) for the electromagnetic form factors of the nucleon[8]. The kinematic resonance model, for example, leads to the expression

$$F_\pi(t) = [(\xi - x)\widetilde{F}(t) + A(1+x)]\varphi(t), \tag{26}$$

where $\varphi(t)$ and ξ have the same form as before, $\widetilde{F}(t)$ is determined by Eq. (25), and the coefficient is

$$A = a[(3+x_0)(1+x_1)/2$$
$$+ \sqrt{-x_1}(x_0 - x_1) - 1 - x_0]/(1+x_1)^2.$$

The second term in (26), proportional to A, gives an insignificant contribution in the case of a narrow resonance, and therefore the new expression for $F_\pi(t)$ is obtained essentially by simple multiplication of formula (17) by $F(t)$ and has a zero at the point $\xi + \Delta\xi \approx \xi < x_r$ ($|\Delta\xi| \ll \xi$).

The model of dynamic resonance is obtained by making the limiting transition $x_0 \to \infty$, at which $x_1 \to -x_0 \to -\infty$, $A \to -a/2$, and $\xi \to x_r + a$. In this case the parameter a has the same meaning as the width γ. The position of the zero shifts by an amount $-\Delta\xi \sim \gamma$ compared with formula (16) and can go over to the other side of x_r, unlike the case of kinematic resonance.

In calculations with two subtractions, an important property of resonant models is retained, namely the presence of a zero in $F_\pi(t)$ at a distance $|\Delta x| \sim \gamma$ from x_r for dynamic resonance and a distance $1 \gtrsim |\Delta x| \gg \gamma$ ahead of the resonance for the second case, and the resultant important difference between the two models[8]. We note that the position of the zero of the form factor $F_\pi(t)$ obtained in the two-meson approximation for the case of resonance does not coincide with the point x_0 (the position of zero for the $\pi\pi$ amplitude), as would follow from the field theory of the intermediate vector ρ meson developed by Gell-Mann and Zachariasen[9].

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