STABLE BETATRON OSCILLATIONS IN A NONLINEAR MAGNETIC FIELD

Yu. F. ORLOV

Physics Institute, Academy of Sciences, Armenian S.S.R.

Submitted to JETP editor March 15, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **43**, 1308-1314 (October, 1962)

We consider the stability of betatron oscillations in an annular axially-symmetrical magnetic field with maximum (or minimum) along the radius r. In a linear approximation with respect to vertical z-oscillations, it is reasonable to use for the r-oscillations a potential well, inside which bands of stability for z exist. The nonlinearity of small z-oscillations does not violate the stability. Acceleration without violation of stability is possible with strong focusing at the onset of the acceleration.

1. GENERAL CONSIDERATIONS

STRONG focusing of small r- and z-deviations from an equilibrium orbit, i.e., focusing with a wavelength $\lambda \ll R$, is impossible in an axially-symmetrical magnetic field. The only possibility of simultaneously stabilizing the r- and z-oscillations is used in weak focusing and is connected with the dependence of the centrifugal force pv/R on $r = R - R_0$. Without account of the centrifugal force, the summary potential energy of the transverse oscillations U(r, z) coincides with the vector potential $A_\varphi(r, z)$ and therefore has no minima, being an analytic function of r + iz.

This theorem does not exclude stability of large nonlinear oscillations. Unfortunately, it has been impossible to consider so far oscillations that are nonlinear in both r and z, but the most interesting case, when the z-oscillations remain linear, is not only feasible in a magnetic field but is even practical. If, as usual, the magnetic field H_z is symmetric about the plane z = 0:

$$H_z(r,z) = H_z(r,-z), \quad H_r(r,z) = -H_r(r,-z),$$

then the equation for r contains only even powers of z, while the equation for z contains only odd powers. Assuming z^2/r^2 to be small, we obtain in the zeroth approximation in z^2/r^2 an equation for r which does not contain z, and a linear equation for z in which r(t) enters as a parameter.

If we neglect the terms of type r/R_0, z/R_0, $\dot{r}^2/v^2$, $\dot{z}^2/v^2$, which are inessential in this analysis, then the equations of motion assume the form

$$\frac{d^2r}{d\varphi^2} + \frac{H_z - H_0}{H_0} R_0 = 0, \qquad \frac{d^2z}{d\varphi^2} - \frac{R_0}{H_0}\frac{\partial H_z}{\partial r} z = 0, \tag{1}$$

and in the zeroth approximation $H_z(r, z) = H_z(r)$.

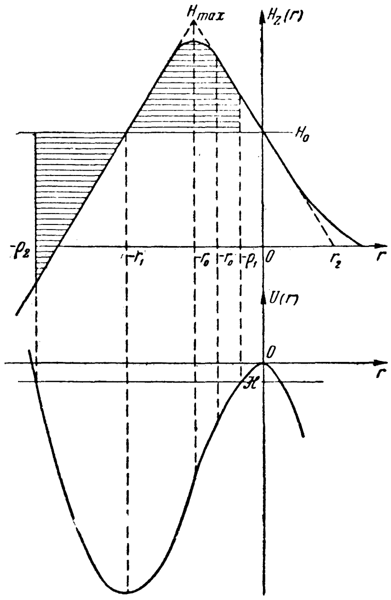

FIG. 1

(Here H_0 is the equilibrium field and $cp = eH_0R_0 \approx eH_0R$).

Let $H_z(r)$ have the form shown in Fig. 1. The potential energy

$$U(r) = \frac{R_0}{H_0}\int dr\,(H_z - H_0) \tag{2}$$

ensures stable motion along r inside the well. Let us show that inside the potential well there are energy levels $\mathcal{H} = U + (dr/d\varphi)^2/2$ (and even an infinite number of such level bands), which correspond to stable z-oscillations. This dynamic stability is guaranteed by the fact that in the case of r-oscillations the particle traverses alternately through gradients $\partial H_z/\partial r$ of opposite sign with a period Φ, which we represent for $\mathcal{H} \to 0$ in the form of the sum (see Fig. 2)

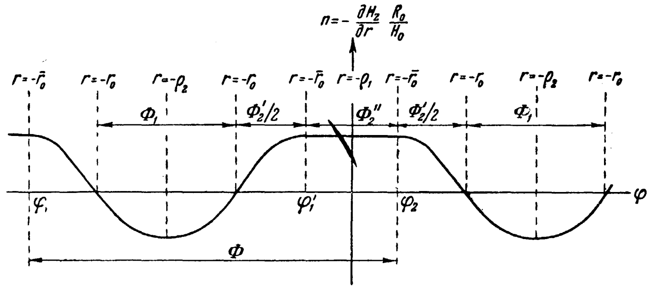

FIG. 2

$$\Phi = 2\int_{\rho_1}^{\rho_2} \frac{dr}{\sqrt{2(\mathscr{H} - U)}} = \Phi_1 + \Phi_2' + \Phi_2''$$

$$\approx 2\int_{\rho_1}^{\bar{r}_0} \frac{dr}{\sqrt{n_0(r^2 - \rho_1^2)}} \rightarrow \frac{2}{\sqrt{n_0}} \ln \frac{\bar{r}_0}{\rho_1}. \quad (3)$$

The fixed point $\bar{r}_0$ is chosen such that in the interval $0 > r > -\bar{r}_0$ we can assume the field to be a linear function of the radius: $H_z \approx H_0 - g_0 r$; this is always possible if $\mathscr{H} \approx -n_0\rho_1^2/2$ is sufficiently small, where $n_0 = g_0R_0/H_0$.

As $\rho_1 \rightarrow 0$, the function Φ_2'' tends logarithmically to infinity, whereas the lengths of the remaining sections and the course of the gradient inside them cease to depend on ρ_1. The stability of the z-oscillations is determined by the value of the trace of the transition matrix over the length of the period of the gradient. This matrix breaks up into a product of two matrices, of which one (from the point φ_1 to the point φ_1') is independent of $\mathscr{H}$, while the other (from φ_1' to φ_2) depends logarithmically on $\mathscr{H}$:

$$\begin{pmatrix} z \\ \frac{dz}{d\varphi} \end{pmatrix}_{\varphi=\varphi_2} = \begin{pmatrix} \cos\left(2\ln\frac{\bar{r}_0}{\rho_1}\right) & \frac{1}{\sqrt{n_0}}\sin\left(2\ln\frac{\bar{r}_0}{\rho_1}\right) \\ -\sqrt{n_0}\sin\left(2\ln\frac{\bar{r}_0}{\rho_1}\right) & \cos\left(2\ln\frac{\bar{r}_0}{\rho_1}\right) \end{pmatrix}$$

$$\times \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} z \\ \frac{dz}{d\varphi} \end{pmatrix}_{\varphi=\varphi_1} \quad (4)$$

If we denote by μ_z the number of z-oscillations over the length of the period Φ, then it follows from (4) that

$$\cos 2\pi\mu_z = \frac{a_{11} + a_{22}}{2}\cos\left(2\ln\frac{\bar{r}_0}{\rho_1}\right)$$

$$+ \frac{1}{2}\left(\frac{a_{21}}{\sqrt{n_0}} - a_{12}\sqrt{n_0}\right)\sin\left(2\ln\frac{\bar{r}_0}{\rho_1}\right). \quad (5)$$

the z-oscillations are stable if $|\cos 2\pi\mu_z| < 1$. According to (5), as $\rho_1 \rightarrow 0$ the trace of the transition matrix goes through zero an infinite number of times, so that the potential well contains an infinite number of stability bands.

2. ACCELERATION OF PARTICLES IN A FIELD WITH NONLINEAR FOCUSING

Let us approximate the field $H_z(r)$ in Fig. 1 by an angle with vertex at the point $-r_0$, choosing for simplicity equal gradients on both slopes of the field. In this simple field model, as can be readily calculated, we have

$$\cos 2\pi\mu_z = \cos\left(2\,\mathrm{Ar\,ch}\frac{r_0}{\rho_1}\right)\mathrm{ch}\left[\pi + 2\arcsin\left(2 - \frac{\rho_1^2}{r_0^2}\right)^{-1/2}\right], \quad (6)^*$$

$$\Phi = \frac{2}{\sqrt{n}}\left[\frac{\pi}{2} + \arcsin\left(2 - \frac{\rho_1^2}{r_0^2}\right)^{-1/2} + \mathrm{Ar\,ch}\frac{r_0}{\sqrt{\rho_1}}\right]. \quad (7)$$

The adiabatic invariant of the radial oscillations is

$$J = 2H_0\int_{\rho_1}^{\rho_2} dr\sqrt{2(\mathscr{H} - U)}$$

$$= H_0\sqrt{n}\,r_0^2\Big[2\sqrt{1 - x^2} - x^2\,\mathrm{Ar\,ch}\frac{1}{x}$$

$$+ \left(1 - \frac{x^2}{2}\right)(\pi + 2\arcsin(2 - x^2)^{-1/2}\Big], \quad (8)$$

where $x = \rho_1/r_0$. At the centers of the stability bands, $2\cosh^{-1}(r_0/\rho_1) = (k + 1/2)\pi$, $1/x = \cosh(2k+1)\pi/4$, and the half-width of the k-th band, on the edges of which $\cos 2\pi\mu_z = \pm 1$, is

$$\Delta\rho_1/\rho_1 = \Delta\mathscr{H}/2\mathscr{H}$$

$$\approx \sqrt{1 - x^2}/2\ \mathrm{ch}\,[\pi + 2\arcsin(2 - x^2)^{-1/2}]. \quad (9)$$

For the main band ($k = 0$) we have $x = 0.755$, $\Delta\rho_1/\rho_1 = 4 \times 10^{-3}$, and $\Phi = 6.67/\sqrt{n}$; for $k \gg 1$ we have $\Delta\rho_1/\rho_1 = 9 \times 10^{-3}$ and $\Phi = (k + 1/2)\pi\sqrt{n}$.

From the relation $\Delta J = \Phi\Delta\mathscr{H}$ and from (9) we obtain the phase volume of the radial oscillations, corresponding to the stability band

$$\Delta r\,\Delta\vartheta_r = \frac{\Delta r}{R}\,\Delta\frac{dr}{d\varphi} = 2\frac{\Delta\rho_1}{\rho_1}\mathscr{H}\frac{\Phi}{R} = \frac{\Delta\rho_1}{\rho_1}\frac{\Phi}{R}\,n r_0^2 x^2. \quad (10)$$

The phase volume of the vertical oscillations is determined by the permissible amplitude z_{per}. Using the well known theory for linear oscillations with periodic coefficients, we can obtain the phase volumes

$$\Delta z\,\Delta\vartheta_z = \frac{2\pi^2 z_{per}^2}{R\Phi\,|\chi|_{max}^2},$$

$$\Delta S\,\Delta\Omega \equiv \Delta r\,\Delta z\,\Delta\vartheta_r\,\Delta\vartheta_z = \frac{\Delta\rho_1}{\rho_1}\frac{2\pi^2 n r_0^2 z_{per}^2 x^2}{R^2\,|\chi|_{max}^2}\ \mathrm{cm}^2\text{-sr} \quad (11)$$

---

*Ar ch = $\cosh^{-1}$; ch = cosh.

where χ is the Floquet function of the vertical oscillations with Wronskian $\Phi(\chi^*\chi'_\varphi - \chi\chi^{*\prime}_\varphi)/2\pi = 2i$. In our example $|\chi|^2_{max} = 2\pi(\sinh\alpha + \cosh\alpha)/\Phi\sqrt{n}$, where $\alpha = \pi + 2\sin^{-1}(2 - x^2)^{-1/2}$. From (9) and (11) we get

$$(\Delta S\Delta\Omega)_k = \beta_k n r_0^2 z_{per}^2/R^2, \qquad \beta_0 = 3.1\cdot 10^{-4},$$
$$\beta_1 = 0.8\cdot 10^{-4}.$$

Before we estimate the phase volume with these formulas, let us consider the acceleration process. To maintain stability during the acceleration process it is sufficient to keep the ratio $\rho_1/r_0 = x$ constant. In this case it follows from (8) that $J \sim H_0\sqrt{n}\, r_0^2$ and from the constancy of the adiabatic invariant we obtain the necessary connection between the time variation of the field $H_{max}(t)$ and the increase in the momentum $p \sim H_0$. Using the relation

$$n = \frac{R_0}{H_0}\frac{H_{max}}{r_0 + r_2}, \qquad r_0 + r_2 = \text{const},$$
$$r_0 = (r_0 + r_2)\frac{H_{max} - H_0}{H_{max}}$$

(see Fig. 1) and introducing $y = H_0/H_{max}$ we write this connection in the form

$$H_0\left(\frac{H_{max}}{H_0}\right)^{1/2}\left(\frac{H_{max} - H_0}{H_{max}}\right)^2 = \text{const},$$
$$\frac{H_{max}(t)}{H_{max}(0)} = \left(\frac{y_0}{y}\right)^{1/2}\left(\frac{1-y_0}{1-y}\right)^2. \qquad (12)$$

It is seen from this formula that the magnetic field should decrease at the start of the acceleration ($y \ll 1$), and reach a minimum at $y = 1/5$, $H_{max}/H_{max}(0) \approx 25\sqrt{5y_0}/16$. The field then increases again and H_{max} returns to the initial value when $y = 1 - y_0^{1/4}$. The quantity $1/y = H_{max}/H_0$ is equal to the ratio n/n_{geom}, where $n_{geom} = R/(r_0 + r_2)$ is the constant "geometrical" index of the falloff of the magnetic field. At the end of the acceleration $n/n_{geom} \sim 1$. However, at the start of the acceleration we have $n \gg n_{geom}$, and this greatly increases the capture efficiency.

The vertical oscillations attenuate with acceleration as $1/H_0\sqrt{n}$. The factor $1/\sqrt{n}$ is brought about by the fact that the spatial frequency of the z-oscillations is of the order $\sqrt{n}$, since $\mu_z = \text{const}$ and $\Phi \sim 1/\sqrt{n}$. At the very onset of acceleration, according to (12), $H_{max} \sim 1/H_0$ and $n \sim 1/H_0^2$, so that the z oscillations do not attenuate. At the end of the acceleration $n \approx \text{const}$ and $z^2 \sim 1/p$.

Taking these remarks into consideration, let us estimate the phase volume encompassed by the acceleration mode. Since the acceleration conserves the ratio ρ_1/r_0, the radial-oscillation volume occupied upon injection does not decrease during the acceleration. The initially occupied phase volume of the z-oscillation also remains constant. Recognizing that

$$n_{inj} = n_{geom} H_{max}/H_0(0) = n_{geom}/y_0, \quad (r_0)_{inj} = R/n_{geom},$$

we obtain

$$(\Delta S\,\Delta\Omega)_k = \beta_k z_{per}^2/y_0 n_{geom}. \qquad (13)$$

A field of the type shown in Fig. 1 can be obtained in a quadrupole lens with asymmetrical right-hand and left-hand poles. The injection energy is limited only by the scattering on the gas, so that the quantity $y_0 = (H_0/H_{max})_{inj}$ is very small. Taking by way of an example a large ring with $R = 2.5\times 10^5$ cm, $r_{0\,inj} = 5.0$ cm, $z_{per} = 1$ cm, $n_{geom} = 5\times 10^{14}$, $E_{inj} = 10$ MeV, and $y_0 = 8\times 10^{-5}$ we obtain $(\Delta S\Delta\Omega)_0 = 10^{-14}$ cm^2 sr, whereas in an accelerator with hard focusing [1], having comparable dimensions, the phase volume is $\sim 10^{-7}$ cm^2 sr [1].

The period of the radial oscillations, $\Phi \sim 2\pi/\sqrt{n}$ and with it also the number of vertical oscillations per revolution, which is equal to $\sim\sqrt{n}/4$, changes very strongly during the acceleration, particularly during the start of the acceleration, up to $y = 1/5$. Therefore the particle passes through a large number of vertical-oscillations resonances. This makes the tolerances one or two orders of magnitude more stringent compared with hard focusing.

We note the interesting fact that in nonlinear focusing small deviations from large nonlinear r-oscillations are unstable. Indeed, if we represent r in the form $r_1(\varphi) + \sigma$, where $r_1(\varphi)$ is an arbitrary trajectory within the potential well, then we obtain for $0 < \rho_1 < r_0$

$$\cos 2\pi\mu_\sigma = \operatorname{ch}\left(2\operatorname{Ar}\operatorname{ch}\frac{r_0}{\rho_1}\right)$$
$$\times\cos\left(\pi + 2\arcsin\left(2 - \frac{\rho_1^2}{r_0^2}\right)^{-1/2}\right). \qquad (14)$$

It is easy to see that the argument of the cosine on the right lies in the range between $3\pi/2$ and 2π, $|\cos 2\pi\mu_\sigma| > 1$, and σ is unstable. This result is obvious, for in the region of nonlinear oscillations the period Φ depends on $\mathcal{H}$, i.e., on σ, and the two solutions, with energies $\mathcal{H}$ and $\mathcal{H} + \Delta\mathcal{H}$, oscillate relative to each other with an amplitude of the order of the well width, independently of the smallness of the difference $\Delta\mathcal{H}$. Since σ has complex frequencies, the sinusoidal perturbations cannot produce resonances in the r-oscillations.

3. EFFECT OF NONLINEARITY OF z-OSCILLATIONS

The effect of terms nonlinear in z on the stability of motion is the principal problem, inasmuch as the stability itself is the product of the nonlinearity. It is necessary to show that the nonlinearity in z does not eliminate the instability (with the exception, of course, of special cases of nonlinear resonances, which arise at certain discrete values of the frequency μ_z).

We demonstrate this with an example of "minimal" nonlinearity, such as a magnetic field quadratic in r and z

$$H_z = ar - b(r^2 - z^2), \quad H_r = az - 2brz \tag{15}$$

with equation of motion

$$HRr'' + ar - br^2 = -bz^2, \tag{16}$$

$$HRz'' - az + 2brz = 0. \tag{17}$$

For simplicity we consider linear motion along the s axis; $H_0 = 0$ and H(r, z) is independent of s.

In the zeroth approximation we discard the z^2 term on the right. In place of s, r, and z it is convenient to introduce the variables

$$q = r/\sqrt{\mathcal{H}}, \quad \zeta = z/\sqrt{\mathcal{H}}, \quad v^2 = s^2 a\sqrt{\mathcal{H}}/HR, \tag{18}$$

where $\mathcal{H}$ is the energy of the radial oscillation; the potential energy is chosen such that it has a minimum at r = 0.

The equations assume the form

$$q'' + q - \varkappa^2 q^2 = -\varkappa^2\zeta^2, \tag{19}$$

$$\zeta'' - \zeta + 2\varkappa^2 q\zeta = 0, \quad \varkappa^2 = b\sqrt{\mathcal{H}}/a, \tag{20}$$

With an integral of motion for q in the zeroth approximation $q = q_0(v)$:

$$(q_0')^2/2 + q_0^2/2 - \varkappa^2 q_0^3/3 = 1. \tag{21}$$

Inside the well we have $0 \le \kappa^2 \le 1/\sqrt{6}$. It is convenient to investigate the solution near the hump, i.e., at $\kappa^2 \sim 1/\sqrt{6}$. After straightforward but cumbersome derivations we obtain in the approximation linear in $\sqrt{\epsilon} = (8/3\sqrt{6})(\kappa^{-2} - \sqrt{6})$ and in the approximation $z^2 = 0$:

$$q_0 = -\sqrt{6}\left(1 + \frac{\sqrt{\varepsilon}}{2}\right) + \frac{\sqrt{6\varepsilon}}{1 - (1 - 2\sqrt{\varepsilon}/3 + 2\varepsilon/9)\sin^2\phi}, \tag{22}$$

where the variable ϕ is connected with v by the relation

$$v - v_0 = 2(1 - \sqrt{\varepsilon}/6)\, F(\alpha, \phi), \quad \alpha = \pi/2 - \beta,$$
$$\beta = \sqrt{2\varepsilon^{1/2}/3}; \tag{23}$$

$F(\alpha, \phi)$ is an elliptic integral of the first kind [2].

Substituting (22) in (20) we get

$$\zeta_0'' + \left(1 + \sqrt{\varepsilon} - \frac{2\sqrt{\varepsilon}}{1 - (1 - 2\sqrt{\varepsilon}/3 + 2\varepsilon/9)\sin^2\phi}\right)\zeta_0 = 0. \tag{24}$$

For sufficiently small values of $\sqrt{\epsilon} \ll 1$ we can confine ourselves to the simpler equation

$$\zeta_0'' + \left(1 - \frac{3}{1 + \delta^2/\beta^2}\right)\zeta_0 = 0, \quad \delta = \phi - \pi/2. \tag{25}$$

The period of variation of $\delta(v)$ is $4K = 4F(\alpha, \pi/2) \sim 4\ln(4/\beta)$, i.e., it increases logarithmically as expected with decreasing $\sqrt{\epsilon}$, owing to the increase in the time spent on the section that focuses in ζ (i.e., when $\delta^2 \gg \beta^2$). The turning point in the q-oscillations near the hump corresponds to $\phi = 0$, $\delta \gg \beta$, while reflection from the steep wall corresponds to the point $\phi = \pi/2$, $\delta = 0$.

By choosing β such that ζ_0 is stable, we can now substitute ζ_0 in the right half of (19), find the addition $\sigma = q - q_0$ resulting from the term ζ_0^2 in (19), and substitute the new $q = q_0 + \sigma$ in (20). The only effect that could lead here to instability in ζ is that σ obviously has a forced oscillation frequency $2\mu_\zeta$, and substitution in (20) yields parametric resonance that is independent of the value of μ_ζ. It must be noted, however, that such a situation with quadratic nonlinearity occurs also in ordinary focusing, and were this effect actually to exist, no accelerator could operate in practice. In fact, in the same approximation, the time average $\bar{\sigma}$ yields in Eq. (20) a frequency shift $\Delta\mu_\zeta$ which is always much larger than the half-width of the parametric resonance.

Such a result is obvious beforehand for an accelerator with linear focusing, but it should be verified here if for no other reason than that the equation for σ without the right half yields an unstable solution. Introducing $\tau = \zeta - \zeta_0$ we have

$$\sigma'' - \left(1 - \frac{3}{1 + \delta^2/\beta^2}\right)\sigma = -\frac{1}{\sqrt{6}}\zeta_0^2, \tag{26}$$

$$\tau'' + \left(1 - \frac{3}{1 + \delta^2/\beta^2}\right)\tau = -\frac{2}{\sqrt{6}}\sigma\zeta_0. \tag{27}$$

We denote by χ as before, the Floquet function for the ζ_0-oscillations with Wronskian $\chi\chi^{*\prime} - \chi^*\chi' = -2iw$. Assume that we have found a solution σ which vanishes when $\zeta_0^2 \to 0$. Then the width of the parametric resonance g for τ and the correction to the frequency $\Delta\mu$ are given by the formula

$$g = \left|\frac{2}{w}\frac{1}{4Kn}\int_0^{4Kn} \sigma\chi^2 dv\right|, \tag{28}$$

$$\Delta\mu = \frac{2}{w}\frac{1}{4Kn}\int_0^{4Kn} \sigma|\chi|^2 dv, \tag{29}$$

where n is a large integer.

Let us let β approach zero (jumping over from one stability band to another), making use of the previously proved existence of an infinite number of stability bands on approaching the hump of the potential energy curve. Then the contribution of the regions $\delta < \beta$ in (28) and (29) tends to zero like $(\ln \beta^{-1})^{-1}$. Indeed, the "time" Δv of sojourn in the region $\delta < \beta$, in accordance with the general theory, tends as $\beta \to 0$ to a constant equal to π, since we have from (23) and (25), using the definition $F(\alpha, \phi)$[2]

$$\Delta v = 2 \int_{\pi/2-\beta}^{\pi/2} d\psi \,[1 - (1 - \beta^2) \sin^2\psi]^{-1/2}$$

$$\approx 2 \int_0^{\beta} \frac{dv}{\sqrt{v^2 + \beta^2}} = \pi.$$

When $\beta \to 0$, χ^2 and ζ_0^2 remain bounded precisely because the length and the gradient in (27) remain constant in the defocusing region $\delta < \beta$ as $\beta \to 0$. The forced solution σ of Eq. (26) also remains bounded, for when $\beta \to 0$ the natural frequencies of σ remain complex and no resonance occurs in (26). Thus, when $\beta \to 0$ the integrals over the sections $\delta < \beta$ vanish in (28) and (29) owing to the increase in the denominator $K \sim \ln(1/\beta)$.

Let us consider the regions $\delta \gg \beta$, numbering them with the index n with increasing v. Inside each such region, according to (26) and (27) and according to the general properties of the Floquet functions, we have

$$\chi_n(v) = (Ae^{i(v+\alpha)} + Be^{-i(v+\beta)})\, e^{2\pi i\mu n}. \quad (30)$$

For ζ we can write

$$\zeta_n = a\,(\chi_n + \chi_n^*). \quad (31)$$

From (26) we get

$$\sigma_n = 2\cdot 6^{-1/2} \operatorname{Re} a^2 e^{4\pi i\mu n} \left(\tfrac{1}{5} A^2 e^{2i(v+\alpha)} + \tfrac{1}{5} B^2 e^{-2i(v+\beta)} + 2ABe^{i(\alpha+\beta)}\right) + 2\cdot 6^{-1/2} a^2 \left(A^2 + B^2 + \tfrac{2}{5} AB \cos(2v + \alpha + \beta)\right). \quad (32)$$

In (30)—(32) the quantities A, B, α, β, μ, and a (chosen real for convenience) are independent of the number n of the region.

Substituting (30), (32) in (28) and (29) we get

$$g = (2/w\sqrt{6})\, a^2 [\tfrac{1}{5}(A^4 + B^4) + 4A^2B^2], \quad (33)$$

$$\Delta\mu = (2/w\sqrt{6}) a^2 (2A^4 + 2B^4 + \tfrac{24}{5} A^2B^2) > g. \quad (34)$$

The inequality $\Delta\mu > g$ is satisfied with a large margin, and we see further that the ratio $\Delta\mu/g$ would be smaller were σ to be stable. Thus, the nonlinearity of the z-oscillations does not introduce any qualitative changes in the focusing mechanism considered above.

[1] Kotov, Kuznetsov, and Rubin, UFN **64**, 197 (1958).

[2] E. Jahnke and F. Emde, Tables of Functions, Dover, New York, 1943.

Translated by J. G. Adashko
228